Derived categories of coherent sheaves and integral functors

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Dedicated to my sister Carina.

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Resumo

Introduzem-se os conceitos de categoria derivada e de functores derivados entre categorias derivadas. A categoria derivada $D(\mathscr{A})$ de uma categoria abeliana $\mathscr{A}$ é obtida a partir da categoria de homotopia de complexos $K(\mathscr{A})$ através de localização formal em relação à classe de quasi-isomorfismos. Apesar de $D(\mathscr{A})$ não ser abeliana, esta categoria possui uma classe de triângulos singulares que desempenham um papel análogo às sequências curtas exatas. Os functores derivados são então definidos como objetos iniciais na categoria de extensões que preservam os triângulos singulares. Utilizando sequências espectrais e functores δ, aplica-se este formalismo a uma categoria abeliana concreta: a categoria de feixes coerentes Coh_X sobre uma variedade projetiva suave X. Por fim, introduz-se o conceito de functor integral. Dadas duas variedades X e Y nas condições anteriores, um functor integral é um certo tipo de functor $D(\mathrm{Coh}_X) \to D(\mathrm{Coh}_Y)$ entre as categorias derivadas de X e Y. Estes functores são utilizados frequentemente tanto em Geometria Algébrica como em Física Matemática devido à sua natureza geométrica.

Palavras-chave: Categorias Derivadas, Functores Derivados, Functores Integrais, Transformadas de Fourier-Mukai, Álgebra Homológica.

Abstract

We provide an introduction to the theory of derived categories and derived functors. To achieve this, we begin by studying the triangulated structure on the homotopy category of complexes over an abelian category $\mathscr{A}$, and define its derived category $D(\mathscr{A})$ by formally inverting quasi-isomorphisms. In this way, the derived category, although not abelian, inherits a canonical structure of a triangulated category, and derived functors are defined as initial objects in the category of extensions that preserve the distinguished triangles. We apply these constructions to the abelian category Coh_X of coherent sheaves on a smooth projective variety X, with the help of tools such as spectral sequences and δ-functors. Finally, we introduce integral functors. Given two such varieties X and Y, these are geometrically motivated functors $D^b(\mathrm{Coh}_X) \to D^b(\mathrm{Coh}_Y)$ between the derived categories, which are extensively used in present-day Algebraic Geometry and Mathematical Physics.

Keywords: Derived Categories, Derived Functors, Integral Functors, Fourier-Mukai Transforms, Homological Algebra.

Notation and necessary background

We assume familiarity with the basics of category theory. Usually, we denote an arbitrary category by $\mathscr{C}$ (or $\mathscr{D}, \ldots$), and use the notation $\mathrm{Obj}(\mathscr{C})$ and $\mathrm{Mor}(\mathscr{C})$ for its class of objects and its class of morphisms (which are also called arrows), respectively. We often write $A \in \mathscr{C}$ as a shorthand for $A \in \mathrm{Obj}(\mathscr{C})$. Given $A, B \in \mathrm{Obj}(\mathscr{C})$, we denote the subclass of morphisms $A \to B$ in $\mathscr{C}$ by $\mathrm{Mor}_{\mathscr{C}}(A, B)$, or simply $\mathrm{Mor}(A, B)$ if the category is understood from context. The composition operation on $\mathrm{Mor}(\mathscr{C})$ is denoted by the symbol $\circ$. A functor F from $\mathscr{C}$ to $\mathscr{D}$ is denoted by $F \colon \mathscr{A} \to \mathscr{B}$, whereas a natural transformation η between F and a second functor $G \colon \mathscr{C} \to \mathscr{D}$ is denoted, when possible, by a double arrow $\eta \colon F \Rightarrow G$. Unless mentioned, all functors are assumed to be covariant. A contravariant functor $F \colon \mathscr{C} \to \mathscr{D}$ is also described as a (covariant) functor $F \colon \mathscr{C}^{\mathrm{opp}} \to \mathscr{D}$, where $\mathscr{C}^{\mathrm{opp}}$ denotes the opposite category of $\mathscr{C}$. Full generality on the definition of a category is often not required for our purposes. Consequently, we can assume that any category $\mathscr{C}$ is, at least, locally small, *i.e.* $\mathrm{Mor}_{\mathscr{C}}(A, B)$ is a set for any $A, B \in \mathrm{Obj}(\mathscr{C})$.

Additive or abelian categories are usually denoted by $\mathscr{A}$ (or $\mathscr{B}, \ldots$). In this case, we use $\mathrm{Hom}_{\mathscr{A}}(A, B)$ for the set of morphisms $A \to B$. The kernel of a morphism $f \colon A \to B$ is denoted by $\ker f \colon \mathrm{Ker}\, f \to A$, where the lowercase symbol is used for the morphism, and the uppercase symbol for the object. Similar notation is used for the cokernel and for the (co)image. Equivalences between additive categories $\mathscr{A} \to \mathscr{B}$ are assumed to be additive functors. For any $A \in \mathscr{A}$, where $\mathscr{A}$ is at least additive, we use the notation $\mathrm{Hom}(A, -) \colon \mathscr{A} \to \mathrm{Ab}$ for the representable functor defined as follows: $B \mapsto \mathrm{Hom}(A, B)$ on objects; if $f \colon B \to C$ is an arrow in $\mathscr{A}$, its image under this functor is the map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ that sends $\varphi \in \mathrm{Hom}(A, B)$ to $(f \circ \varphi) \in \mathrm{Hom}(A, C)$. There is an analogous contravariant relative, $\mathrm{Hom}(-, A) \colon \mathscr{A}^{\mathrm{opp}} \to \mathrm{Ab}$.

We use the following notation for well known categories: Ab for the category of abelian groups, Rings for the category of commutative rings, Mod_A for the category of modules over a commutative ring A, and Vec_k for the category of vector spaces over a field k. In this text, we always assume rings to be commutative and unital.

We assume familiarity with the core concepts of modern Algebraic Geometry, at the level of [Vak17], Parts I-III. Throughout the text, we use $\mathrm{Op}(X)$ to denote the category of open subsets of a topological space X. This is the category whose objects are open subsets $U \subseteq X$, and that has a single morphism $V \to U$ (the inclusion) if and only if $V \subseteq U$. A presheaf on X with values in a concrete category $\mathscr{C}$ is hence viewed as a functor $\mathscr{F} \colon \mathrm{Op}(X)^{\mathrm{opp}} \to \mathscr{C}$. If $U \in \mathrm{Op}(X)$, use two notations indiscriminately for the sections over U: $\mathscr{F}(U)$ and $\Gamma(U, \mathscr{F})$. If $V \subseteq U$, the restriction morphism from U to V is denoted $\mathrm{res}_{U,V}$. If $f \in \mathscr{F}(U)$, we often use the shorthand notation $f|_V$ for $\mathrm{res}_{U,V}(f)$. The stalk of a sheaf $\mathscr{F}$ at a point p is denoted $\mathscr{F}_p$. The category of sheaves on X with values in $\mathscr{C}$ is denoted $\mathscr{C}_X$. Lastly, we specify notation we use for certain sheaves on X: if $i \colon U \hookrightarrow X$ is the inclusion of an open subset, given $\mathscr{F} \in \mathscr{C}_X$, we denote its restriction sheaf by $\mathscr{F}|_U$, [Vak17, 2.2.8]; given $\mathscr{G} \in \mathrm{Ab}_U$, its extension by zero is denoted $i_!(\mathscr{F}) \in \mathrm{Ab}_X$, [Vak17, 2.7.G]. The constant sheaf associated to $G \in \mathrm{Ab}$ is denoted $\underline{G}$, [Vak17, 2.2.E], and the skyscraper at $p \in X$ by the pushforward $j_{p,*}(\underline{G})$ by under inclusion $j \colon \{p\} \hookrightarrow X$, [Vak17, 2.2.9 and 2.2.12].

Contents

Chapter 1

Introduction

1.1 Brief history and motivation

The birth of Category Theory is usually attributed to a 1945 article by Mac Lane and Eilenberg entitled *General theory of natural equivalences*. Motivated by the relationship between a vector space and its bidual, the authors introduced ***categories*** as auxiliary objects to define the concepts of ***functors*** and ***natural transformations***, [EM45]. This formalism was built on the premise that

> *Mathematical objects are determined by – and understood by – the network of relationships they enjoy with all the other objects of their species.* (B. Mazur in [Maz17])

A ***chain complex*** in an additive category is simply a sequence of objects and morphisms, with the property that consecutive arrows should compose to the zero map. From such a complex, one can retrieve data, namely objects called the ***(co)homology groups***. These groups reflect interactions between first neighbors in the sequence. ***Homological algebra*** is the study of the information retained in these objects, and how this information can be translated into mathematical invariants. As an example, given a topological space, one can build a chain complex of abelian groups that encodes the number of "holes" of each dimension in the space, which is a topological invariant. This area of Mathematics also provides "*obstructions to carrying out various kinds of constructions; when the obstructions are zero, the construction is possible*", [Wei94]. Additionally, working with complexes and taking (co)homology is sometimes the only palpable way of doing actual calculations in problems requiring a high level of abstraction.

This field reached maturity in the mid-fifties with the publication of Cartan and Eilenberg's *Homological Algebra*, [CE56], where the authors formalized the central notions of exact sequences, injective and projective resolutions, and gave a first definition of derived functors. Given a short exact sequence of, say, modules over a commutative ring A,

$$0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0,$$

we can apply to it the representable functor $\mathrm{Hom}_A(P, -)$, to get a complex of abelian groups

$$0 \longrightarrow \mathrm{Hom}_A(P, M) \longrightarrow \mathrm{Hom}_A(P, N) \longrightarrow \mathrm{Hom}_A(P, N) \longrightarrow 0$$

which may not be exact at $\mathrm{Hom}_A(P, N)$ anymore[1]. However, one can always find abelian groups $\mathrm{Ext}^i_A(P, -)$ that augment the complex above to a sequence

[1] Hence, we say that $\mathrm{Hom}_A(P, -)$ is left exact.

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_A(P,M) \longrightarrow \mathrm{Hom}_A(P,N) \longrightarrow \mathrm{Hom}_A(P,K) \\ \hookrightarrow \mathrm{Ext}^1_A(P,M) \longrightarrow \mathrm{Ext}^1_A(P,N) \longrightarrow \mathrm{Ext}^1_A(P,K) \\ \hookrightarrow \mathrm{Ext}^2_A(P,M) \longrightarrow \cdots \end{array}$$

which is now exact. The assignments $\mathrm{Ext}^i_A(P,-)\colon \mathrm{Mod}_A \to \mathrm{Ab}$ were known at the time as right iterated satellites[2] of the functor $\mathrm{Hom}_A(P,-)$, and mathematicians knew how to compute them. Indeed, to compute $\mathrm{Ext}^i_A(P,M)$, one takes an injective resolution of M, that is, an exact sequence

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow I^3 \longrightarrow \dots,$$

where I^i are injective A-modules, removes the M term and applies $\mathrm{Hom}_A(P,-)$ to get the complex

$$0 \longrightarrow \mathrm{Hom}_A(P,I^0) \longrightarrow \mathrm{Hom}_A(P,I^1) \longrightarrow \mathrm{Hom}_A(P,I^2) \longrightarrow \mathrm{Hom}_A(P,I^3) \longrightarrow \dots .$$

This way, $\mathrm{Ext}^i_A(P,M)$ is precisely the cohomology of this sequence at $\mathrm{Hom}_A(P,I^i)$. This definition makes sense since any injective resolution of M is unique up to homotopy, and hence $\mathrm{Ext}^i_A(P,M)$ is determined up to canonical isomorphism, while also being functorial in M. Cartan and Eilenberg generalized this notion to arbitrary additive functors with source in the category of modules over a ring, and also described a way to compose the iterated satellites via ***spectral sequences***. Mac Lane wrote in a book review in the *Bulletin of the American Mathematical Society* that "*The authors' approach in this book can best be described in philosophical terms and as monistic: everything is unified.*", [Lan56].

One year after the publication of [CE56], and without previous access to this celebrated book, Grothendieck published the article *Sur quelques points d'algèbre homologique* in the *Tôhoku Mathematical Journal*. In *Tôhoku*, as it is nowadays called, Grothendieck drew a comparison between modules over a ring and sheaves of abelian groups, and noted that one can develop their homology theory in a similiar way. Furthermore, this article axiomatized ***abelian categories*** as we know them today, while also introducing the concept of ***equivalence of categories***.

In the early sixties, Grothendieck realized that the derived functors of Cartan and Eilenberg were too limited to allow several manipulations which arise naturally in the context of general abelian categories. In fact, he noted that one usually does not work with complexes that are defined up to homotopy, but only that are defined up to a weaker equivalence, called ***quasi-isomorphism***. One example of this behaviour occurs in the abelian category of sheaves of abelian groups on a topological space. Indeed, given such a sheaf $\mathscr{F}$, one can construct a resolution by flasque sheaves. The issue is that two such resolutions of $\mathscr{F}$ need not be homotopy equivalent, but only quasi-isomorphic. For Grothendieck, the solution to this problem was to invent a new category, called the ***derived category***.

A proper definition of the derived category of an abelian category first[3] appeared in Verdier's 1967

[2]These, of course, are now known as the right (*higher*) derived functors of $\mathrm{Hom}_A(P,-)$.

[3]A fair share of the theory was first made publicly available in a book by Hartshorne, consisting of lecture notes from a seminar given by Grothendieck at Harvard University in 1963/64, [Har66]. Verdier's thesis was published later in the nineties in *Société Mathématique de France*'s journal *Astérisque*, [Ver96].

PhD thesis, titled appropriately *Des catégories dérivées des catégories abéliennes*, which he conducted under the supervision of Grothendieck. There, we read the first sentence of the 250-page monograph: "*Nous proposons dans ce travail un formalisme de l'hyperhomologie*". Motivated by the fact that two *quasi*-isomorphic complexes give two *isomorphic* Cartan-Eilenberg satellite functors, he proposed to build a theory of derived functors that would be invariant under quasi-isomorphism, and not only under homotopy equivalence. To every abelian category $\mathscr{A}$, one would take the classical category $\mathrm{Com}_{\mathscr{A}}$ of complexes over $\mathscr{A}$, and construct a new one, its derived category $D(\mathscr{A})$, by formally inverting quasi-isomorphisms. Despite losing the structure of an abelian category during this procedure, $D(\mathscr{A})$ would carry a structure of ***distinguished triangles***, which would play the role of short exact sequences. Under this formalism, given a left exact functor $F\colon \mathscr{A} \to \mathscr{B}$ between abelian categories, he defined its ***total right derived functor*** $RF\colon D^*(\mathscr{A}) \to D^*(\mathscr{B})$ between the derived categories. Analogously to how an exact functor of abelian categories sends short exact sequences to short exact sequences, RF would send distinguished triangles in $D^*(\mathscr{A})$ to distinguished triangles in $D^*(\mathscr{B})$.

Derived categories found important roles in several areas of Mathematics in the years to follow, but especially in Algebraic Geometry. This field underwent a revolution during the mid 20th century, largely due to Grothendieck's overall of the subject building upon the concept of ***schemes***. These objects were defined in his eight-fascicle 1500-page long *Éléments de géométrie algébrique*, published from 1960 through 1967. The study of schemes is, to a large extent, the study of sheaves defined over them. Being a (locally) ringed space, a scheme X carries a structure sheaf of rings $\mathscr{O}_X$, called its ***sheaf of regular functions***. Due to this inherent structure, there is a natural choice of sheaves to consider on a scheme, sheaves of $\mathscr{O}_X$-modules. Among these type of sheaves, there are two important variants, that of ***quasi-coherent sheaves***, and that of ***coherent sheaves***, which carry special relevance due to their algebraic nature[4]. Given a scheme X, the category Coh_X of coherent sheaves on X is an abelian category. A morphism of schemes $f\colon X \to Y$, induces two useful functors $f_*\colon \mathrm{Coh}_X \to \mathrm{Coh}_Y$ and $f^*\colon \mathrm{Coh}_Y \to \mathrm{Coh}_X$ between them, which, in most cases, are not exact. It is then natural to use the formalism of derived functors to help solve the complications that non-exactness brings.

The use of Verdier's derived functors has been ubiquitous in the study of coherent sheaves on (projective) varieties since the eighties. In [Muk81], Mukai used compositions of these functors to construct geometrically motivated equivalences between the derived categories of dual abelian varieties. In the article, the author draws a comparison between these type of equivalences and the ones provided by classical Fourier transforms in L^2-spaces. Any functor of the type Mukai introduced became thereafter baptized an ***integral functor*** and, in the case of the functor being an equivalence, a ***Fourier-Mukai transform***. In the 2000s, Orlov showed that, under certain conditions, *any* equivalence of categories between the derived categories of coherent sheaves on smooth projective varieties is in fact a Fourier-Mukai transform (Corollary 5.2.9).

In the present day, Fourier-Mukai transforms are extensively used in Mathematical Physics as a powerful source of dualities. An example of such a duality is the Homological Mirror Symmetry conjecture, which relates the algebrogeometric properties of a variety with the symplectic geometry of its dual.

[4]As we will see in Chapter 4, the study of quasi-coherent sheaves on an affine scheme $\operatorname{Spec} A$ is equivalent to the study of modules over the ring A.

1.2 Objectives and thesis outline

Our goal is to provide an academic, yet concise, introduction to the derived category of coherent sheaves on a regular projective scheme over a field. To our knowledge, there is no reference that introduces this theory in a self-contained detailed manner, starting from the basics of triangulated categories up until the definition of integral functors. Therefore, our discussion is not meant to be broad, but somewhat deep. Our writing style is, hopefully, pedagogical in nature. Our main references are [GM03], [Wei94], [Ver96] and [Nee01] for homological algebra results, and [Vak17], [Har77] and [Sta21] for algebraic geometry ones. We remark that, due to the nature of the subject, most of the statements about derived functors have a *dual* analogue. If the context makes it clear what the dual statements are, these will not be explicitly written.

The text is divided into five chapters, the first one consisting of these introductory notes. Due to guidelines of the university, there are also four complementary appendices.

The second chapter introduces the derived category of an abelian category, using the theory developed by Verdier and Grothendieck. We start by discussing the formal procedure of localizing a category, and introduce what we mean by a triangulated category. The derived category is constructed by formally inverting certain arrows in the homotopy category of complexes. Appendix A focuses on the interplay between an abelian and a triangulated structure on a category. A particularly long proof is provided in Appendix B.

Chapter 3 introduces derived functors. Our main theorem is an existence result for derived functors (Theorem 3.1.10), which relies on the existence of classes of adapted objects to the functor we try to derive. The classical derived functors of Cartan-Eilenberg are retrieved from the "total" ones in Section 3.3. We introduce δ-functors in Subsection 3.3.1, which are an abstraction of the higher derived functors. Shortly after, we show that injective objects are adapted to right derive any left exact functor, and explore how the derived category of an abelian category $\mathscr{A}$ relates to the derived category of a full abelian subcategory $\mathscr{B} \subseteq \mathscr{A}$. We end the chapter showing how the derived functor formalism handles composition. We introduce spectral sequences in Appendix C and show how they were classically used to compose higher derived functors.

The fourth chapter is devoted to recalling important results about abelian categories of sheaves defined on a scheme. To be specific, given a scheme X, we deal with the categories of sheaves of $\mathscr{O}_X$-modules, quasicoherent sheaves and coherent sheaves on X, and probe each category for classes adapted to four half-exact functors: global sections, pushforward, tensor product and pullback. We take the approach of requiring the least conditions on X as possible. However, some conditions are standard to impose, namely Noetherianess (Section 4.2). Section 4.3 is arguably the most important part of this text, which is where we construct the derived functors of the aforementioned half-exact functors. Most of the statements in this chapter are given without proof, or referred to Appendix D.

Lastly, Chapter 5 is a brief introduction to integral functors and Fourier-Mukai transforms. We state no more than the main definitions and most basic properties. Our last remark concerns how knowing the category of coherent sheaves on a scheme X determines X up to isomorphism.

Chapter 2

The derived category

2.1 Formal localization of categories

Let $\mathscr{C}$ be any category and $\mathcal{S}$ any subclass of morphisms in $\mathscr{C}$. Motivated by the construction of the localization of a ring with respect to a multiplicative subset, our goal is to define a category $\mathscr{C}[\mathcal{S}^{-1}]$, together with a functor $Q\colon \mathscr{C} \to \mathscr{C}[\mathcal{S}^{-1}]$ which maps arrows in $\mathcal{S}$ to isomorphisms $\mathscr{C}[\mathcal{S}^{-1}]$. Moreover, Q should be initial with respect to functors out of $\mathscr{C}$ that have this property. In other words, Q should satisfy the following universal property: for any category $\mathscr{D}$ with a functor $F\colon \mathscr{C} \to \mathscr{D}$ sending morphisms in $\mathcal{S} \subseteq \mathscr{C}$ to isomorphisms in $\mathscr{D}$, there exists a unique morphism $G\colon \mathscr{C}[\mathcal{S}^{-1}] \to \mathscr{D}$ making the following diagram commute:

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & \mathscr{D} \\ {\scriptstyle Q}\downarrow & \nearrow_{\exists\,!\,G} & \\ \mathscr{C}[\mathcal{S}^{-1}] & & \end{array} \quad . \tag{2.1}$$

As a motivation for the definitions that will follow, consider the informal discussion below. Further details can be found in [GM03, III.2].

A quiver (or directed multigraph) is a 4-tuple (V, E, s, t) consisting of two sets V (the set of vertices) and E (the set of edges), and two morphisms $s, t\colon E \to V$ such that, for each edge $e \in E$, $s(e)$ is the initial vertex of e and $t(e)$ is its target vertex, [Die16]. A standard way of defining a (small) category $\mathscr{C}$ is by considering a directed multigraph Γ, equipped with an associative composition operation on edges and with a distinguished self-loop for each vertex. The set of objects of $\mathscr{C}$ is then defined to be the vertex set of Γ, a morphism in $\mathscr{C}$ is a directed edge of Γ, composition of morphisms is defined, and so are the identity morphisms on each object, by using the self-loops.

With this in mind, an intuitive first approach to the construction of a pair $(\mathscr{C}[\mathcal{S}^{-1}], Q)$ as above is to construct a new directed multigraph Γ' from Γ, such that morphisms in $\mathcal{S}$ "become invertible". Informally, we describe this procedure in the following steps:

- We define the vertex set of Γ' to be the same as the vertex set of the quiver associated to $\mathscr{C}$, Γ.
- We introduce variables x_s, one for each morphism $s \in \mathcal{S}$.
- Let E' be the union of the edge set of Γ and a set consisting of a directed edge for each new variable x_s. The directed edge corresponding to x_s has the same vertices as the edge of the quiver of $\mathscr{C}$ corresponding to s, but with the opposite orientation.
- A path in E' is a finite sequence of elements of E' such that the target vertex of any edge coincides with the source vertex of the next edge. Two paths in E' that agree on their source and target

vertices are said to be equivalent if they can be concatenated by repeated use of the following two equivalences:

  - Consecutive edges coming from Γ (*i.e.* corresponding to morphisms in $\mathscr{C}$) can be replaced by their composition;
  - A path $X \xrightarrow{s} Y \xrightarrow{x_s} X$ (respectively, $Y \xrightarrow{x_s} X \xrightarrow{s} Y$) can be replaced by the edge $X \xrightarrow{\mathrm{id}_X} X$ (respectively, $Y \xrightarrow{\mathrm{id}_Y} Y$).

- With these definitions, there is an associative composition operation on the set of paths in E'. We define the set of edges of Γ' to be the set of equivalence classes of paths in E'.

As expected, we define $\mathscr{C}[\mathcal{S}^{-1}]$ to be the category corresponding to the directed multigraph Γ'. There is a natural functor $Q\colon \mathscr{C} \to \mathscr{C}[\mathcal{S}^{-1}]$ that sends each morphism $X \xrightarrow{f} Y$ in $\mathscr{C}$ to the equivalence class of its corresponding path (of lenght 1) in Γ'. It can be shown that this pair $(\mathscr{C}[\mathcal{S}^{-1}], Q)$ has the universal property described in diagram (2.1), [GM03, III.2]. However, this construction is cumbersome to work with. For instance, note that, with this description of $\mathscr{C}[\mathcal{S}^{-1}]$, a morphism $A \to B$ is an equivalence classes of paths of the form

$$A \xrightarrow{f_1} X_1 \xrightarrow{x_{s_1}} X_2 \xrightarrow{f_2} X_3 \xrightarrow{x_{s_2}} X_4 \longrightarrow \ldots \longrightarrow X_k \xrightarrow{f_k} B \ . \tag{2.2}$$

In particular, given any two morphisms $A \to B$ in $\mathscr{C}[\mathcal{S}^{-1}]$, there is no guarantee that they can be represented by paths of the same length. This poses an obstacle if we want to do addition on morphisms. More precisely, if $\mathscr{C}$ has the structure of an additive category, $\mathscr{C}[\mathcal{S}^{-1}]$ may not inherit this structure.

Consider the path (2.2). If are able to somehow "shift" all the edges corresponding to morphisms in $\mathcal{S}$ to the left or to the right of the path, then we can represent this (and hence all) morphism(s) in the localization by equivalence classes of paths of length two. This discussion motivates the following definition.

Definition 2.1.1. Let $\mathscr{C}$ be any category and $\mathcal{S}$ a subclass of morphisms of $\mathscr{C}$. We say that $\mathcal{S}$ is a ***localizing class*** if the following axioms hold:

A1) $\mathcal{S}$ is multiplicatively closed, *i.e.* for every $A \in \mathscr{C}$, $\mathrm{id}_A \in \mathcal{S}$, and if $f, g \in \mathcal{S}$ are two composable morphisms, their composition is also in $\mathcal{S}$.

A2) Given an arbitrary morphism $X \to Y$ (in $\mathscr{C}$) and a morphism $Z \to Y$ in $\mathcal{S}$, there exists an object $T \in \mathscr{C}$, together with morphisms $T \dashrightarrow Z$ (in $\mathscr{C}$) and $T \dashrightarrow X$ in $\mathcal{S}$, such that the diagram

$$\begin{array}{ccc} T & \dashrightarrow & Z \\ {\scriptstyle \mathcal{S}\ni}\downarrow & & \downarrow{\scriptstyle \in\mathcal{S}} \\ X & \longrightarrow & Y \end{array}$$

commutes.

A3) The dual statement of A2) also holds, *i.e.* any solid diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ {\scriptstyle \mathcal{S}\ni}\big\downarrow & & \big\downarrow{\scriptstyle \in\mathcal{S}} \\ Z & \dashrightarrow & T \end{array}$$

can be completed to a commutative square by the dashed arrows.

A4) Given two morphisms $f, g\colon X \to Y$, there exists a morphism $Y \xrightarrow{s\in\mathcal{S}} T$ such that $s \circ f = s \circ g$ if and only if there exists a morphism $U \xrightarrow{s'\in\mathcal{S}} X$ such that $f \circ s' = g \circ s'$.

Axiom A2 (respectively, A3) above is known as the ***right*** (respectively, ***left***) ***Ore condition***. Note that, together with axiom A1, they enable one to "shift morphisms in $\mathcal{S}$" to the left (respectively, right) in the description of $\mathscr{C}[\mathcal{S}^{-1}]$ discussed above. In fact, suppose we have a morphism $X \to Z$ in $\mathscr{C}[\mathcal{S}^{-1}]$ given by the equivalence class of a path $X \xrightarrow{f} Y \xrightarrow{x_s} Z$. By axiom A2, we can find morphisms $g\colon T \to Z$ in $\mathscr{C}$ and $s'\colon T \to X$ in $\mathcal{S}$ such that $s \circ g = f \circ s'$. But then, as edges in $\mathscr{C}[\mathcal{S}^{-1}]$,

$$\left[\ X \xrightarrow{\ f\ } Y \xrightarrow{\ x_s\ } Z\ \right] \sim \left[\ X \xrightarrow{\ x_{s'}\ } T \xrightarrow{\ s'\ } X \xrightarrow{\ f\ } Y \xrightarrow{\ x_s\ } Z\ \right] \sim$$
$$\sim \left[\ X \xrightarrow{\ x_{s'}\ } T \xrightarrow{\ g\ } Z \xrightarrow{\ s\ } Y \xrightarrow{\ x_s\ } Z\ \right] \sim \left[\ X \xrightarrow{\ x_{s'}\ } T \xrightarrow{\ g\ } Z\ \right],$$

proving the claim. The next definition follows naturally from this discussion.

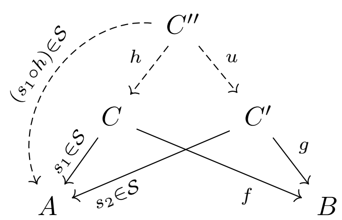
Definition/Proposition 2.1.2. Let $\mathscr{C}$ be a category and $\mathcal{S} \subseteq \mathrm{Mor}(\mathscr{C})$ a localizing class. We define the ***localization of*** $\mathscr{C}$ ***with respect to*** $\mathcal{S}$ as the category $\mathscr{C}[\mathcal{S}^{-1}]$ such that:

i) $\mathrm{Obj}(\mathscr{C}[\mathcal{S}^{-1}]) = \mathrm{Obj}(\mathscr{C})$.

ii) Morphisms $A \to B$ are equivalence classes of roof diagrams from A to B. More precisely, up to equivalence, a morphism $A \to B$ is a pair (a ***roof***) $(s\colon C \to A, f\colon C \to B)$,

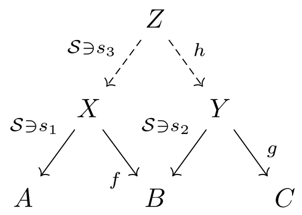
$$\begin{array}{ccc} & C & \\ {\scriptstyle \mathcal{S}\ni s}\swarrow & & \searrow{\scriptstyle f} \\ A & & B \end{array} .$$

We denote the equivalence class of this pair by fs^{-1}, which is indicative of the fact that going from A to B is equal to going through the "inverse" of s and then through f. Two roofs from A to B, (s_1, f) and (s_2, g), are equivalent if we can complete the solid diagram



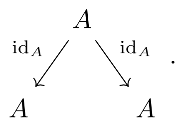
by the dashed arrows (which are in $\mathscr{C}$) so that the two squares commute (*i.e.* $s_1 \circ h = s_2 \circ u$ and $f \circ h = g \circ u$) and $s_1 \circ h \in \mathcal{S}$.

iii) Given two morphisms $A \xrightarrow{fs_1^{-1}} B$ and $B \xrightarrow{gs_2^{-1}} C$, the composition morphism is given in the following way: fill in the solid diagram



by completing the solid diagram $X \xrightarrow{f} B \xleftarrow{s_2} Y$ to the shown commutative square (which is always possible, by axiom A2 of Definition 2.1.1). The composition of the equivalence classes is defined as the equivalence class of the roof $(s_1 \circ s_3, g \circ h)$.

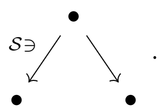
iv) The identity morphism $A \to A$ is defined as the equivalence class of the roof $(\mathrm{id}_A, \mathrm{id}_A)$:



Proof. There are several things to check here, namely: the relation on roofs defined above is an equivalence relation; the composition of equivalence classes is well-defined (*i.e.* does not depend on the chosen completion); composition is associative; the identity morphisms satisfy the required axioms. The proof can be found in [GM03, III.8]. □

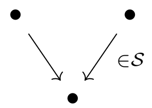
Remark 2.1.3*.* A few comments are to be made regarding Definition 2.1.1 and Definition/Proposition 2.1.2. If $\mathscr{C}$ is a category and $\mathcal{S}$ is a subclass of morphisms of $\mathscr{C}$, we say that $\mathcal{S}$ is a *right Ore system* if axioms A1, A2 and the necessary direction of axiom A4 of Definition 2.1.1 hold. Dually, $\mathcal{S}$ is called a *left Ore system* if axioms A1, A3 and the sufficient condition of axiom A4 of said definition hold. In this way, $\mathcal{S}$ is a localizing class (as in Definition 2.1.1) if it is both a left and right Ore system.

A detailed look at the proof of Definition/Proposition 2.1.2 shows that, in order to define what we called "the localization of $\mathscr{C}$ with respect to $\mathcal{S}$", one actually just needs that $\mathcal{S}$ is a right Ore system. Consequently, we call $\mathscr{C}[\mathcal{S}^{-1}]$ a *right calculus of fractions*, since morphisms are (equivalence classes of) *spans* or *left $\mathcal{S}$-roofs*,



Denote this construction temporarily by $\mathscr{C}[\mathcal{S}^{-1}]_{\mathrm{right}}$. We will show in Proposition 2.1.5 that $\mathscr{C}[\mathcal{S}^{-1}]_{\mathrm{right}}$ comes equipped with a functor $Q_{\mathrm{right}} \colon \mathscr{C} \to \mathscr{C}[\mathcal{S}^{-1}]_{\mathrm{right}}$ satisfying the universal property of diagram (2.1).

Dually, if we start with a left Ore system $\mathcal{S}$, we built its *left calculus of fractions* $\mathscr{C}[\mathcal{S}^{-1}]_{\mathrm{left}}$. Its morphisms are now (equivalence classes of) *cospans* or *right $\mathcal{S}$-roofs*,



It can be shown that $\mathscr{C}[\mathcal{S}^{-1}]_{\mathrm{left}}$ is also equipped with a functor $Q_{\mathrm{left}}\colon \mathscr{C} \to \mathscr{C}[\mathcal{S}^{-1}]_{\mathrm{left}}$ that satisfies the same universal property as Q_{right}. Therefore, if $\mathcal{S}$ is a localizing class, $\mathscr{C}[\mathcal{S}^{-1}]_{\mathrm{right}}$ and $\mathscr{C}[\mathcal{S}^{-1}]_{\mathrm{left}}$ are canonically isomorphic, and calling these objects *the* localization $\mathscr{C}[\mathcal{S}^{-1}]$ makes sense. Further details can be found in [Sta21, Tag 04VB].

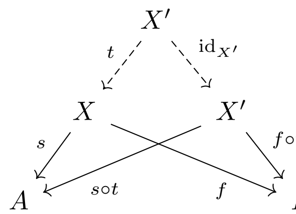
The natural question to ask now is: why are we only interested in localizing classes and two-sided calculus of fractions? The answer to this question will not be clear until Section 3.2, where we use the localization formalism to construct left and right derived functors. Indeed, if $F\colon \mathscr{C} \to \mathscr{D}$ is a functor, the construction of its "left derived" (respectively, "right derived") counterpart (whatever this means at the moment) will require $\mathscr{C}$ to have a right (respectively, left) calculus of fractions. Since we are interested in constructing both "left and right counterparts", we want $\mathscr{C}$ to have a localizing class.

The next proposition establishes some useful properties of this construction, namely the possibility of finding a "common denominator" of a finite collection of morphisms in the localization.

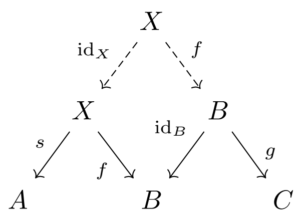
Proposition 2.1.4. Let $\mathscr{C}$, $\mathcal{S}$ and $\mathscr{C}[\mathcal{S}^{-1}]$ be as in Definition/Proposition 2.1.2.

a) Cancellation laws:

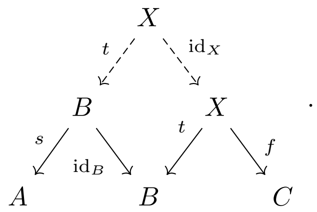
i) If $f\colon X \to B$ is a morphism in $\mathscr{C}$, $s\colon X \to A$ is a map in $\mathcal{S}$, and $t\colon X' \to X$ is a map in $\mathscr{C}$ such that $s \circ t \in \mathcal{S}$, then $(f \circ t)(s \circ t)^{-1} = fs^{-1}$,



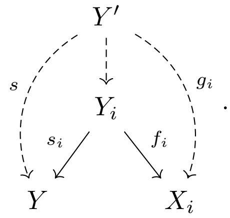
ii) If $f\colon X \to B$ and $g\colon B \to C$ are morphisms in $\mathscr{C}$, and $s\colon X \to A$ is a morphism in $\mathcal{S}$, then $(g\mathrm{id}_B^{-1}) \circ (fs^{-1}) = (g \circ f)s^{-1}$,



iii) If $f\colon X \to C$ is a morphism in $\mathscr{C}$, and $s\colon B \to A$ and $t\colon X \to B$ are morphisms in $\mathcal{S}$, then $(ft^{-1}) \circ (\mathrm{id}_B s^{-1}) = f(s \circ t)^{-1}$,



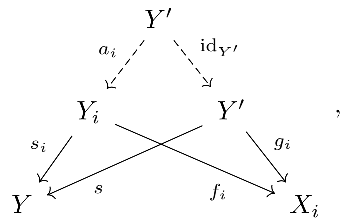
b) Finding common denominators: if $\{Y \xrightarrow{f_i s_i^{-1}} X_i\}$ is a finite collection of morphisms of $\mathscr{C}[\mathcal{S}^{-1}]$ with the same source, there exists a finite collection of morphisms $\{g_i \colon Y' \to X_i\}$ of $\mathscr{C}$ and a morphism $s \colon Y' \to Y$ of $\mathcal{S}$ such that, for each i, $f_i s_i^{-1} = g_i s^{-1}$,



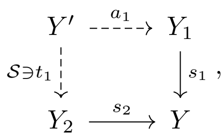
c) Equality of morphisms with same denominator: if $f, g \colon Y' \to Y$ are two morphisms of $\mathscr{C}$ and $s \colon X \to Y'$ is a morphism of $\mathcal{S}$, then $fs^{-1} = gs^{-1}$ in $\mathscr{C}[\mathcal{S}^{-1}]$ if and only if there exists a morphism $a \colon Y'' \to Y'$ in $\mathscr{C}$ such that $s \circ a \in \mathcal{S}$ and $f \circ a = g \circ a$.

Proof. The details of the proofs of assertions a) i)-iii) are omitted (the diagrams are essentially the proof).

For b), suppose that $s_i \colon Y \to Y_i$ and $f_i \colon Y_i \to Y$. Then, it suffices to prove that there exist $s \colon Y' \to Y$ in $\mathcal{S}$ and $a_i \colon Y' \to Y_i$ such that $s = s_i \circ a_i$:

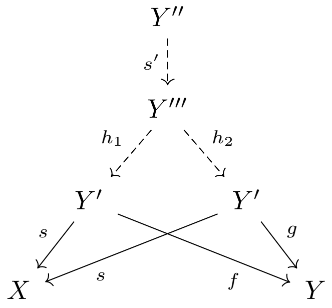


where we set $g_i \coloneqq f_i \circ a_i$. Let us show that we can do this when we have two roofs $f_1 s_1^{-1}$ and $f_2 s_2^{-1}$. We can fill the square



and so, we take $s = s_2 \circ t_1 \in \mathcal{S}$ and $a_2 = t_1$. Now, it is clear that we can use this procedure to reduce the case of finding a common denominator of n morphisms to the case of finding a common denominator of $n-1$ morphisms, so we are done by induction.

For c), the condition is clearly sufficient since $fs^{-1} = (f \circ a)(a \circ s)^{-1} = (g \circ a)(a \circ s)^{-1} = gs^{-1}$, by part a) i). Conversely, suppose $fs^{-1} = gs^{-1}$, *i.e.* there exist $h_1 \colon Y''' \to Y'$ and $h_2 \colon Y''' \to Y'$ such that $s \circ h_1 = s \circ h_2 \in \mathcal{S}$ and $f \circ h_1 = g \circ h_2$, as in the following diagram:



By axiom A4 of Definition 2.1.1, there exists $s' \colon Y'' \to Y'''$ in $\mathcal{S}$ such that $h_1 \circ s' = h_2 \circ s'$. Then, define $a = h_1 \circ s' = h_2 \circ s'$. $\square$

Finally, the next proposition validates that our construction of $\mathscr{C}[\mathcal{S}^{-1}]$ has the universal property mentioned at the begining of this section.

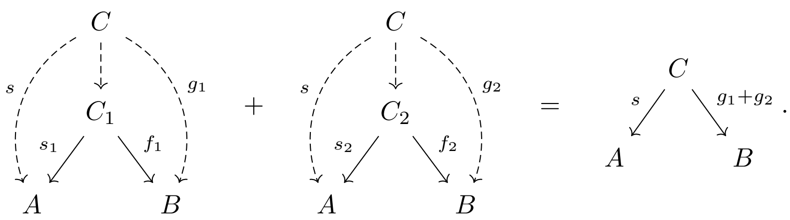
Proposition 2.1.5. Let $\mathscr{C}$, $\mathcal{S}$ and $\mathscr{C}[\mathcal{S}^{-1}]$ be as in Definition/Proposition 2.1.2. There is a functor $Q \colon \mathscr{C} \to \mathscr{C}[\mathcal{S}^{-1}]$ defined by $Q(X) = X$ for every $X \in \mathrm{Obj}(\mathscr{C})$, and $Q(f \colon X \to Y) = f\mathrm{id}_X^{-1}$. This functor sends every morphism in $\mathcal{S}$ to an isomorphism in $\mathscr{C}[\mathcal{S}^{-1}]$. Moreover, Q is initial with respect to this property, as in diagram (2.1).

Proof. Q behaves well with respect to composition by Proposition 2.1.4 a) ii) and $Q(\mathrm{id}_X) = \mathrm{id}_X \mathrm{id}_X^{-1} =: \mathrm{id}_X^{\mathscr{C}[\mathcal{S}^{-1}]}$. If $s \colon X \to Y$ is a morphism in $\mathcal{S}$ then it is easy to check that $s\mathrm{id}_X^{-1}$ has inverse $\mathrm{id}_X s^{-1}$. Then, if G exists as in diagram (2.1), $G(fs^{-1}) = G((f\mathrm{id}_X^{-1}) \circ (\mathrm{id}_X s^{-1})) = G(Q(f)) \circ G(Q(s))^{-1} = F(f) \circ F(s)^{-1}$. Therefore, we define G by this expression. $\square$

Having already established that it is possible to find common denominators, if we have an addition operation on the morphisms in $\mathscr{C}$, we can extend it to an addition operation on morphisms in the localization $\mathscr{C}[\mathcal{S}^{-1}]$. This is the content of the next proposition.

Proposition 2.1.6. Let $\mathscr{C}$ be an additive category and $\mathcal{S} \subseteq \mathrm{Mor}(\mathscr{C})$ a localizing class. Then, there is a canonical structure of an additive category on $\mathscr{C}[\mathcal{S}^{-1}]$.

i) For every $A, B \in \mathrm{Obj}(\mathscr{C})$, $\mathrm{Mor}_{\mathscr{C}[\mathcal{S}^{-1}]}(A, B)$ is an abelian group for the operation defined as follows: we define the sum of two morphisms $A \xrightarrow{f_1 s_1^{-1}} B$ and $A \xrightarrow{f_2 s_2^{-1}} B$ by first finding a common denominator using Proposition 2.1.4 b); if $f_1 s_1^{-1} = g_1 s^{-1}$ and $f_2 s_2^{-1} = g_2 s^{-1}$ (for $s \in \mathcal{S}$), we define the sum to be $(g_1 + g_2)s^{-1}$:



This definition does not depend on the choice of C, g_1, g_2 and s. Moreover, the composition operation of $\mathscr{C}[\mathcal{S}^{-1}]$ is distributive over this addition operation.

ii) The zero object of $\mathrm{Mor}_{\mathscr{C}[\mathcal{S}^{-1}]}(A, B)$ is the zero object of $\mathscr{C}$.

iii) The product of two objects $A, B \in \mathrm{Obj}(\mathscr{C})$ in $\mathrm{Mor}_{\mathscr{C}[\mathcal{S}^{-1}]}(A, B)$ is the triple $(A \times B, p_1 \mathrm{id}_{A\times B}^{-1}, p_2 \mathrm{id}_{A\times B}^{-1})$, where $A \times B$ is the product in $\mathscr{C}$ and $p_1 \colon A \times B \to A$, $p_2 \colon A \times B \to B$ are the natural projection morphisms in $\mathscr{C}$.

In addition, the natural functor $Q \colon \mathscr{C} \to \mathscr{C}[\mathcal{S}^{-1}]$ of Proposition 2.1.5 is additive.

Proof. We leave (almost all) the details of the proof omitted – see [Mil21, pgs. 26-35]. Note that the identity element for the sum in $\mathrm{Mor}_{\mathscr{C}[\mathcal{S}^{-1}]}(A, B)$ is the morphism

$$A \xleftarrow{\mathrm{id}_A} A \xrightarrow{0} B \underset{\text{Prop 2.1.4 a) i)}}{=} A \xleftarrow{s} C \xrightarrow{0} B \quad \text{for any } s \in \mathcal{S}.$$

□

Remark 2.1.7. Notice that if $\mathcal{S}$ is a localizing class of an additive category $\mathscr{C}$, axiom A4 of Definition 2.1.1 can be replaced by:

A4)' Given $f \colon X \to Y$, $\exists\, Y \xrightarrow{s \in \mathcal{S}} T$ such that $s \circ f = 0$ if and only if $\exists\, U \xrightarrow{s' \in \mathcal{S}} X$ such that $f \circ s' = 0$.

We finish this section with the proposition below, a result that will be used extensively throughout this text.

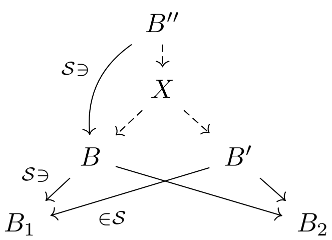
Proposition 2.1.8. Let $\mathscr{C}$ be a category and $\mathcal{S}$ a localizing class in $\mathscr{C}$. If $\mathscr{B}$ is a full subcategory of $\mathscr{C}$, consider the class $\mathcal{S}_{\mathscr{B}} = \mathcal{S} \cap \mathrm{Mor}(\mathscr{B})$. Suppose that $\mathcal{S}_{\mathscr{B}}$ is a localizing class in $\mathscr{B}$. Then, the natural functor $\mathscr{B}[\mathcal{S}_{\mathscr{B}}^{-1}] \to \mathscr{C}[\mathcal{S}^{-1}]$ is fully faithful if one of following conditions hold: given any $X \in \mathrm{Obj}(\mathscr{C})$, and

i) any morphism $X \to B$ in $\mathcal{S}$ with $B \in \mathrm{Obj}(\mathscr{B})$, there exists $B' \in \mathrm{Obj}(\mathscr{B})$ and a morphism $B' \to X$, such that the composition $B' \to X \to B$ is in $\mathcal{S}$;

ii) any morphism $B \to X$ in $\mathcal{S}$ with $B \in \mathrm{Obj}(\mathscr{B})$, there exists $B' \in \mathrm{Obj}(\mathscr{B})$ and a morphism $X \to B'$, such that the composition $B \to X \to B'$ is in $\mathcal{S}$.

Proof. Condition i) is adapted the description of the localizations as right calculi of fractions, while condition ii) is adapted to description as left calculi of fractions (Remark 2.1.3). We prove the claim assuming condition i).

In order to prove faithfulness, suppose $B_1 \xleftarrow{\in \mathcal{S}} B \to B_2$ and $B_1 \xleftarrow{\in \mathcal{S}} B' \to B_2$ are two morphisms in $\mathscr{B}[\mathcal{S}_{\mathscr{B}}^{-1}]$ that are equivalent in $\mathscr{C}[\mathcal{S}^{-1}]$. Then, there exists $X \in \mathrm{Obj}(\mathscr{C})$ and arrows $X \to B$ and $X \to B'$ such that the composition $X \to B \to B_1$ is in $\mathcal{S}$. By hypothesis, there exists $\mathrm{Obj}(\mathscr{B}) \ni B'' \to X$ as in the diagram



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and the morphisms are equivalent in $\mathscr{B}[\mathcal{S}_{\mathscr{B}}^{-1}]$. Regarding fullness, suppose $B_1 \xleftarrow{s \in \mathcal{S}} X \xrightarrow{f} B_2$ is a morphism in $\mathscr{C}[\mathcal{S}^{-1}]$ with source and target in $\mathscr{B}$. Then there exists $B' \in \mathrm{Obj}(\mathscr{B})$ and $B' \xrightarrow{t} X$, and $(f \circ t)(s \circ t)^{-1} = fs^{-1}$ by Proposition 2.1.4 a) i). □

2.2 Triangulated categories

Definition 2.2.1. Let $(\mathscr{D}, T)$ be a pair consisting of an additive category $\mathscr{D}$ and an additive automorphism $T \colon \mathscr{D} \to \mathscr{D}$. A ***triangle*** of the pair $(\mathscr{D}, T)$ is a sextuplet (A, B, C, f, g, h), where $A, B, C \in \mathscr{D}$, $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$ and $h \in \mathrm{Hom}(C, T(A))$, which we denote by

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \xrightarrow{\ h\ } T(A).$$

A ***morphism of triangles*** $(A, B, C, f, g, h) \to (D, E, F, x, y, z)$ is given by morphisms $A \xrightarrow{\alpha} D$, $B \xrightarrow{\beta} E$ and $C \xrightarrow{\gamma} F$, such that the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & T(A) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ D & \xrightarrow{x} & E & \xrightarrow{y} & F & \xrightarrow{z} & T(D) \end{array}$$

is commutative. This morphism is an ***isomorphism*** if α, β and γ are isomorphisms.

Definition 2.2.2. Let $\mathscr{D}$ be an additive category. The structure of a ***triangulated category*** on $\mathscr{D}$ is the data of an additive automorphism $T \colon \mathscr{D} \to \mathscr{D}$, called the ***shift functor***, and a class $\mathcal{T}$ of triangles of the pair $(\mathscr{D}, T)$ that are required to satisfy the following axioms:

A1) i) Any triangle of the form $A \xrightarrow{\mathrm{id}_A} A \to 0 \to T(A)$ is in $\mathcal{T}$.

ii) Any triangle isomorphic to a triangle in $\mathcal{T}$ is in $\mathcal{T}$ itself.

iii) For each morphism $f \colon A \to B$ there exists a triangle in $\mathcal{T}$ of the form

$$A \xrightarrow{\ f\ } B \longrightarrow C \longrightarrow T(A)\ .$$

A2) The triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$ is in $\mathcal{T}$ if and only if the triangle $B \xrightarrow{g} C \xrightarrow{h} T(A) \xrightarrow{-T(f)} T(B)$ is in $\mathcal{T}$.

A3) Every solid diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & T(A) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \vdots{\scriptstyle \gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & T(D) \end{array}$$

whose rows are triangles in $\mathcal{T}$ can be completed *(not necessarily uniquely)* by γ to a morphism of triangles.

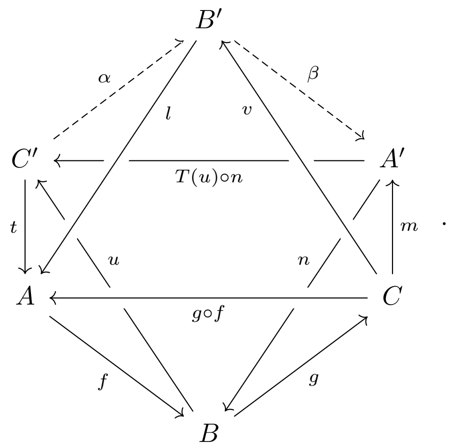
A4) Octahedron axiom: let $f\colon A \to B$ and $g\colon B \to C$ be two morphisms; using A1 iii) complete f, $g \circ f$ and g to triangles in $\mathcal{T}$:

$$A \xrightarrow{f} B \xrightarrow{u} C' \xrightarrow{t} T(A),$$
$$A \xrightarrow{g\circ f} C \xrightarrow{v} B' \xrightarrow{l} T(A),$$
$$B \xrightarrow{g} C \xrightarrow{m} A' \xrightarrow{n} T(B).$$

Then, we require the existence of morphisms $\alpha\colon C' \to B'$ and $\beta\colon B' \to A'$ fitting into a triangle

$$C' \overset{\alpha}{\dashrightarrow} B' \overset{\beta}{\dashrightarrow} A' \xrightarrow{T(u)\circ n} T(C'),$$

which is in $\mathcal{T}$. Moreover, these morphisms are required to satisfy certain commutativity conditions, which are best explained by observing the following diagram:



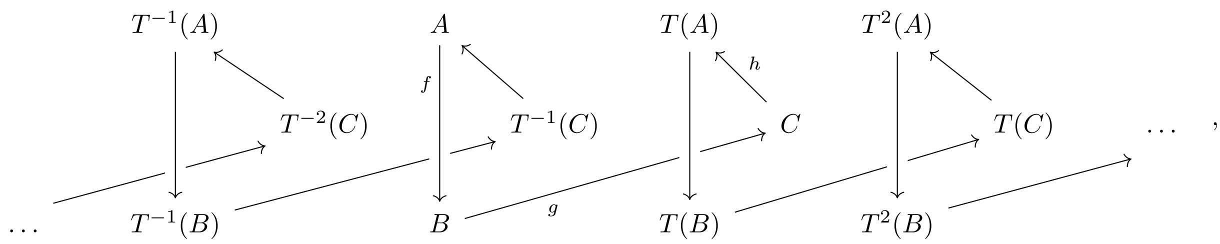
Note that we write the maps t, l, n and $T(u) \circ n$ as if their targets would be A, A, B and C', respectively. This diagram is constructed by first filling in the bottom cap of the octahedron – its left face is the the triangle induced by f, its right face is the the triangle induced by g, the front face is completed by commutativity, as is the rear face (if u would be replaced by $T(u)$, of course). Finally, the front face of the upper cap is the triangle induced by the composition $g \circ f$. Then, this axiom requires that arrows α and β exist so that the rear face of the upper cap is a triangle in $\mathcal{T}$, and, moreover, they should form commutative triangles in all the other faces that contain them.

A triangle in the class $\mathcal{T}$ is said to be a ***distinguished triangle***.

Remark 2.2.3. Axiom A4 was included in the definition above for completeness. However, we will never use this axiom explicitly when dealing with derived categories and derived functors. For that reason, when we need to prove that a class of triangles is distinguished, we will only deal with axioms A1-A3, referring the interested reader to the literature for the octahedron axiom.

These axioms supply a lot of information. For example, starting with the distinguished triangle $A \xrightarrow{\mathrm{id}_A} A \to 0 \to T(A)$, we have that $0 \to A \xrightarrow{\mathrm{id}_A} A \to 0$ is also distinguished by A2.

By the same axiom, it is also easy to show that, given any distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$, we can include it into a helix of morphisms



such that any three consecutive morphisms form a distinguished triangle. With these observations, it is easy to draw a comparison with chain complexes. In fact, consider the next proposition.

Proposition 2.2.4. Let $(\mathscr{D}, T)$ be a triangulated category and $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$ a distinguished triangle. Then $g \circ f = 0$. Consequently, the composition of any two consecutive morphisms in a distinguished triangle is trivial.

Proof. Consider the morphism of distinguished triangles

$$\begin{array}{ccccccc} A & \xrightarrow{\mathrm{id}_A} & A & \longrightarrow & 0 & \longrightarrow & T(A) \\ \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle f} & & \vdots & & \downarrow{\scriptstyle \mathrm{id}_{T(A)}} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & T(A) \end{array},$$

whose existence is guaranteed by axioms A1 i) and A3 of Definition 2.2.2. The more general claim is obtained by "rotating" the triangle to the left or to the right (*i.e.* using axiom A2). ☐

We can go one step further and equip our triangulated category $\mathscr{D}$ with a functor to an abelian category, enabling us to define a notion of cohomology in $\mathscr{D}$, under certain conditions.

Definition 2.2.5. Let $(\mathscr{D}, T)$ be a triangulated category and $\mathscr{A}$ an abelian category. An additive (covariant[1]) functor $F \colon \mathscr{D} \to \mathscr{A}$ is called a ***cohomological functor*** if, for every distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$$

in $\mathscr{D}$, the complex

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact in $\mathscr{A}$.

Corollary 2.2.6. If $F \colon \mathscr{D} \to \mathscr{A}$ is a cohomological functor, then any distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$$

[1]The definition of a cohomological contravariant functor is the expected one.

in $\mathscr{D}$ gives rise to a long exact sequence

$$\cdots \longrightarrow F(T^{-1}(C)) \xrightarrow{-F(T^{-1}(h))} F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \xrightarrow{-F(T(f))} F(T(A)) \longrightarrow \cdots .$$

Example 2.2.7. Let $(\mathscr{D}, T)$ be a triangulated category and $D \in \mathscr{D}$ any object. Then the additive functors $\operatorname{Hom}(D, -) \colon \mathscr{D} \to \mathrm{Ab}$ and $\operatorname{Hom}(-, D) \colon \mathscr{D} \to \mathrm{Ab}$ are cohomological.

Let us prove the case of $\operatorname{Hom}(D, -)$. Suppose $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$ is distinguished. By Proposition 2.2.4, we just need to show that $\operatorname{Ker}(g \circ -) \subseteq \operatorname{Im}(f \circ -)$. Let $\phi \colon D \to B$ be such that $g \circ \phi = 0$. Consider the distinguished triangles $D \to 0 \to T(D) \xrightarrow{-\mathrm{id}_{T(D)}} T(D)$ (axioms A1-i and A2 of Definition 2.2.2) and $B \xrightarrow{g} C \xrightarrow{h} T(A) \xrightarrow{-T(f)} T(B)$ (axiom A2). Then, by axiom A3, we can fill in the solid diagram:

$$\begin{array}{ccccccc} D & \longrightarrow & 0 & \longrightarrow & T(D) & \xrightarrow{-\mathrm{id}_{T(D)}} & T(D) \\ \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle T(\phi)} \\ B & \xrightarrow{g} & C & \xrightarrow{h} & T(A) & \xrightarrow{-T(f)} & T(B) \end{array},$$

and so $T(\phi) = T(f) \circ \psi \implies \phi = f \circ T^{-1}(\psi)$.

The next proposition consolidates several useful properties of the set of distinguished triangles.

Proposition 2.2.8. Let $(\mathscr{D}, T)$ be a triangulated category.

i) Given a morphism of distinguished triangles

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & T(A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & T(A') \end{array},$$

if two of the morphisms f, g and h are isomorphisms, then so is the third.

ii) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$ is a distinguished triangle, then f is an isomorphism if and only if $C \cong 0$.

iii) Let $f \colon A \to B$ be a morphism in $\mathscr{D}$. Complete f to distinguished triangles

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$$
$$A \xrightarrow{f} B \xrightarrow{g'} C' \xrightarrow{h'} T(A)$$

using axiom A1 i) of Definition 2.2.2. Then $C \cong C'$.

Proof. Assertions ii) and iii) follow easily from i). For i), it suffices to prove that if f and g are isomorphisms, so is h (since we can just rotate the triangle). Since the functor $\operatorname{Hom}(C', -) \colon \mathscr{D} \to \mathrm{Ab}$ is cohomological (Example 2.2.7), we have a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccccc}
\mathrm{Hom}(C',A) & \longrightarrow & \mathrm{Hom}(C',B) & \longrightarrow & \mathrm{Hom}(C',C) & \longrightarrow & \mathrm{Hom}(C',T(A)) & \longrightarrow & \mathrm{Hom}(C',T(B)) \\
\Big\downarrow{\scriptstyle f\circ -} & & \Big\downarrow{\scriptstyle g\circ -} & & \Big\downarrow{\scriptstyle h\circ -} & & \Big\downarrow{\scriptstyle T(f)\circ -} & & \Big\downarrow{\scriptstyle T(g)\circ -} \\
\mathrm{Hom}(C',A') & \longrightarrow & \mathrm{Hom}(C',B') & \longrightarrow & \mathrm{Hom}(C',C') & \longrightarrow & \mathrm{Hom}(C',T(A')) & \longrightarrow & \mathrm{Hom}(C',T(B'))
\end{array}$$

and $h\circ-$ is an isomorphism by the 5-lemma. Therefore, there exists a unique morphism $\phi \in \mathrm{Hom}(C', C)$ such that $h \circ \phi = \mathrm{id}_{C'}$. We get a left inverse by a similar argument applied to the long exact sequence associated to the cohomological functor $\mathrm{Hom}(-, C) \colon \mathscr{D} \to \mathrm{Ab}$. □

Remark 2.2.9. It follows from assertion iii) of the proposition above and from axiom A2 of Definition 2.2.2 that any distinguished triangle is determined (up to isomorphism) by any one of its maps. For example, the data of the octahedron axiom is determined (up to isomorphism) by the two maps $f\colon A \to B$ and $g\colon B \to C$, [Wei94, Rem. 10.2.2, pag. 375]. However, a distinguished triangle is not determined up to *unique* isomorphism, namely because we don't require the dashed morphism in axiom A3 of Definition 2.2.2 to be unique (we will see a concrete example of this non-uniqueness in Example 2.3.25). For reasons that will become apparent in Subsection 2.3.2, an object C fitting into a distinguished triangle $A \to B \to C \to T(A)$ is usually called a *cone* of $A \to B$. Under this terminology, the fact that distinguished triangles are not determined up to unique isomorphism is often stated in the literature as saying that "the choice of cones is not functorial". This statement can be understood as follows. If $(\mathscr{D}, T)$ is a triangulated category, given a morphism $f\colon A \to B$, we always have a "cone" C, by axiom A2 of Definition 2.2.2. We consider the category of morphisms $\mathrm{Mor}(\mathscr{D})$, whose objects are morphisms in $\mathscr{D}$, and whose morphisms are commuting squares of morphisms. If we try to define an assignment $\mathrm{cone}\,(-) : \mathrm{Mor}(\mathscr{D}) \to \mathscr{D}$, which sends an object of $\mathrm{Mor}(\mathscr{D})$ (such as f) to a "cone" (such as C), and sends a morphism

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\big\downarrow & & \big\downarrow \\
D & \xrightarrow{g} & E
\end{array}$$

to a morphism $\mathrm{cone}\,(f) \to \mathrm{cone}\,(g)$ in the conditions of axiom A3 of Definition 2.2.2, this assignment will not define a functor in general, as we can not expect it to respect composition, [Fri14, pag. 12]. More details can be found in [Ste]. The choice of words "not functorial" is not the best since, for the triangulated categories we will consider, we actually can define a "cone" *functor* (Lemma 2.3.22). What we will still *not* have is that providing f determines $\mathrm{cone}\,(f)$ up to unique isomorphism.

After verifying that any distinguished triangle gives rise to a double infinite complex, a fitting question one may ask is if it is possible to endow an abelian category with the structure of triangulated category and, conversely, in what situation is a triangulated category abelian. Consider the following definition.

Definition 2.2.10 ([GM03, III.3]). Let $\mathscr{A}$ be an abelian category. We say that $\mathscr{A}$ is ***semisimple*** if, for any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the following three equivalent conditions hold:

i) There exists a right-inverse for the epimorphism g (which we call a *section*).

ii) There exists a left-inverse for the monomorphism f (which we call a *retraction*).

iii) The short exact sequence *splits*, *i.e.* is isomorphic to the short exact sequence

$$0 \longrightarrow A \xrightarrow{\binom{1}{0}} A \oplus C \xrightarrow{(0\;1)} C \longrightarrow 0 \ .$$

Under this terminology, the next proposition makes the connection between triangulated and abelian categories precise.

Proposition 2.2.11. Let $\mathscr{C}$ be an additive category.

i) If $\mathscr{C}$ is triangulated (with automorphism T) and abelian, then $\mathscr{C}$ is semisimple. Moreover, any distinguished triangle is isomorphic to a triangle of the form

$$A \xrightarrow{f} B \xrightarrow{g} T(\operatorname{Ker} f) \oplus \operatorname{Coker} f \xrightarrow{h} T(A) \tag{2.3}$$

for natural maps g, h.

ii) Conversely, if $\mathscr{C}$ is abelian and semisimple, $\mathscr{C}$ has the structure of a triangulated category, by picking any automorphism T and setting a triangle to be distinguished if it is isomorphic to a triangle of the form (2.3).

We refer the reader to Appendix A for the proof of Proposition 2.2.11, as well as some additional remarks on triangulated categories. We finish this section with two definitions. The first one gives a name to the functors between triangulated categories that preserve the additional structure we impose on the underlying additive categories. The second clarifies the notion of a subobject in the category of triangulated categories.

Definition 2.2.12. Let $(\mathscr{D}, T)$ and $(\mathscr{N}, S)$ be triangulated categories. An additive functor $F\colon \mathscr{D} \to \mathscr{N}$ is called ***exact*** if:

i) There is a functor isomorphism $F \circ T \overset{\cong}{\Longrightarrow} S \circ F$.

ii) Any distinguished triangle $A \to B \to C \to T(A)$ in $\mathscr{D}$ is mapped via F to a distinguished triangle $F(A) \to F(B) \to F(C) \to S(F(A))$ in $\mathscr{N}$, where $F(T(A))$ is identified with $S(F(A))$ via the isomorphism in i).

Definition 2.2.13 ([Nee01, 1.5])**.** Let $(\mathscr{D}, T)$ be a triangulated category. If $\mathscr{D}'$ is a full additive subcategory of $\mathscr{D}$, we say that $(\mathscr{D}', T)$ is a ***full triangulated subcategory*** if $T(\mathscr{D}') = \mathscr{D}'$ (*i.e.* $\mathscr{D}'$ is invariant under shift), and for any distinguished triangle $A \to B \to C \to T(A)$ in $\mathscr{D}$, with $A, B \in \operatorname{Obj}(\mathscr{D}')$, C is isomorphic to an object in $\mathscr{D}'$.

2.3 The homotopy category of complexes

Throughout this section, let $\mathscr{A}$ be an abelian category. We recall the basic definitions and propositions regarding the homotopy category of complexes over $\mathscr{A}$. We do not provide proofs for most of the statements in this section. If necessary, we refer the reader to [GM03], sections III.1 and III.3.

Definition 2.3.1. A ***complex*** in $\mathscr{A}$ is a collection of objects $\{A^i\}_{i\in\mathbb{Z}}$ of $\mathscr{A}$ together with morphisms $d^i\colon A^i \to A^{i+1}$ (called ***differentials***) such that $d^{i+1}\circ d^i = 0$ for every i. We denote a complex by $(A^\bullet, d)$, or simply by $A^\bullet$ if there is no risk for confusion.

A morphism of complexes $f\colon (A^\bullet, d_A) \to (B^\bullet, d_B)$ (also called a ***chain map***) is a collection of morphisms $\{f^i\colon A^i \to B^i\}_{i\in\mathbb{Z}}$ of $\mathscr{A}$ such that $f^{i+1}\circ d_A^i = d_B^i \circ f^i$ for every i.

We denote the ***category of complexes*** over $\mathscr{A}$ by $\mathrm{Com}_{\mathscr{A}}$.

Remark 2.3.2. There is a natural functor $\mathscr{A} \to \mathrm{Com}_{\mathscr{A}}$ taking an object $A \in \mathscr{A}$ to the complex with the only non-zero term being A in degree 0 (and, obviously, with all differentials begin trivial). A complex in the image of this map is said to be ***concentrated*** in degree 0.

Remark 2.3.3. $\mathrm{Com}_{\mathscr{A}}$ has a canonical structure of an abelian category.

Definition/Proposition 2.3.4. For every i, there is a well-defined functor $H^i\colon \mathrm{Com}_{\mathscr{A}} \to \mathscr{A}$ called the i***-th cohomology functor***. For each $(A^\bullet, d_A) \in \mathrm{Com}_{\mathscr{A}}$, $H^i(A^\bullet) = \mathrm{Ker}\, d^i/\mathrm{Im}\, d^{i-1}$. Given a chain map $f\colon (A^\bullet, d_A) \to (B^\bullet, d_B)$, the map on cohomology $H^i(f)\colon H^i(A^\bullet) \to H^i(B^\bullet)$ arises from the composition

$$\mathrm{Ker}\, d_A^i \xrightarrow{f|_{\ker d_A^i}} \mathrm{Ker}\, d_B^i \longrightarrow H^i(B^\bullet).$$

Definition 2.3.5. If $A^\bullet \in \mathrm{Com}_{\mathscr{A}}$ is such that $H^i(A^\bullet) = 0$ for every i, we say that $A^\bullet$ is ***acyclic***.

Definition 2.3.6. A chain map $f\colon A^\bullet \to B^\bullet$ is said to be a ***quasi-isomorphism*** (or ***quis*** for short) if the induced maps on cohomology $H^i(f)\colon H^i(A^\bullet) \to H^i(B^\bullet)$ are isomorphisms for all i.

Example 2.3.7. There is a functor $\tau_{\le 0}\colon \mathrm{Com}_{\mathscr{A}} \to \mathrm{Com}_{\mathscr{A}}$ that assigns to a complex $A^\bullet$, the subcomplex $\tau_{\le 0}(A^\bullet)$ given as

$$\begin{array}{lccccccccc}
A^\bullet: & \dots & \xrightarrow{d^{-2}} & A^{-1} & \xrightarrow{d^{-1}} & A^0 & \xrightarrow{d^0} & A^1 & \xrightarrow{d^1} & \dots \\
 & & & \| & & \uparrow & & \uparrow & & \\
\tau_{\le 0}(A^\bullet): & \dots & \xrightarrow{d^{-2}} & A^{-1} & \xrightarrow{d^{-1}} & \mathrm{Ker}\, d^0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}.$$

Notice that we have a canonical map $\tau_{\le 0}(A^\bullet) \to A^\bullet$. Dually, we define $\tau_{\ge 0}\colon \mathrm{Com}_{\mathscr{A}} \to \mathrm{Com}_{\mathscr{A}}$ (with a canonical map $A^\bullet \to \tau_{\ge 0}(A^\bullet)$) as

$$\begin{array}{lccccccccc}
A^\bullet: & \dots & \xrightarrow{d^{-2}} & A^{-1} & \xrightarrow{d^{-1}} & A^0 & \xrightarrow{d^0} & A^1 & \xrightarrow{d^1} & \dots \\
 & & & \downarrow & & \downarrow & & \| & & \\
\tau_{\ge 0}(A^\bullet): & \dots & \longrightarrow & 0 & \longrightarrow & A^0/\mathrm{im}\, d^{-1} & \xrightarrow{d^0} & A^1 & \xrightarrow{d^1} & \dots
\end{array}.$$

These subcomplexes have the following properties:

- $H^i(\tau_{\le 0}(A^\bullet)) = 0$ for $i > 0$, and $H^i(\tau_{\le 0}(A^\bullet)) \cong H^i(A^\bullet)$ for $i \le 0$;
- $H^i(\tau_{\ge 0}(A^\bullet)) = 0$ for $i < 0$, and $H^i(\tau_{\ge 0}(A^\bullet)) \cong H^i(A^\bullet)$ for $i \ge 0$.

We conclude that, if $A^\bullet$ is such that $H^i(A^\bullet) = 0$ for all $i > 0$, then the natural map $\tau_{\le 0}(A^\bullet) \to A^\bullet$ is a quis. Dually, if $B^\bullet$ is such that $H^i(B^\bullet) = 0$ for all $i < 0$, the natural map $B^\bullet \to \tau_{\ge 0}(B^\bullet)$ is a quis.

Definition/Proposition 2.3.8. Given any $(A^\bullet, d_A), (B^\bullet, d_B) \in \mathrm{Com}_{\mathscr{A}}$, we define a relation $\sim$ on the abelian group $\mathrm{Hom}_{\mathrm{Com}_{\mathscr{A}}}(A^\bullet, B^\bullet)$ by setting $f \sim g$ if and only if there exists a collection[2] of maps $h = \{h^i \colon A^i \to B^{i-1}\}_{i\in\mathbb{Z}}$ such that $f^i - g^i = d_B^{i-1} \circ h^i + h^{i+1} \circ d_A^i$ for every i.

This is an equivalence relation that behaves well with respect to addition of complexes, whose equivalence classes are called ***homotopy classes***. Two chain maps that belong to the same equivalence class are said to be ***homotopic***. A chain map that is in the same equivalence class of the trivial map is said to be ***nulhomotopic***.

The composition of homotopy classes of chain maps is well-defined. We define the ***homotopy category of complexes*** in $\mathscr{A}$, denoted $K(\mathscr{A})$, as the category whose objects are the same as $\mathrm{Com}_{\mathscr{A}}$, but with $\mathrm{Hom}_{K(\mathscr{A})}(A^\bullet, B^\bullet)$ consisting of homotopy classes of chain maps in $\mathrm{Hom}_{\mathrm{Com}_{\mathscr{A}}}(A^\bullet, B^\bullet)$, *i.e.* $\mathrm{Hom}_{K(\mathscr{A})}(A^\bullet, B^\bullet) \coloneqq \mathrm{Hom}_{\mathrm{Com}_{\mathscr{A}}}(A^\bullet, B^\bullet)/\sim$.

An isomorphism $\phi \in \mathrm{Hom}_{K(\mathscr{A})}(A^\bullet, B^\bullet)$ is called a ***homotopy equivalence***.

Remark 2.3.9. $K(\mathscr{A})$ inherits a structure of an additive category from $\mathrm{Com}_{\mathscr{A}}$.

Proposition 2.3.10. If two chain maps $f, g \colon A^\bullet \to B^\bullet$ are homotopic, then $H^i(f) = H^i(g)$ for every i. Therefore, we have well-defined functors $H^i \colon K(\mathscr{A}) \to \mathscr{A}$.

Corollary 2.3.11. Any homotopy equivalence $f \colon A^\bullet \to B^\bullet$ is a quis.

2.3.1 Cones

Definition 2.3.12. We define a functor $[1] \colon \mathrm{Com}_{\mathscr{A}} \to \mathrm{Com}_{\mathscr{A}}$ such that, if $(A^\bullet, d_A) \in \mathrm{Com}_{\mathscr{A}}$, $[1](A^\bullet) \coloneqq (A^\bullet[1], d_{A^\bullet[1]})$ is the complex with $(A^\bullet[1])^i \coloneqq A^{i+1}$ and $d^i_{A^\bullet[1]} \coloneqq -d_A^{i+1}$, for every i. For $f \in \mathrm{Hom}_{\mathrm{Com}_{\mathscr{A}}}(A^\bullet, B^\bullet)$, $f[1] \colon A^\bullet[1] \to B^\bullet[1]$ is the chain map that, in degree i, is equal to f^{i+1}. We call this functor the ***left shift*** (by 1).

Definition/Proposition 2.3.13. The functor $[1] \colon \mathrm{Com}_{\mathscr{A}} \to \mathrm{Com}_{\mathscr{A}}$ is an automorphism, with inverse the functor $[-1] \colon \mathrm{Com}_{\mathscr{A}} \to \mathrm{Com}_{\mathscr{A}}$. Given $(A^\bullet, d_A) \in \mathrm{Com}_{\mathscr{A}}$, this functor is defined by $(A^\bullet[-1])^i = A^{i-1}$ and $d^i_{A^\bullet[-1]} = -d_A^{i-1}$, for every i. For $f \in \mathrm{Hom}_{\mathrm{Com}_{\mathscr{A}}}(A^\bullet, B^\bullet)$, $f[-1] \colon A^\bullet[-1] \to B^\bullet[-1]$ is the chain map that, in degree i, is equal to f^{i-1}. We call this functor the ***right shift*** (by 1).

Remark 2.3.14. We define the left (respectively, right) shift by n, $[n] \colon \mathrm{Com}_{\mathscr{A}} \to \mathrm{Com}_{\mathscr{A}}$ (respectively $[-n] \colon \mathrm{Com}_{\mathscr{A}} \to \mathrm{Com}_{\mathscr{A}}$) as the n-fold composition of the functor $[1]$ (respectively, $[-1]$). These obviously extend to well defined automorphisms in $K(\mathscr{A})$.

Remark 2.3.15. It is easy to see that $H^i(A^\bullet[n]) = H^{i+n}(A^\bullet)$, for every $i, n \in \mathbb{Z}$.

As mentioned in Remark 2.3.3, the category $\mathrm{Com}_{\mathscr{A}}$ is abelian (and so, in particular, it is additive). One might wonder if it is possible to use the automorphism $[1] \colon \mathrm{Com}_{\mathscr{A}} \to \mathrm{Com}_{\mathscr{A}}$ of Definition 2.3.12 to define a structure of a triangulated category on $\mathrm{Com}_{\mathscr{A}}$, by, of course, specifying a class of distinguished triangles. In general, this is not possible.

[2]This collection is not a chain map.

Example 2.3.16. Consider the category of abelian groups Ab. $\mathrm{Com}_{\mathrm{Ab}}$ does not have the structure of a triangulated category (for any automorphism) since it is abelian, but not semisimple (Proposition 2.2.11). In fact, for any prime p, consider the group homomorphism $f\colon \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ viewed as a chain map between complexes concentrated in degree 0. Consider the kernel of this map, $\ker f\colon \mathrm{Ker}\, f \to \mathbb{Z}/p^2\mathbb{Z}$. This is a monomorphism and, if $\mathrm{Com}_{\mathrm{Ab}}$ were semisimple, this would split (Definition 2.2.10), implying that $\mathbb{Z}/p^2\mathbb{Z}$ is decomposable, a false statement.

Definition 2.3.17. Given a chain map $f\colon A^\bullet \to B^\bullet$, the ***cone*** of f is the chain complex $(\mathrm{cone}\,(f)\,, d_{\mathrm{cone}(f)})$, where $(\mathrm{cone}\,(f))^i := A^{i+1} \oplus B^i$ and the differential is

$$d^i_{\mathrm{cone}(f)} := \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}$$

for every i.

Remark 2.3.18. If A, B are objects in $\mathscr{A}$ (viewed as complexes concentrated in degree 0) and $f\colon A \to B$ is a morphism (viewed as a chain map), $\mathrm{cone}\,(f)$ has information about the (co)kernel of f. In fact, $H^{-1}(\mathrm{cone}\,(f)) \cong \mathrm{Ker}\, f$ and $H^0(\mathrm{cone}\,(f)) = \mathrm{Coker}\, f$.

Definition/Proposition 2.3.19. Given any chain map $f\colon A^\bullet \to B^\bullet$, we have two natural morphisms

$$\tau_f\colon B^\bullet \to \mathrm{cone}\,(f)$$
$$\pi_f\colon \mathrm{cone}\,(f) \to A^\bullet[1]$$

which are given, for each degree i, by the canonical injection $B^i \hookrightarrow A^{i+1} \oplus B^i$, and the canonical projection $A^{i+1} \oplus B^i \twoheadrightarrow A^{i+1}$, respectively.

The next proposition says that we can fit these morphisms into a short exact sequence in $\mathrm{Com}_{\mathscr{A}}$.

Proposition 2.3.20. Given any chain map $f\colon A^\bullet \to B^\bullet$, there is a short exact sequence of chain complexes

$$0 \longrightarrow B^\bullet \xrightarrow{\tau_f} \mathrm{cone}\,(f) \xrightarrow{\pi_f} A^\bullet[1] \longrightarrow 0.$$

Consequently, we have a long exact sequence

$$\cdots \longrightarrow H^i(A^\bullet) \xrightarrow{\partial^{i-1}} H^i(B^\bullet) \longrightarrow H^i(\mathrm{cone}\,(f)) \longrightarrow H^{i+1}(A^\bullet) \xrightarrow{\partial^i} H^{i+1}(B^\bullet) \longrightarrow \cdots$$

where the connecting homomorphism is actually $\partial^i = H^{i+1}(f)$.

Proof. It is easy to check that the sequence is exact. In the long exact sequence we have used that $H^i(A^\bullet[1]) = H^{i+1}(A^\bullet)$.

By the Snake Lemma, for any $x \in \operatorname{Ker} d_A^{i+1}$, and some $y \in B^i$, we have:

$$
\begin{aligned}
\partial^i(x) &= \left[(\tau_f^{i+1})^{-1} \circ d^i_{\mathrm{cone}(f)} \circ (\pi_f^i)^{-1}\right](x) \mod \operatorname{Im} d_B^i \\
&= \left[(\tau_f^{i+1})^{-1} \circ d^i_{\mathrm{cone}(f)}\right](x, y) \mod \operatorname{Im} d_B^i \\
&= (\tau_f^{i+1})^{-1}(-d_A^{i+1}(x), f^{i+1}(x) + d_B^i(y)) \mod \operatorname{Im} d_B^i \\
&= f^{i+1}(x) + d_B^i(y) \mod \operatorname{Im} d_B^i \\
&= f^{i+1}(x) \mod \operatorname{Im} d_B^i.
\end{aligned}
$$

□

We obtain from this proposition the following important characterization of quasi-isomorphisms.

Corollary 2.3.21. A chain map $f\colon A^\bullet \to B^\bullet$ is a quasi-isomorphism if and only if $\mathrm{cone}\,(f)$ is acyclic.

The following property says assigning the cone of a chain map is a functorial procedure[3].

Lemma 2.3.22. Any solid commutative diagram in $\mathrm{Com}_{\mathscr{A}}$

$$
\begin{array}{ccccccc}
A^\bullet & \xrightarrow{f} & B^\bullet & \xrightarrow{\tau_f} & \mathrm{cone}\,(f) & \xrightarrow{\pi_f} & A^\bullet[1] \\
\downarrow h & \circlearrowleft & \downarrow w & & \downarrow \begin{pmatrix} h[1] & 0 \\ 0 & w \end{pmatrix} & & \downarrow h[1] \\
C^\bullet & \xrightarrow{g} & D^\bullet & \xrightarrow{\tau_g} & \mathrm{cone}\,(g) & \xrightarrow{\pi_g} & C^\bullet[1]
\end{array}
$$

can be completed by the dashed arrow (so that all squares are commutative). Moreover, $\mathrm{cone}\,(-)\colon \mathrm{Hom}(\mathrm{Com}_{\mathscr{A}}) \to \mathrm{Com}_{\mathscr{A}}$ is a functor.

If we have a chain map $f\colon A^\bullet \to B^\bullet$, we get a complex $\mathrm{cone}\,(f)$. In particular, we can take the cone of the natural injection $\tau_f\colon B^\bullet \to \mathrm{cone}\,(f)$ to get the complex $\mathrm{cone}\,(\tau_f)$, or the cone of the natural projection $\pi_f\colon \mathrm{cone}\,(f) \to A^\bullet[1]$ to get the complex $\mathrm{cone}\,(\pi_f)$. Then, we get natural maps

$$
\begin{aligned}
&\tau_{(\tau_f)}\colon \mathrm{cone}\,(f) \to \mathrm{cone}\,(\tau_f)\,, && \tau_{(\pi_f)}\colon A^\bullet[1] \to \mathrm{cone}\,(\pi_f)\,, \\
&\pi_{(\tau_f)}\colon \mathrm{cone}\,(\tau_f) \to B^\bullet[1], && \pi_{(\pi_f)}\colon \mathrm{cone}\,(\pi_f) \to \mathrm{cone}\,(f)\,[1].
\end{aligned}
$$

The following proposition asserts that these operations are closely related to the left and right shifts.

Proposition 2.3.23. Let $f\colon A^\bullet \to B^\bullet$ be a chain map. There exist homotopy equivalences

$$
\begin{aligned}
\alpha\colon A^\bullet[1] &\to \mathrm{cone}\,(\tau_f) \\
\beta\colon \mathrm{cone}\,(\pi_f) &\to B^\bullet[1]
\end{aligned}
$$

whose homotopy inverses are the natural projection $\mathrm{cone}\,(\tau_f) \xrightarrow{(0\ 1\ 0)} A^\bullet[1]$, and the natural inclusion $B^\bullet[1] \xrightarrow{(0\ 1\ 0)^T} \mathrm{cone}\,(\tau_f)$, respectively. Moreover, α and β fit into the following diagram:

[3] Recall the discussion of Remark 2.2.9.

$$
\begin{array}{ccccccccc}
 & & A^{\bullet}[1] & \xrightarrow{\tau_{(\pi_f)}} & \operatorname{cone}(\pi_f) & \xrightarrow{\pi_{(\pi_f)}} & \operatorname{cone}(f)[1] & & \\
 & & \Big\| & \circlearrowleft_{\mathrm{Com}_{\mathscr{A}}} & \Big\downarrow{\scriptstyle \beta} & \circlearrowleft_{K(\mathscr{A})} & \Big\| & & \\
\operatorname{cone}(f) & \xrightarrow{\pi_f} & A^{\bullet}[1] & \xrightarrow[-f[1]]{f[1]} & B^{\bullet}[1] & \xrightarrow{\tau_{f[1]}} & \operatorname{cone}(f)[1] & . & \\
\Big\| & \circlearrowleft_{K(\mathscr{A})} & \Big\downarrow{\scriptstyle \alpha} & \circlearrowleft_{\mathrm{Com}_{\mathscr{A}}} & \Big\| & & & & \\
\operatorname{cone}(f) & \xrightarrow{\tau_{(\tau_f)}} & \operatorname{cone}(\tau_f) & \xrightarrow{\pi_{(\tau_f)}} & B^{\bullet}[1] & & & &
\end{array}
$$

The upper and lower rectangles are commutative $K(\mathscr{A})$ (but not in $\mathrm{Com}_{\mathscr{A}}$) when taken separately. The whole diagram is commutative in $K(\mathscr{A})$ up to the sign of $f[1]$.

Proof. We deal first with the existence of the homotopy equivalences. Denote by ξ the natural projection $\operatorname{cone}(\tau_f) \xrightarrow{(0\ 1\ 0)} A^{\bullet}[1]$, and by ι the natural inclusion $B^{\bullet}[1] \xrightarrow{(0\ 1\ 0)^T} \operatorname{cone}(\tau_f)$. Note the explicit expressions for the differentials:

$$
d^i_{\operatorname{cone}(\tau_f)} = \begin{pmatrix} -d_B^{i+1} & 0 & 0 \\ 0 & -d_A^{i+1} & 0 \\ 1 & f^{i+1} & d_B^i \end{pmatrix}, \qquad d^i_{\operatorname{cone}(\pi_f)} = \begin{pmatrix} d_A^{i+2} & 0 & 0 \\ -f^{i+2} & -d_B^{i+1} & 0 \\ 1 & 0 & -d_A^{i+1} \end{pmatrix}.
$$

Define $\alpha^i = \left(\,-f^{i+1}\ 1\ 0\,\right)^T$ and $\beta^i = \left(\,0\ 1\ f^{i+1}\,\right)$. The signs of f^{i+1} in the first entry of α^i and the last entry of β^i are chosen to guarantee that α and β are chain maps. It is immediate to check that $\xi \circ \alpha = \mathrm{id}_{A^{\bullet}[1]}$ and $\beta \circ \iota = \mathrm{id}_{B^{\bullet}[1]}$. Note that

$$
\alpha \circ \xi - \mathrm{id}_{\operatorname{cone}(\tau_f)} = \begin{pmatrix} -1 & -f[1] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \iota \circ \beta - \mathrm{id}_{\operatorname{cone}(\pi_f)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & f[1] \\ 0 & 0 & -1 \end{pmatrix}.
$$

Consider the collections of maps $h = \{h^i \colon \operatorname{cone}(\tau_f)^i \to \operatorname{cone}(\tau_f)^{i-1}\}$ and $t = \{t^i \colon \operatorname{cone}(\pi_f)^i \to \operatorname{cone}(\pi_f)^{i-1}\}$ given by

$$
h^i = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad t^i = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

By direct computation, $\alpha^i \circ \xi^i - \mathrm{id}_{\operatorname{cone}(\tau_f)^i} = d^{i-1}_{\operatorname{cone}(\tau_f)} \circ h^i - h^{i+1} \circ d^i_{\operatorname{cone}(\tau_f)}$, so that $\alpha \circ \xi \overset{h}{\sim} \mathrm{id}_{\operatorname{cone}(\tau_f)}$. Similarly, $\iota \circ \beta \overset{t}{\sim} \mathrm{id}_{\operatorname{cone}(\pi_f)}$.

Regarding the commutativity of the diagram, the squares marked $\circlearrowleft_{\mathrm{Com}_{\mathscr{A}}}$ commute by direct computation of the compositions. Since $\pi_f = \xi \circ \tau_{(\tau_f)}$ we have that $\alpha \circ \pi_f = \alpha \circ \xi \circ \tau_{(\tau_f)} \overset{h}{\sim} \tau_{(\tau_f)}$. Similarly, $\tau_{f[1]} = \pi_{(\pi_f)} \circ \iota$ and $\tau_{f[1]} \circ \beta \overset{t}{\sim} \pi_{(\pi_f)}$. $\square$

2.3.2 The homotopy category is triangulated

So far, we have introduced:

- The homotopy category $K(\mathscr{A})$ of an abelian category $\mathscr{A}$, which has a canonical structure of an additive category, inherited from $\mathrm{Com}_{\mathscr{A}}$.
- A natural shift automorphism $[1]\colon K(\mathscr{A}) \to K(\mathscr{A})$.
- A cone functor $\mathrm{cone}(-)\colon \mathrm{Hom}(K(\mathscr{A})) \to K(\mathscr{A})$. Moreover, for any morphism $f\colon A^\bullet \to B^\bullet$ in $K(\mathscr{A})$, there are natural morphisms $\tau_f\colon B^\bullet \to \mathrm{cone}(f)$ and $\pi_f\colon \mathrm{cone}(f) \to A^\bullet[1]$ which compose to the zero map.

Comparing this list with Definition 2.2.2 and Proposition 2.2.4, we can ask if one is able to define a structure of a triangulated category on $K(\mathscr{A})$, using the cone construction. This is indeed possible.

Proposition 2.3.24. The pair $(K(\mathscr{A}), [1])$ has the structure of a triangulated category, by specifying the distinguished triangles to be triangles isomorphic to ones of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau_f} \mathrm{cone}(f) \xrightarrow{\pi_f} A^\bullet[1] \ .$$

Proof. We verify the axioms of Definition 2.2.2 one by one:

A1) Assertions ii) and iii) are clear. For i), we need to show that $A^\bullet \xrightarrow{\mathrm{id}_{A^\bullet}} A^\bullet \to 0 \to A^\bullet[1]$ is distinguished. Consider the following diagram:

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{\mathrm{id}_{A^\bullet}} & A^\bullet & \longrightarrow & 0 & \longrightarrow & A^\bullet[1] \\ \| & & \| & & \downarrow & & \| \\ A^\bullet & \xrightarrow{\mathrm{id}_{A^\bullet}} & A^\bullet & \xrightarrow{\tau_{\mathrm{id}_{A^\bullet}}} & \mathrm{cone}(\mathrm{id}_{A^\bullet}) & \xrightarrow{\pi_{\mathrm{id}_{A^\bullet}}} & A^\bullet[1] \end{array} .$$

Showing that $\mathrm{cone}(\mathrm{id}_{A^\bullet})$ is homotopy equivalent to the zero object is the same as showing that the identity map on $\mathrm{cone}(\mathrm{id}_{A^\bullet})$ is nulhomotopic. It is easy to check that the homotopy $\{h^i\colon \mathrm{cone}(\mathrm{id}_{A^\bullet})^i \to \mathrm{cone}(\mathrm{id}_{A^\bullet})^{i-1}\}_i$ given by

$$h^i = \begin{pmatrix} 0 & \mathrm{id}_{A^i} \\ 0 & 0 \end{pmatrix}$$

works.

A2) We show that if $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau_f} \mathrm{cone}(f) \xrightarrow{\pi_f} A^\bullet[1]$ is distinguished, so is $B^\bullet \xrightarrow{\tau_f} \mathrm{cone}(f) \xrightarrow{\pi_f} A^\bullet[1] \xrightarrow{-f[1]} B^\bullet[1]$ (the converse statement is similar). Proposition 2.3.23 provides an isomorphism between such triangles:

$$\begin{array}{ccccccc} B^\bullet & \xrightarrow{\tau_f} & \mathrm{cone}(f) & \xrightarrow{\pi_f} & A^\bullet[1] & \xrightarrow{-f[1]} & B^\bullet[1] \\ \| & & \| & & \downarrow\cong & & \| \\ B^\bullet & \xrightarrow{\tau_f} & \mathrm{cone}(f) & \xrightarrow{\tau(\tau_f)} & \mathrm{cone}(\tau_f) & \xrightarrow{\pi(\tau_f)} & B^\bullet[1] \end{array} .$$

A3) The proof of this axiom is just Lemma 2.3.22.

A4) Again, we refer to the literature for the octahedron axiom. See, for example, [GM03, IV.1.14].

□

We finish this section with a few remarks.

Recall that we have used Proposition 2.2.11 in Example 2.3.16 in order to show that, while $\mathrm{Com}_{\mathscr{A}}$ is abelian, the pair $(\mathrm{Com}_{\mathscr{A}}, [1])$ is not triangulated. Using the converse statement of the proposition, we note that, if $\mathscr{A}$ is not semisimple, $K(\mathscr{A})$ is not abelian. This "inverse" relationship can be explained in a broader sense than what was discussed here. Even if $\mathscr{A}$ is abelian, the failure of the existence of (co)kernels in the triangulated category $K(\mathscr{A})$ (or limits and colimits in general) arises from not requiring the dashed arrow in axiom A3 of Definition 2.2.2 to be unique (Remark 2.2.9). More details can be found in [Ste]. Let us illustrate a concrete example of this non-uniqueness behavior.

Example 2.3.25. Consider the category of abelian groups Ab and the triangulated category $(K(\mathrm{Ab}), [1])$. Let $\mathbb{Z}$ be viewed as a complex concentrated in degree 0. Then $\mathbb{Z}[1]$ is a complex concentrated in degree -1. Consider the following diagram:

$$\begin{array}{ccccccc} \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}[1] & \xrightarrow{-1} & \mathbb{Z}[1] \\ \downarrow & & \downarrow & & \vdots & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z}[1] & \xrightarrow{1} & \mathbb{Z}[1] & \longrightarrow & 0 \end{array} .$$

The rows are distinguished triangles by the axioms of Definition 2.2.2. There are two dashed arrows that make the diagram complete to a morphism of distinguished triangles in $K(\mathrm{Ab})$, namely the zero map and the identity $\mathrm{id}_{\mathbb{Z}[1]}$. These maps are not homotopy equivalent since $\mathbb{Z}[1]$ is not the zero object.

2.4 Definition of the derived category

Again, let $\mathscr{A}$ be an abelian category throughout this section.

Definition 2.4.1. The ***derived category*** of $\mathscr{A}$ is the localization of $K(\mathscr{A})$ with respect to the localizing class of quasi-isomorphisms.

We check that the class of quasi-isomorphisms in $K(\mathscr{A})$ is indeed a localizing class, by verifying the axioms of Definition 2.1.1.

Proposition 2.4.2. $\mathcal{S} = \{f \in \mathrm{Mor}(K(\mathscr{A})) \colon f \text{ is a quis}\}$ is a localizing class.

Proof. A1) Follows directly from functoriality of the cohomology functors as in Proposition 2.3.10.

A2) Consider the diagram in $K(\mathscr{A})$

$$\begin{array}{ccc} & & A^\bullet \\ & & \downarrow f \\ C^\bullet & \xrightarrow{g} & B^\bullet \end{array}$$

where g is any chain map and f is a quis. We want to complete this diagram to a commuting square so that the left downwards morphism is a quis. Consider the following distinguished triangles[4]:

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau_f} \operatorname{cone}(f) \xrightarrow{\pi_f} A^\bullet[1],$$
$$B^\bullet \xrightarrow{\tau_f} \operatorname{cone}(f) \xrightarrow{\tau_{(\tau_f)}} \operatorname{cone}(\tau_f) \xrightarrow{\pi_{(\tau_f)}} B^\bullet[1].$$

Shift the second triangle to the right. By Proposition 2.3.23, the diagram to the right is commutative in $K(\mathscr{A})$. In the diagram, $\xi = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$ is the natural projection, in the notation of Proposition 2.3.23.

$$\begin{array}{ccc}
\operatorname{cone}(\tau_f)[-1] & \xrightarrow[\cong]{\xi[-1]} & A^\bullet \\
\downarrow{\scriptstyle -\pi_{(\tau_f)}[-1]} & & \downarrow{\scriptstyle f} \\
B^\bullet & = & B^\bullet \\
\downarrow{\scriptstyle \tau_f} & & \downarrow{\scriptstyle \tau_f} \\
\operatorname{cone}(f) & = & \operatorname{cone}(f) \\
\downarrow{\scriptstyle \tau_{(\tau_f)}} & & \downarrow{\scriptstyle \pi_f} \\
\operatorname{cone}(\tau_f) & \xrightarrow[\cong]{\xi} & A^\bullet[1]
\end{array}$$

Now take the distinguished triangle associated to the morphism $(\tau_f \circ g)\colon C^\bullet \to \operatorname{cone}(f)$, *i.e.*

$$C^\bullet \xrightarrow{\tau_f \circ g} \operatorname{cone}(f) \xrightarrow{\tau_{(\tau_f \circ g)}} \operatorname{cone}(\tau_f \circ g) \xrightarrow{\pi_{(\tau_f \circ g)}} C^\bullet[1]$$

and shift it to the right. By Lemma 2.3.22, we can complete the diagram to the left to a commutative diagram via the dashed arrow.

$$\begin{array}{ccc}
\operatorname{cone}(\tau_f \circ g)[-1] & \overset{\gamma[-1]}{\dashrightarrow} & \operatorname{cone}(\tau_f)[-1] \\
\downarrow{\scriptstyle -\pi_{(\tau_f \circ g)}[-1]} & & \downarrow{\scriptstyle -\pi_{(\tau_f)}[-1]} \\
C^\bullet & \xrightarrow{g} & B^\bullet \\
\downarrow{\scriptstyle \tau_f \circ g} & & \downarrow{\scriptstyle \tau_f} \\
\operatorname{cone}(f) & = & \operatorname{cone}(f) \\
\downarrow{\scriptstyle \tau_{(\tau_f \circ g)}} & & \downarrow{\scriptstyle \tau_{(\tau_f)}} \\
\operatorname{cone}(\tau_f \circ g) & \overset{\gamma}{\dashrightarrow} & \operatorname{cone}(\tau_f)
\end{array}$$

Finally, join the two diagrams to obtain the diagram on right side. We prove the assertion if we show that the top left downwards morphism $-\pi_{(\tau_f \circ g)}[-1]$ is a quis, or, equivalently, that the morphism

$$\operatorname{cone}(\tau_f \circ g) \xrightarrow{\pi_{(\tau_f \circ g)}} C^\bullet[1]$$

is a quis.

$$\begin{array}{ccccc}
\operatorname{cone}(\tau_f \circ g)[-1] & \overset{\gamma[-1]}{\dashrightarrow} & \operatorname{cone}(\tau_f)[-1] & \xrightarrow[\cong]{\xi[-1]} & A^\bullet \\
\downarrow{\scriptstyle -\pi_{(\tau_f \circ g)}[-1]} & & \downarrow{\scriptstyle -\pi_{(\tau_f)}[-1]} & & \downarrow{\scriptstyle f} \\
C^\bullet & \xrightarrow{g} & B^\bullet & = & B^\bullet \\
\downarrow{\scriptstyle \tau_f \circ g} & & \downarrow{\scriptstyle \tau_f} & & \downarrow{\scriptstyle \tau_f} \\
\operatorname{cone}(f) & = & \operatorname{cone}(f) & = & \operatorname{cone}(f) \\
\downarrow{\scriptstyle \tau_{(\tau_f \circ g)}} & & \downarrow{\scriptstyle \tau_{(\tau_f)}} & & \downarrow{\scriptstyle \pi_f} \\
\operatorname{cone}(\tau_f \circ g) & \overset{\gamma}{\dashrightarrow} & \operatorname{cone}(\tau_f) & \xrightarrow[\cong]{\xi} & A^\bullet[1]
\end{array}$$

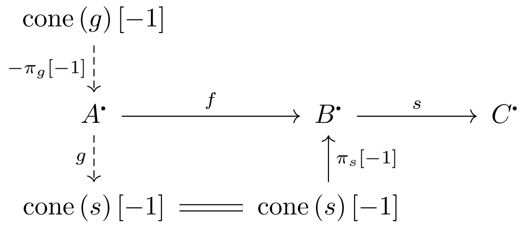
By Corollary 2.3.21, this the same as showing that $\operatorname{cone}\left(\pi_{(\tau_f \circ g)}\right)$ is acyclic. But, again by Proposition 2.3.23, $\operatorname{cone}\left(\pi_{(\tau_f \circ g)}\right)$ is homotopy equivalent to $\operatorname{cone}(f)$, which is acyclic by hypothesis.

A3) This is similar to axiom A2.

A4) Since $K(\mathscr{A})$ is additive, it suffices to prove the equivalent axiom A4)', stated in the Remark 2.1.7. We will prove the direct implication, *i.e.* if $f\colon A^\bullet \to B^\bullet$ is a chain map and there exists a quis

[4]The fact that $K(\mathscr{A})$ is triangulated is not necessary for the proof. We just use this notion to make the diagram easier to digest.

$s\colon B^\bullet \to C^\bullet$ such that $s \circ f$ is nulhomotopic, then there exists a quis $s'\colon D^\bullet \to A^\bullet$ such that $f \circ s'$ is nulhomotopic. Let $h = \{h^i\colon A^i \to C^{i-1}\}_i$ be a homotopy so that $s^i \circ f^i = d_C^{i-1} \circ h^i + h^{i+1} \circ d_A^i$. Then, we can define a map of complexes $g\colon A^\bullet \to \operatorname{cone}(s)[-1]$ given by $g^i = \begin{pmatrix} f^i & -h^i \end{pmatrix}^T$, so that the bottom square is commutative:

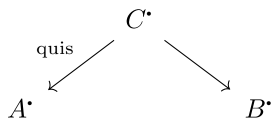


Now, since the composition of any two consecutive morphisms in a distinguished triangle is zero[5]., $g \circ \pi_g[-1]$ is nulhomotopic. Then $f \circ \pi_g[-1]$ is also nulhomotopic by commutativity of the square. Finally, $\pi_g[-1]$ is a quis because $\operatorname{cone}(\pi_g) \cong \operatorname{cone}(s)$ by Proposition 2.3.23.

□

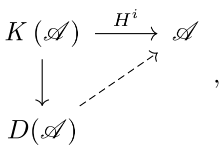
Some remarks are timely.

- Explicitly, the objects in $D(\mathscr{A})$ are chain complexes $A^\bullet$ with terms in $\mathscr{A}$, and morphisms $A^\bullet \to B^\bullet$ are equivalence classes of roofs



  where the arrows themselves are homotopy classes of chain maps.

- By the definition of quasi-isomorphism (Definition 2.3.6) and the universal property of the localization (Proposition 2.1.5), we have well-defined cohomology functors on the derived category, defined as the unique functors (up to unique natural isomorphism) making the diagram



  commute.

- In particular, any two quasi-isomorphic complexes in $K(\mathscr{A})$ become isomorphic in $D(\mathscr{A})$. However, as we will see in Example 2.4.10, there exist isomorphic complexes $A^\bullet$ and $B^\bullet$ in $D(\mathscr{A})$, such that there are no quis $A^\bullet \to B^\bullet$ or $B^\bullet \to A^\bullet$.

- As is the case of general localization of additive categories (Proposition 2.1.6), $D(\mathscr{A})$ inherits a structure of an additive category from $K(\mathscr{A})$.

[5] Again, the fact that $K(\mathscr{A})$ is triangulated is not necessary for this to hold.

These remarks allow us to further characterize the structure of the derived category. Recall from Proposition 2.1.5 that if $Q\colon K(\mathscr{A}) \to D(\mathscr{A})$ is the natural localization functor, and $f\colon A^\bullet \to B^\bullet$ is a chain map, $Q(f) = f\mathrm{id}_{A^\bullet}^{-1}$.

Corollary 2.4.3 (of Proposition 2.1.4 c)**.** A chain map $f\colon A^\bullet \to B^\bullet$ (or, more precisely, $Q(f) = f\mathrm{id}_{A^\bullet}^{-1}$) is 0 in $D(\mathscr{A})$ if and only if there exists a quis $g\colon C^\bullet \to A^\bullet$ such that $f \circ g$ is homotopy equivalent to zero.

In particular,

Corollary 2.4.4. A complex $A^\bullet$ is 0 in $D(\mathscr{A})$ if and only if it is acyclic.

2.4.1 The derived category is triangulated

The derived category $D(\mathscr{A})$ of an abelian category $\mathscr{A}$ was defined in the last section to be the localization of the homotopy category $K(\mathscr{A})$ with respect to the set of quasi-isomorphisms. $K(\mathscr{A})$ has a structure of a triangulated category by Proposition 2.3.24, so it is natural to ask if we can extend this to a triangulated structure on $D(\mathscr{A})$. In this section, we prove that this is indeed possible.

We prove the general result for the localization of a triangulated category $(\mathscr{D}, T)$ first, and then focus on the specific example of $(K(\mathscr{A}), [1])$.

Definition 2.4.5. Let $(\mathscr{D}, T)$ be a triangulated category and $\mathcal{S}$ be a localizing class in $\mathscr{D}$. We say that $\mathcal{S}$ is ***compatible with the triangulation*** if the following conditions hold:

a1) If s is a morphism of $\mathscr{D}$, $s \in \mathcal{S}$ if and only if $T(s) \in \mathcal{S}$.

a2) Axiom A3 of Definition 2.2.2 is "well-behaved with respect to localization"; more precisely, every solid diagram with rows consisting of distinguished triangles

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & T(A) \\ \downarrow{\scriptstyle \alpha\in\mathcal{S}} & & \downarrow{\scriptstyle \beta\in\mathcal{S}} & & \vdots{\scriptstyle \gamma\in\mathcal{S}} & & \downarrow{\scriptstyle T(\alpha)\in\mathcal{S}} \\ D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & T(D) \end{array}$$

and with $\alpha, \beta \in \mathcal{S}$ can be completed *(not necessarily uniquely)* by $\gamma \in \mathcal{S}$ to a morphism of triangles.

Theorem 2.4.6. Let $(\mathscr{D}, T)$ be a triangulated category and $\mathcal{S}$ a localizing class in $\mathscr{D}$ compatible with the triangulation. Denote by $Q\colon \mathscr{D} \to \mathscr{D}[\mathcal{S}^{-1}]$ the natural functor of the localization.

The functor $T_{\mathcal{S}}\colon \mathscr{D}[\mathcal{S}^{-1}] \to \mathscr{D}[\mathcal{S}^{-1}]$ defined by $T_{\mathcal{S}}(A) = T(A)$ for $A \in \mathrm{Obj}(\mathscr{D}[\mathcal{S}^{-1}]) = \mathrm{Obj}(\mathscr{D})$, and $T_{\mathcal{S}}(fs^{-1}) = T(f)(T(s))^{-1}$, is well-defined with respect to equivalence of roofs and is an automorphism.

Then, the pair $(\mathscr{D}[\mathcal{S}^{-1}], T_{\mathcal{S}})$ has the structure of a triangulated category if we define a triangle in $\mathscr{D}[\mathcal{S}^{-1}]$ to be distinguished if it is isomorphic to the image under Q of a distinguished triangle of $\mathscr{D}$.

Proof. Due to the length of this proof, we refer the reader to Appendix B. □

Having settled the general statement for arbitrary triangulated categories, we apply our efforts to the homotopy category and its localization, the derived category.

Proposition 2.4.7. Let $\mathscr{A}$ be an abelian category and $(K(\mathscr{A}), [1])$ its homotopy category, equipped with the triangulated structure introduced in Proposition 2.3.24. The set of quasi-isomorphisms $\mathcal{S}$ in $K(\mathscr{A})$ is compatible with the triangulation. Therefore, by Theorem 2.4.6, we have a canonical structure of a triangulated category in $D(\mathscr{A})$.

Proof. We prove that $\mathcal{S}$ satisfies the conditions of Definition 2.4.5. Axiom a1 is clear from the definition of quasi-isomorphism (Definition 2.3.6). For axiom a2, if we are given the solid diagram

$$\begin{array}{ccccccc} A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & A^\bullet[1] \\ \downarrow{\scriptstyle \text{quis}} & & \downarrow{\scriptstyle \text{quis}} & & \vdots & & \downarrow{\scriptstyle \text{quis}} \\ D^\bullet & \longrightarrow & E^\bullet & \longrightarrow & F^\bullet & \longrightarrow & D^\bullet[1] \end{array}$$

where the rows are distinguished triangles in $K(\mathscr{A})$, we can find a dashed arrow completing the diagram to a morphism of distinguished triangles (since $K(\mathscr{A})$ is triangulated). Any such arrow is a quis. In fact, extend the completed diagram on its right side by adding the square

$$\begin{array}{ccc} A^\bullet[1] & \longrightarrow & B^\bullet[1] \\ \downarrow{\scriptstyle \text{quis}} & & \downarrow{\scriptstyle \text{quis}} \\ D^\bullet[1] & \longrightarrow & E^\bullet[1], \end{array}$$

which maintains commutativity. Apply the cohomology functor $H^i\colon K(\mathscr{A}) \to \mathscr{A}$ and use the 5-lemma. □

2.4.2 Short exact sequences

Let $\mathscr{A}$ be an abelian category. We have constructed two additive categories, $K(\mathscr{A})$ and $D(\mathscr{A})$, both of which have the structure of a triangulated category, while not being abelian in general.

By the long cohomology sequence of Proposition 2.3.20 and the definition of distinguished triangles in $K(\mathscr{A})$ (Proposition 2.3.24), the cohomology functors $H^i\colon K(\mathscr{A}) \to \mathscr{A}$ are cohomological, in the sense of Definition 2.2.5. Therefore, any distinguished triangle in $K(\mathscr{A})$,

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau_f} \operatorname{cone}(f) \xrightarrow{\pi_f} A^\bullet[1],$$

induces a long exact sequence in cohomology. The same thing happens for the cohomology functors $H^i\colon D(\mathscr{A}) \to \mathscr{A}$.

A pertinent question to ask is if there is a converse to this behaviour. Namely, if any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ in $\mathrm{Com}_{\mathscr{A}}$ induces a distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ in $K(\mathscr{A})$ or $D(\mathscr{A})$. The next example shows the answer is negative for $K(\mathscr{A})$.

Example 2.4.8. Consider $\mathscr{A} = \mathrm{Ab}$ and the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0, \tag{2.4}$$

viewed as a short exact sequence of complexes concentrated in degree 0. If (2.4) was a distinguished triangle in $K(\mathrm{Ab})$, there would exist a homotopy equivalence $\mathbb{Z}/2 \xrightarrow{\cong} \operatorname{cone}\left(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\right)$ by Proposition 2.2.8 iii), but this is impossible. In fact, there is no non-zero chain map between these complexes,

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{Z}/2 \\ \downarrow & & \downarrow \\ \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} \end{array},$$

and the zero map between these complexes is not a quasi-isomorphism.

The advantage of working with $D(\mathscr{A})$ as opposed to $K(\mathscr{A})$ is that there is a one-to-one correspondence between short exact sequences in $\mathrm{Com}_{\mathscr{A}}$ and distinguished triangles in $D(\mathscr{A})$.

Proposition 2.4.9. Let $0 \to A^{\bullet} \xrightarrow{f} B^{\bullet} \xrightarrow{g} C^{\bullet} \to 0$ be a short exact sequence in $\mathrm{Com}_{\mathscr{A}}$. Then, there is a natural quasi-isomorphism $\alpha\colon \operatorname{cone}(f) \to C^{\bullet}$, and so the triangle

$$A^{\bullet} \xrightarrow{\quad f \quad} B^{\bullet} \xrightarrow{\quad g \quad} C^{\bullet} \xrightarrow{\pi_f \circ \alpha^{-1}} A^{\bullet}[1]$$

is distinguished in $D(\mathscr{A})$, where α^{-1} is the inverse of α in $D(\mathscr{A})$, and $\pi_f\colon \operatorname{cone}(f) \to A^{\bullet}[1]$ is the natural projection. Moreover, any distinguished triangle in $D(\mathscr{A})$ is isomorphic to one obtained in this way.

Proof. Define $\alpha^i\colon A^{i+1} \oplus B^i \xrightarrow{(0\ g^i)} C^i$. Since g is surjective, so is α. The kernel of α is the complex $A^{\bullet}[1] \oplus \operatorname{Ker} g = A^{\bullet}[1] \oplus \operatorname{Im} f = \operatorname{cone}\left(A^{\bullet} \xrightarrow{f} \operatorname{Im} f\right)$. Since $A^{\bullet} \xrightarrow{f} \operatorname{Im} f$ is an isomorphism, its cone is acyclic. The short exact sequence of complexes

$$0 \longrightarrow \operatorname{cone}\left(A^{\bullet} \xrightarrow{f} \operatorname{Im} f\right) \longrightarrow \operatorname{cone}(f) \longrightarrow C^{\bullet} \longrightarrow 0$$

gives rise to a long exact sequence on cohomology, and hence α is a quis. Then,

$$\begin{array}{ccccccc} A^{\bullet} & \xrightarrow{f} & B^{\bullet} & \xrightarrow{g} & C^{\bullet} & \xrightarrow{\pi_f \circ \alpha^{-1}} & A^{\bullet}[1] \\ \| & & \| & & \cong \downarrow \alpha^{-1} & & \| \\ A^{\bullet} & \xrightarrow{f} & B^{\bullet} & \xrightarrow{\tau_f} & \operatorname{cone}(f) & \xrightarrow{\pi_f} & A^{\bullet}[1] \end{array}$$

clearly provides an isomorphism in $D(\mathscr{A})$ with a distinguished triangle, which is in the image of the natural functor $K(\mathscr{A}) \to D(\mathscr{A})$.

Now, for the converse direction, if $A^{\bullet} \xrightarrow{f} B^{\bullet} \xrightarrow{\tau_f} \operatorname{cone}(f) \xrightarrow{\pi_f} A^{\bullet}[1]$ is a distinguished triangle, there is an isomorphism

$$\begin{array}{ccccccc} A^{\bullet} & \xrightarrow{\quad f \quad} & B^{\bullet} & \xrightarrow{\quad \tau_f \quad} & \operatorname{cone}(f) & \xrightarrow{\quad \pi_f \quad} & A^{\bullet}[1] \\ \| & & \cong \downarrow & & \| & & \| \\ A^{\bullet} & \xrightarrow{\tau_{\pi_f}[-1]} & \operatorname{cone}(\pi_f)[-1] & \xrightarrow{\pi_{\pi_f}[-1]} & \operatorname{cone}(f) & \xrightarrow{\quad \pi_f \quad} & A^{\bullet}[1] \end{array},$$

by Proposition 2.3.23, and the bottom distinguished triangle is the one obtained by the procedure above from the short exact sequence

$$0 \longrightarrow A^{\bullet} \xrightarrow{\tau_{\pi_f}[-1]} \operatorname{cone}(\pi_f)[-1] \xrightarrow{\pi_{\pi_f}[-1]} \operatorname{cone}(f) \longrightarrow 0\ .$$

$\square$

We take this opportunity to clarify that, although quasi-isomorphic complexes are canonically isomorphic in the derived category, not every pair of isomorphic complexes in the derived category has a quasi-isomorphism between its elements.

Example 2.4.10. Consider $\mathscr{A} = \mathrm{Ab}$, and the chain complexes

$$\begin{array}{llllllllll}
A^\bullet : & \ldots \longrightarrow 0 \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow 0 \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow 0 \longrightarrow \ldots , \\
C^\bullet : & \ldots \longrightarrow 0 \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow 0 \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow 0 \longrightarrow \ldots , \\
B^\bullet : & \ldots \longrightarrow 0 \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow 0 \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow 0 \longrightarrow \ldots .
\end{array}$$

By Example 2.4.8, there are no quasi-isomorphisms $A^\bullet \to B^\bullet$ or $B^\bullet \to A^\bullet$. However, there is a right roof

$$\begin{array}{ccc}
A^\bullet & & B^\bullet \\
& {\scriptstyle \mathrm{quis}} \searrow \quad \swarrow {\scriptstyle \mathrm{quis}} & \\
& C^\bullet &
\end{array}$$

in $K(\mathrm{Ab})$. Therefore, $A^\bullet \cong B^\bullet$ in $D(\mathrm{Ab})$.

Chapter 3

Derived functors

Recall that an additive functor $F\colon \mathscr{A} \to \mathscr{B}$ between abelian categories is said to be exact if it preserves short exact sequences. Since taking cohomology commutes with exact functors [Vak17, 1.6.H], F is exact if and only if the image of an acyclic complex under F is again acyclic.

Such an additive functor F has a natural extension to an additive functor $\mathrm{Com}_{\mathscr{A}} \to \mathrm{Com}_{\mathscr{B}}$, by applying F term-wise. It is easy to show that this extension is well-defined with respect to homotopy of complexes, hence defining an additive functor $K(F)\colon K(\mathscr{A}) \to K(\mathscr{B})$.

A natural next step in this reasoning is asking: can we extend this even further, to an additive functor $D(F)\colon D(\mathscr{A}) \to D(\mathscr{B})$ at the level of derived categories? More precisely, denoting by $Q_A\colon K(\mathscr{A}) \to D(\mathscr{A})$ the natural functor of the localization (and similarly for Q_B), can we define an arrow $D(\mathscr{A}) \dashrightarrow D(\mathscr{B})$ such that the diagram

$$\begin{array}{ccccc} K(\mathscr{A}) & & \xrightarrow{\quad Q_B\circ K(F) \quad} & & D(\mathscr{B}) \\ & {\scriptstyle Q_A}\searrow & & \nearrow & \\ & & D(\mathscr{A}) & & \end{array} \tag{3.1}$$

is commutative? Well, if such a $D(F)$ exists, F must send zero objects in $D(\mathscr{A})$ to zero objects in $\mathscr{B}$. By Corollary 2.4.4, this is equivalent to requiring $K(F)$ to send acyclic complexes in $\mathrm{Com}_{\mathscr{A}}$ to acyclic complexes in $\mathrm{Com}_{\mathscr{B}}$. By the remark in the paragraph above, this occurs if and only if F is exact. Now, F being exact is also a sufficient condition for $D(F)$ to be well-defined by Proposition 2.1.5. In fact, it is trivial to verify that

Lemma 3.0.1. For any chain map $f\colon A^{\bullet} \to B^{\bullet}$ we have a canonical isomorphism $K(F)(\mathrm{cone}\,(f)) \cong \mathrm{cone}\,(K(F)(f))$.

Therefore, if F is exact, $K(F)$ sends quasi-isomorphisms in $K(\mathscr{A})$ to quasi-isomorphisms in $K(\mathscr{B})$. These two observations also show that, under this hypothesis, the functor $D(F)$ is exact in the sense of triangulated categories (Definition 2.2.12).

As we will see in Chapter 4, a fair share of the most important functors used in Algebraic Geometry are additive functors between abelian categories that are not exact, but only left or right exact. The derived versions of such functors are exact functors (of triangulated categories) between the derived categories of the abelian categories where they are defined, and aim to provide an extension of the form of diagram (3.1). Since strict commutativity of such a diagram is impossible if F is not exact, this condition is relaxed to a suitable universal property.

3.1 Definition of derived functors

Let $F\colon \mathscr{A} \to \mathscr{B}$ be an additive functor between abelian categories. If F is left-exact, one defines its right derived functor RF. Dually, F is right-exact, one defines its left derived functor LF. These functors will not be defined in the whole derived category $D(\mathscr{A})$, but only for specific full triangulated subcategories of $D(\mathscr{A})$ (recall Definition 2.2.13).

Definition/Proposition 3.1.1. Let $\mathscr{A}$ be an abelian category. $K(\mathscr{A})$ has the following full additive subcategories:

i) $K^+(\mathscr{A})$, whose objects are complexes $A^\bullet$ which are ***bounded below***, *i.e.* $\exists\, N$ such that $A^i = 0$ for all $i \leq N$.

ii) $K^-(\mathscr{A})$, with objects complexes $A^\bullet$ that are ***bounded above***, *i.e.* $\exists\, N$ such that $A^i = 0$ for all $i \geq N$.

iii) $K^b(\mathscr{A})$, whose objects are complexes which are ***bounded***, *i.e.* both bounded below and bounded above.

Definition/Proposition 3.1.2. Let $\mathscr{A}$ be an abelian category and $\mathcal{S}$ be the class of quasi-isomorphisms in $K(\mathscr{A})$. For each $* = \{+, -, b\}$, define $\mathcal{S}_* \coloneqq \mathcal{S} \cap \mathrm{Mor}(K^*(\mathscr{A}))$. Then,

i) the full additive subcategory $K^*(\mathscr{A})$ and the class $\mathcal{S}_*$ are in the conditions of Proposition 2.1.8. Therefore, $D^*(\mathscr{A}) \coloneqq K^*(\mathscr{A})[\mathcal{S}_*^{-1}]$ is a full additive subcategory of $D(\mathscr{A})$.

ii) $(K^*(\mathscr{A}), [1])$ is a full triangulated subcategory of $(K(\mathscr{A}), [1])$. From this, it follows that $\mathcal{S}_*$ is compatible with the triangulation on $K^*(\mathscr{A})$ (as in Definition 2.4.5), which implies there is a canonical structure of a triangulated category in $D^*(\mathscr{A})$ by Theorem 2.4.6.

iii) Finally, $D^*(\mathscr{A})$ is a full triangulated subcategory of $D(\mathscr{A})$, and the canonical functor $D^*(\mathscr{A}) \to D(\mathscr{A})$ is exact.

Proof. First, note that $\mathcal{S}_*$ is a localizing class in $K^*(\mathscr{A})$. Indeed, by the proof of Proposition 2.3.24, it suffices to show that the cone of a morphism in $K^*(\mathscr{A})$ is a complex in $K^*(\mathscr{A})$, which is clearly true. This fact also proves statement ii).

That being said, we prove statement i) for the case $* = +$. Let $B^\bullet$ be bounded below chain complex and $B^\bullet \xrightarrow{f} A^\bullet$ is a quis an arbitrary chain complex $A^\bullet$. Then, there exists $N > 0$, such that $H^i(B^\bullet) \cong H^i(A^\bullet) = 0$ for $i < N$. Then, similarly to Example 2.3.7, there exists a truncation functor $\tau_{\geq N}(-)$ and the natural map $A^\bullet \to \tau_{\geq N}(A^\bullet)$ is a quis. This shows condition ii) of Proposition 2.1.8. In the case $* = -$, one would use a truncation functor of the type $\tau_{\leq N}(-)$ to show condition i) of Proposition 2.1.8, and in the case $* = b$, one would use both truncation functors.

Regarding iii), the fact that $D^*(\mathscr{A}) \subseteq D(\mathscr{A})$ is a full triangulated subcategory is immediate from ii) and by our definition of distinguished triangles in $D(\mathscr{A})$. Exactness of $D^*(\mathscr{A}) \to D(\mathscr{A})$ is also trivial. $\square$

The basic idea of the construction of the derived version of an additive functor $F\colon \mathscr{A} \to \mathscr{B}$ is to apply F term-wise like we did for $K(F)\colon K(\mathscr{A}) \to K(\mathscr{B})$, but not to every complex in $D^*(\mathscr{A})$. Instead, given $A^\bullet \in D^*(\mathscr{A})$, we choose a special representative $R^\bullet$ in the equivalence class of complexes in $K^*(\mathscr{A})$ that are quasi-isomorphic to $A^\bullet$ (and any two such representatives are canonically isomorphic in $D^*(\mathscr{A})$, by construction), and say that the value of the derived functor of F at $A^\bullet$ equals to the object obtained by term-wise application of F to $R^\bullet$. We make precise what we mean.

Definition/Proposition 3.1.3. Let $\mathscr{A}$ be an abelian category and $\mathcal{R}$ be a subclass of $\mathrm{Obj}(\mathscr{A})$ with the following property: for each $A \in \mathscr{A}$, there exists $R \in \mathcal{R}$ together with a monomorphism $A \hookrightarrow R$. Then, given any $A^\bullet \in K^+(\mathscr{A})$, there exists a bounded below complex $R^\bullet$ of objects of $\mathcal{R}$, and a quis $q\colon A^\bullet \to R^\bullet$. Such a quis q is called a ***quasi-resolution***[1] of $A^\bullet$.

Proof. We assume that $A^\bullet$ is such that $A^i = 0$ for $i < 0$, and that we are working over the category of modules of a ring, thanks to the Freyd-Mitchell Embedding Theorem[2]. Recall that given a diagram $Y \xleftarrow{f} X \xrightarrow{g} Z$, if f is a monomorphism, its fibered coproduct

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & Z \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \iota_Z} \\ Y & \xrightarrow{\quad \iota_Y \quad} & Y \sqcup_X Z \end{array}$$

is such that ι_Z is also a monomorphism. We set $R^i = 0$ for all $i < 0$. We will construct objects $R^i \in \mathcal{R}$, differentials $d_R\colon R^i \to R^{i+1}$ and maps $q^i\colon A^i \to R^0$ by induction on i. After this procedure, we will show that the chain map q is indeed a quis. For $i = 0$, consider the following diagram:

$$\begin{array}{ccccccccc} & & R^0 & \xrightarrow{\;:=d_R^0\;} & R^1 & & & & \\ & & \Vert & & \uparrow{\scriptstyle x} & & & & \\ & & R^0 & \xrightarrow{\;\iota_{R^0}\;} & R^0 \sqcup_{A^0} A^1 & & & & \\ & & {\scriptstyle q^0}\uparrow & & \uparrow{\scriptstyle \iota_{A^1}} & & & & \\ 0 & \longrightarrow & A^0 & \xrightarrow{\quad d_A^0 \quad} & A^1 & \xrightarrow{\quad d_A^1 \quad} & A^2 & \longrightarrow & \dots\,. \end{array}$$

The monomorphism q^0 exists by hypothesis. We take the fibered coproduct of q^0 and d_A^0 to get the arrows ι_{R^0}, and ι_{A^1} (which is a monomorphism). Again, by hypothesis, we get the monomorphism x. Finally, we define $d_R^0 := x \circ \iota_{R^0}$, and $q^1 := x \circ \iota_{A^1}$ (which is again a monomorphism).

For step $i+1$, consider the diagram:

$$\begin{array}{ccccccccccc} & & R^{i-1} & \xrightarrow{d_R^{i-1}} & R^i & & \xrightarrow{\qquad\qquad :=d_R^i \qquad\qquad} & & R^{i+1} \\ & & \Vert & & \Vert & & & & \uparrow{\scriptstyle x} \\ & & R^{i-1} & \xrightarrow{d_R^{i-1}} & R^i & \dashrightarrow & \mathrm{Coker}\,(d_R^{i-1}) & \dashrightarrow & \mathrm{Coker}\,(d_R^{i-1}) \sqcup_{A^i} A^{i+1} \\ & & {\scriptstyle q^{i-1}}\uparrow & \circlearrowleft & {\scriptstyle q^i}\uparrow & & \uparrow & & \uparrow{\scriptstyle \iota_{A^{i+1}}} \\ \dots \longrightarrow A^{i-2} & \xrightarrow[d_A^{i-2}]{} & A^{i-1} & \xrightarrow[d_A^{i-1}]{} & A^i & = & A^i & \xrightarrow[\qquad d_A^i \qquad]{} & A^{i+1} \end{array}$$

[1] This terminology is not standard in the literature. Its use will be justified ahead, in Remark 3.3.5.

[2] This theorem asserts that every small abelian category admits an exact fully faithful functor to Mod_A (where A is not necessarily commutative). A sketch of the proof of this statement is in [Wei94, 1.6]. This result enables us to use standard diagram-chasing techniques on $\mathscr{A}$.

We start step $i+1$ with the diagram filled in up until A^i, with the square marked with $\circlearrowleft$ already commutative. We take the cokernel of d_R^{i-1} to get $\operatorname{coker}(d_R^{i-1})\colon R^i \to \operatorname{Coker}(d_R^{i-1})$. We consider the fibered coproduct of the morphisms $\operatorname{coker}(d_R^{i-1}) \circ q^i$ and d_A^i. Finally, by hypothesis, we get the monomorphism x. We then define $q^{i+1} := x \circ \iota_{A^{i+1}}$, and $d_R^i := x \circ \iota_{\operatorname{Coker}(d_R^{i-1})} \circ \operatorname{coker}(d_R^{i-1})$. By construction, $d_R^i \circ d_R^{i-1} = 0$ and $d_R^i \circ q^i = q^{i+1} \circ d_A^i$.

In order to see that q is a quis, observe the diagram above and note that:

- $H^i(q)$ is surjective: let $\alpha \in \operatorname{Ker}(d_R^i)$ be a representative; since x is injective, $\beta := \operatorname{coker}(d_R^{i-1})(\alpha)$ is zero in the fibered coproduct; therefore, there exists $\gamma \in A^i$ such that $(\operatorname{coker}(d_R^{i-1}) \circ q^i)(\gamma) = = d_A^i(\gamma) = 0$; then γ is a suitable preimage of α under $H^i(q)$.

- $H^{i+1}(q)$ is injective: let $\alpha \in \operatorname{Ker}(d_A^{i+1})$ be a representative of an element in $H^{i+1}(A^\bullet)$; suppose that $q^{i+1}(\alpha) = 0$ in $H^{i+1}(R^\bullet)$; since x is injective, this means that there exists $\beta \in R^i$ such that $\iota_{A^{i+1}}(\alpha) = (\iota_{\operatorname{Coker}(d_R^{i-1})} \circ \operatorname{coker}(d_R^{i-1}))(\beta)$; hence there exists $\gamma \in A^i$ such that

$$(\iota_{\operatorname{Coker}(d_R^{i-1})} \circ \operatorname{coker}(d_R^{i-1}) \circ q^i)(\gamma) = (\iota_{\operatorname{Coker}(d_R^{i-1})} \circ \operatorname{coker}(d_R^{i-1}))(\beta) = \iota_{A^{i+1}}(\alpha) = (\iota_{A^{i+1}} \circ d_A^i)(\gamma)$$

  and hence α is zero in $H^{i+1}(A^\bullet)$.

□

Definition 3.1.4. Let $\mathscr{A}, \mathscr{B}$ be abelian categories and $F\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor. We say that a subclass $\mathcal{R}$ of $\operatorname{Obj}(\mathscr{A})$ is ***F-adapted*** if:

A1) $\mathcal{R}$ is closed under direct sums;

A2) F (or, more precisely, $K^+(F)$) maps any bounded below acyclic complex with terms in $\mathcal{R}$ into an acyclic complex with terms in $\mathscr{B}$;

A3) every object in $\mathscr{A}$ is a subobject of an object in $\mathcal{R}$ (that is, $\mathcal{R}$ is in the conditions of Definition/Proposition 3.1.3).

Remark 3.1.5. Dually, we have a definition of an F-adapted class of objects for a right exact functor F. By the discussion at the start of this chapter, if F is exact, the class of all objects of $\mathscr{A}$ is F-adapted.

Proposition 3.1.6. Let $F\colon \mathscr{A} \to \mathscr{B}$ a left exact functor, $\mathcal{R}$ a class of F-adapted objects of $\mathscr{A}$ and $\mathcal{S}_\mathcal{R}$ the class of quis in $K^+(\mathcal{R})$. Then:

i) $\mathcal{S}_\mathcal{R}$ is a localizing class of morphisms in $K^+(\mathcal{R})$;

ii) the canonical exact functor

$$K^+(\mathcal{R})[{\mathcal{S}_\mathcal{R}}^{-1}] \to D^+(\mathscr{A})$$

is an equivalence of categories.

Proof. For i), an attentive reading of the proof of Proposition 2.4.2 makes apparent that it suffices to show that the cone of a morphism between objects in $\mathcal{R}$ is again an object in $\mathcal{R}$. This follows from axiom A1 of Definition 3.1.4.

Now, we claim that assertion ii) follows from Definition/Proposition 3.1.3. In fact, if any $A^\bullet \in K^+(\mathscr{A})$ admits a quasi-resolution $q\colon A^\bullet \to R^\bullet$ by a bounded below complex $R^\bullet$ of objects of $\mathcal{R}$, the canonical functor $K^+(\mathcal{R})[\mathcal{S}_\mathcal{R}{}^{-1}] \to D^+(\mathscr{A})$ is fully faithful, by Proposition 2.1.8. Moreover, it is also essentially surjective since, according to Remark 2.1.3 and the proof of Proposition 2.1.5, the right roof in $K^+(\mathscr{A})$

$$\begin{array}{ccccc} R^\bullet & & & & A^\bullet \\ & \searrow^{\mathrm{id}_{R^\bullet}} & & \swarrow^{q} & \\ & & R^\bullet & & \end{array}$$

defines an isomorphism $R^\bullet \to A^\bullet$ in $D^+(\mathscr{A})$. This concludes the proof. □

Remark 3.1.7. Note that, according to the proposition above, giving a quasi-resolution of $A^\bullet \in K^+(\mathscr{A})$ by a bounded below complex of objects in $\mathcal{R}$ can be done functorially. Indeed, given $A^\bullet, B^\bullet \in K^+(\mathscr{A})$ with a map $f\colon A^\bullet \to B^\bullet$, given a quis $q\colon A^\bullet \to R^\bullet$ as above, since the class of quis in $K^+(\mathscr{A})$ is a localizing class by Definition/Proposition 3.1.2, we can find x and t to make the square

$$\begin{array}{ccccc} A^\bullet & \xrightarrow[\text{quis}]{q} & R^\bullet & & \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle x} & & \\ B^\bullet & \xrightarrow[\text{quis}]{t} & C^\bullet & & \\ & & & \searrow^{s,\ \text{quis}} & \\ & & & & T^\bullet \end{array},$$

commute, where $C^\bullet \in K^+(\mathscr{A})$ does not necessarily have its terms in $\mathcal{R}$. However, by Definition/Proposition 3.1.3, we can find a quis such as s, where now $T^\bullet \in K^+(\mathcal{R})$. We conclude that, given a solid diagram

$$\begin{array}{ccc} A^\bullet & \xrightarrow{\text{quis}} & R^\bullet_A \\ {\scriptstyle f}\downarrow & & \downarrow \\ B^\bullet & \xrightarrow[\text{quis}]{} & R^\bullet_B \end{array},$$

with $R^\bullet_A, R^\bullet_B \in K^+(\mathcal{R})$, there is always a dashed arrow completing the diagram to a commutative square. A priori, such a dashed arrow need not be unique. However, this is the case for certain classes $\mathcal{R}$, as we will see in Remark 3.4.12.

We can now state the definition of a derived functor.

Definition 3.1.8. Let $F\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor between abelian categories. Denote by $Q_A\colon K^+(\mathscr{A}) \to D^+(\mathscr{A})$ the natural map of the localization (and similarly for Q_B).

The ***right derived functor*** of F is a pair (RF, η) consisting of:

- an exact functor of triangulated categories $RF\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$,
- together with a natural transformation $\eta\colon Q_B \circ K^+(F) \Rightarrow RF \circ Q_A$,

which we represent diagrammatically by

$$\begin{array}{ccc} K^+(\mathscr{A}) & \xrightarrow{Q_B \circ K^+(F)} & D^+(\mathscr{B}) \\ & {\scriptstyle Q_A} \searrow \quad \overset{\eta}{\Downarrow} \quad \nearrow {\scriptstyle RF} & \\ & D^+(\mathscr{A}) & \end{array} .$$

The pair (RF, η) is required to satisfy the following universal property: for any other pair

$$(G\colon D^+(\mathscr{A}) \to D^+(\mathscr{B}), \gamma\colon Q_B \circ K^+(F) \Rightarrow G \circ Q_A),$$

there exists a unique natural transformation $\varepsilon\colon RF \Rightarrow G$ such that the diagram

$$\begin{array}{ccc} Q_B \circ K^+(F) & \overset{\gamma}{\Longrightarrow} & G \circ Q_A \\ & {\scriptstyle \eta} \searrow \qquad \nearrow {\scriptstyle \varepsilon \circ Q_A} & \\ & RF \circ Q_A & \end{array}$$

commutes.

At first sight, this definition may be difficult to digest. In the notation of Definition 3.1.8, and as discussed in the beginning of this chapter, the idea of the derived functor RF is to *extend* the morphism $Q_B \circ K^+(F)$ to a morphism $D^+(\mathscr{A}) \to D^+(\mathscr{B})$.

What we mean by an extension of a functor is the following: if $H\colon \mathscr{C} \to \mathscr{D}$ is a functor between arbitrary categories $\mathscr{C}$ and $\mathscr{D}$, a ***(left) extension*** of H with respect to a functor $Q\colon \mathscr{C} \to \mathscr{C}'$ is a functor $E\colon \mathscr{C}' \to \mathscr{D}$, endowed with a natural transformation $\theta\colon E \circ Q \Rightarrow H$. We can define the category $\mathscr{E}_Q^H$ of (left) extensions of H with respect to Q in an obvious way: objects are (left) extensions (such as E), with morphisms consisting of natural transformations between such (left) extensions. In this setting, a ***(left) Kan extension*** of H with respect to Q is an initial object in $\mathscr{E}_Q^H$, [Hin20]. Such an object is denoted $\mathrm{Lan}_Q H$. Kan extensions are, in a certain way, "*the most universal of the universal constructions*"[3], [Leh14]. Having accepted these definitions, it is immediate to see that, while RF is not a left Kan extension of the composition $Q_B \circ K^+(F)$ with respect to the localization functor Q_A, it is an initial object in the full subcategory of $\mathscr{E}_{Q_A}^{Q_B \circ K^+(F)}$ consisting of left extensions *that are exact* as functors of triangulated categories.

As with all objects defined by universal properties, if a right derived functor (RF, η) exists, then it is uniquely defined, up to unique isomorphism. In particular, if F is exact, $(D^+(F), \mathrm{id})$ is its right derived functor, where $D^+(F)$ is the map $D^+(\mathscr{A}) \to D^+(\mathscr{B})$ induced by $K^+(F)$, as we saw in the the introduction of this chapter.

Remark 3.1.9. Fixing abelian categories $\mathscr{A}$ and $\mathscr{B}$, we can define a functor

$$R\colon \{\text{left exact functors } \mathscr{A} \to \mathscr{B} \text{ that admit a right derived functor}\} \to \{\text{exact functors } D^+(\mathscr{A}) \to D^+(\mathscr{B})\}$$

[3]Indeed, one can show that the existence of (co)limits and adjoints is equivalent to the existence of specific Kan extensions. For example, given a diagram $H\colon \mathscr{I} \to \mathscr{C}$, its colimit exists if and only if H has a left Kan extension $\mathrm{Lan}_Q H$ with respect to the unique functor $Q\colon \mathscr{I} \to \mathbf{1}$ to the terminal category $\mathbf{1}$; in this case, $\varinjlim H$ is the value of $\mathrm{Lan}_Q H$ at the unique object of $\mathbf{1}$, [Lan78, X.7.1].

that sends F to RF, and a natural transformation $\tau\colon F \Rightarrow G$ to a canonical natural transformation $R\tau\colon RF \Rightarrow RG$. Denoting by $K^+(\tau)$ the extension of τ to a natural transformation $K^+(F) \Rightarrow K^+(G)$, we define $R\tau$ as follows. If

$$\eta\colon Q_B \circ K^+(F) \Rightarrow RF \circ Q_A,$$
$$\gamma\colon Q_B \circ K^+(G) \Rightarrow RG \circ Q_A$$

are the natural transformations associated to RF and RG, respectively, $R\tau$ is the unique natural transformation (by the universal property of Definition 3.1.8) fitting into the diagram

$$\begin{array}{ccccc} Q_B \circ K^+(F) & & \xRightarrow{\gamma\circ Q_B(K^+(\tau))} & & RG \circ Q_A \\ & \overset{\eta}{\searrow} & & \overset{R\tau\circ Q_A}{\nearrow} & \\ & & RF \circ Q_A & & \end{array} .$$

The next theorem asserts that the existence of right derived functors follows from the existence of adapted classes.

Theorem 3.1.10. If $F\colon \mathscr{A} \to \mathscr{B}$ is a left exact functor between abelian categories, and $\mathscr{A}$ admits a class of F-adapted objects $\mathcal{R}$, then the right derived functor of F exists.

The proof of this theorem is the subject of the next section.

3.2 Existence of derived functors

We split the proof of Theorem 3.1.10 into four parts:

1) Construction of the functor RF;
2) Showing that RF is exact;
3) Constructing the natural transformation η;
4) Verifying the universal property of the pair (RF, η).

The following reasoning is adapted from [GM03, III.6].

3.2.1 Construction of RF

Let $\mathcal{S}_{\mathcal{R}}$ be the class of quasi-isomorphisms $q\colon A^\bullet \to B^\bullet$ such that $A^\bullet, B^\bullet \in K^+(\mathcal{R})$. Then, by Proposition 3.1.6, the canonical functor $\Psi\colon K^+(\mathcal{R})[{\mathcal{S}_{\mathcal{R}}}^{-1}] \to D^+(\mathscr{A})$ is an equivalence of categories. Denote by $\Phi\colon D^+(\mathscr{A}) \to K^+(\mathcal{R})[{\mathcal{S}_{\mathcal{R}}}^{-1}]$ a quasi-inverse to Ψ. On objects, this map assigns to each $A^\bullet \in D^+(\mathscr{A})$ a complex $R^\bullet \in K^+(\mathcal{R})$ with a quis $A^\bullet \to R^\bullet$. We have two natural isomorphisms of functors:

$$\alpha\colon \mathrm{id}_{K^+(\mathcal{R})[{\mathcal{S}_{\mathcal{R}}}^{-1}]} \Rightarrow \Phi \circ \Psi$$
$$\beta\colon \mathrm{id}_{D^+(\mathscr{A})} \Rightarrow \Psi \circ \Phi \quad .$$

If $q \in \mathcal{S}_{\mathcal{R}}$, $\mathrm{cone}\,(q) \in K^+(\mathcal{R})$ by axiom A1 of Definition 3.1.4. By axiom A2 of the same definition and Lemma 3.0.1, $\mathrm{cone}\,(K^+(F)(f)) \cong K^+(F)(\mathrm{cone}\,(f))$ is acyclic, and hence $K^+(F)$ maps $\mathcal{S}_{\mathcal{R}}$ into the class of quasi-isomorphisms in $K^+(\mathscr{B})$. Therefore, by Proposition 2.1.5, $K^+(F)$ determines a well-defined functor $\overline{F}$ (unique up to unique isomorphism) making the bottom square of the diagram

$$\begin{array}{ccc} D^+(\mathscr{A}) & \xrightarrow{:=RF} & D^+(\mathscr{B}) \\ {\scriptstyle\Phi} \downarrow \uparrow {\scriptstyle\Psi} & & \| \\ K^+(\mathcal{R})[\mathcal{S}_{\mathcal{R}}{}^{-1}] & \xrightarrow{\overline{F}} & D^+(\mathscr{B}) \\ {\scriptstyle Q_{\mathcal{R}}} \uparrow & & \uparrow {\scriptstyle Q_B} \\ K^+(\mathcal{R}) & \xrightarrow{K^+(F)} & K^+(\mathscr{B}) \end{array} \tag{3.2}$$

commute. Moreover, it is clear that $\overline{F}$ is an exact functor of triangulated categories. We set $RF := \overline{F} \circ \Phi$.

3.2.2 Exactness of RF

It suffices to show that Φ is an exact functor. Condition i) of Definition 2.2.12 is easy to show: if $A^\bullet \in D^+(\mathscr{A})$, there exists $R^\bullet \in K^+(\mathcal{R})$ such that $A^\bullet \cong \Psi(R^\bullet)$, and so, $\Phi(A^\bullet) \cong R^\bullet$.; then:

$$\Phi(A^\bullet[1]) \cong \Phi(\Psi(R^\bullet)[1]) \cong \Phi(\Psi(R^\bullet[1])) \cong R^\bullet[1] \cong \Phi(A^\bullet)[1].$$

Now, for condition ii) of Definition 2.2.12, let Δ^d be a distinguished triangle in $D^+(\mathscr{A})$. We want to show that $\Phi(\Delta^d) \cong Q_{\mathcal{R}}(\delta^d)$, for δ^d a distinguished triangle in $K^+(\mathcal{R})$. Since Ψ is full, $\Delta^d \cong \Psi(\Delta)$, where Δ is a (not necessarily distinguished) triangle in $K^+(\mathcal{R})[\mathcal{S}_{\mathcal{R}}{}^{-1}]$, and so we need to show that $\Delta \cong Q_{\mathcal{R}}(\delta^d)$. This is equivalent to showing $\Psi(\Delta) \cong \Psi(Q_{\mathcal{R}}(\delta^d))$ in $D^+(\mathscr{A})$ by fully faithfulness of Ψ.

Let Δ be the triangle $B^\bullet \xrightarrow{fs^{-1}} C^\bullet \xrightarrow{gt^{-1}} D^\bullet \xrightarrow{hk^{-1}} B^\bullet[1]$, where $B^\bullet, C^\bullet, D^\bullet \in K^+(\mathcal{R})$ and $s, t, k \in \mathcal{S}_{\mathcal{R}}$. By hypothesis $\Psi(\Delta)$ is distinguished in $D^+(\mathscr{A})$. If fs^{-1} is the equivalence class of the roof $B^\bullet \xleftarrow{s} T^\bullet \xrightarrow{f} C^\bullet$, the diagram

$$\begin{array}{ccccccc} T^\bullet & \xrightarrow{f\mathrm{id}_{T^\bullet}^{-1}} & C^\bullet & \xrightarrow{\tau_f \mathrm{id}_{C^\bullet}^{-1}} & \mathrm{cone}\,(f) & \xrightarrow{\pi_f \mathrm{id}_{\mathrm{cone}(f)}^{-1}} & T^\bullet[1] \\ \downarrow {\scriptstyle s\mathrm{id}_{T^\bullet}^{-1}} & & \| & & \downarrow & & \downarrow {\scriptstyle s[1]\mathrm{id}_{T^\bullet[1]}^{-1}} \\ B^\bullet & \xrightarrow{fs^{-1}} & C^\bullet & \xrightarrow{gt^{-1}} & D^\bullet & \xrightarrow{hk^{-1}} & B^\bullet[1] \end{array}$$

can be completed by the dashed arrow to a morphism of distinguished triangles in $D^+(\mathscr{A})$, according to axiom A3 of Definition 2.2.2. Since s is a quis, $s\mathrm{id}_{T^\bullet}^{-1}$ is an isomorphism in $D^+(\mathscr{A})$ and, by Proposition 2.2.8 i), the diagram above provides the desired isomorphism.

3.2.3 Construction of η

We want to define a natural transformation η, fitting into the diagram

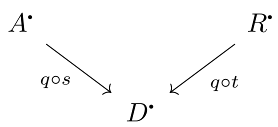
$$\begin{array}{ccc} K^+(\mathscr{A}) & \xrightarrow{Q_B \circ K^+(F)} & D^+(\mathscr{B}) \\ & {\scriptstyle Q_A} \searrow \quad \Downarrow {\scriptstyle\eta} \quad \nearrow {\scriptstyle RF} & \\ & D^+(\mathscr{A}) & \end{array} . \tag{3.3}$$

Let $A^\bullet \in K^+(\mathscr{A})$. According to our definition of RF, such a natural transformation η is

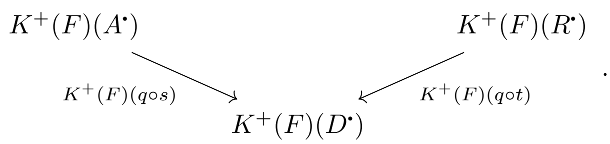
($\star$) a collection of maps $\eta(A^\bullet)$ in $D^+(\mathscr{B})$ (natural with respect to morphisms in $K^+(\mathscr{A})$), from $K^+(F)(A^\bullet)$ to $K^+(F)(R^\bullet)$, where $R^\bullet = \Phi(Q_A(A^\bullet))$ is a complex of F-adapted objects which is quasi-isomorphic to $A^\bullet$.

We can think of $\eta(A^\bullet)$ as measuring the difference, in $D^+(\mathscr{B})$, between the complex $K^+(F)(A^\bullet)$ that one obtains from $A^\bullet$ by term-wise application of F, and its image $RF(A^\bullet)$ under the construction of the right derived functor. Indeed, as already asserted, when F is exact, $RF(A^\bullet) = K^+(F)(A^\bullet)$, and η is just the identity.

Using the natural isomorphism β, we have an isomorphism $\beta(A^\bullet)\colon A^\bullet \xrightarrow{\cong} \Psi(R^\bullet) = R^\bullet$ in $D^+(\mathscr{A})$. Being an isomorphism in the localization, $\beta(A^\bullet)$ can be represented by a right roof $A^\bullet \xrightarrow{s} C^\bullet \xleftarrow{t} R^\bullet$, where s, t are quis, and $C^\bullet \in K^+(\mathscr{A})$ (see Remark 2.1.3). We can find a quis $q\colon C^\bullet \to D^\bullet$ to a $D^\bullet \in K^+(\mathcal{R})$ as in Remark 3.1.7, and the starting right roof is equivalent to the roof



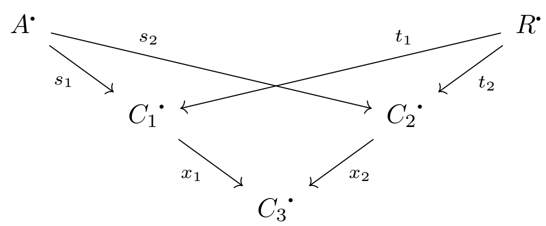
by the dual statement of Proposition 2.1.4 a) i). Applying $K^+(F)$ to this roof we get



Now, since both $D^\bullet$ and $R^\bullet$ are objects in $K^+(\mathcal{R})$, $K^+(F)(q \circ t)$ is a quis in $K^+(\mathscr{B})$, by axiom A2 of Definition 3.1.4. Therefore, this roof determines a morphism $K^+(F)(A^\bullet) \to K^+(F)(R^\bullet)$ in $D^+(\mathscr{B})$. We define $\eta(A^\bullet)$ to be this morphism.

Note that:

- $\eta(A^\bullet)$ does not depend on the choice of representative for the morphism $\beta(A^\bullet)$. Indeed, if $A^\bullet \xrightarrow{s_1} C_1^\bullet \xleftarrow{t_1} R^\bullet$ and $A^\bullet \xrightarrow{s_2} C_2^\bullet \xleftarrow{t_2} R^\bullet$ are two representatives for the morphism, with s_i, t_i quis and $C_i^\bullet \in K^+(\mathcal{R})$ for $i = 1, 2$, then we can find a commutative diagram



  such that $C_3^\bullet \in K^+(\mathcal{R})$ and $x_2 \circ t_2$ is a quis. By applying $K^+(F)$ to the diagram, we see that the two morphisms $K^+(F)(A^\bullet) \to K^+(F)(R^\bullet)$ one obtains from (t_1, s_1) and from (t_2, s_2) are the same morphism one obtains from $(x_2 \circ t_2, x_1 \circ s_1)$.

- The naturality of the collection $\{\eta(A^\bullet)\colon A^\bullet \in K^+(\mathscr{A})\}$ follows from the naturality of the transformation β, [GM03, III.6.10].

3.2.4 Universal property

We want to show that if $G\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ is another exact functor together with a natural transformation $\gamma\colon Q_B \circ K^+(F) \Rightarrow G \circ Q_A$, then there exists a unique natural transformation $\varepsilon\colon RF \Rightarrow G$ such that the following diagram of natural transformations is commutative:

$$\begin{array}{ccc} Q_B \circ K^+(F) & \xRightarrow{\ \gamma\ } & G \circ Q_A \\ & {\scriptstyle\eta}\searrow \quad \nearrow{\scriptstyle\varepsilon\circ Q_A} & \\ & RF \circ Q_A & \end{array} \quad . \tag{3.4}$$

Once again, let $A^\bullet \in K^+(\mathscr{A})$, $R^\bullet = \Phi(Q_A(A^\bullet))$, and $A^\bullet \xrightarrow{s} C^\bullet \xleftarrow{t} R^\bullet$ be a right roof representing the isomorphism $\beta(A^\bullet)$, with $C^\bullet \in K^+(\mathcal{R})$ and s, t quis. We will apply two functors to this roof:

i) Applying $Q_B \circ K^+(F)$ we get a diagram in $D^+(\mathscr{B})$:

$$K^+(F)(A^\bullet) \xrightarrow{Q_B(K^+(F)(s))} K^+(F)(C^\bullet) \xleftarrow[\cong]{Q_B(K^+(F)(t))} K^+(F)(R^\bullet) \ .$$

Note that the morphism in the right side is an isomorphism in $D^+(\mathscr{B})$ because, as discussed in the previous section, $K^+(F)(t)$ is a quis in $K^+(\mathscr{B})$.

ii) Applying $G \circ Q_A$ we get the diagram in $D^+(\mathscr{B})$:

$$G(A^\bullet) \xrightarrow[\cong]{G(Q_A(s))} G(C^\bullet) \xleftarrow[\cong]{G(Q_A(t))} G(R^\bullet) \ .$$

Here both morphisms are isomorphisms because s, t are quis and G is a functor.

Using the natural transformation γ, we can relate the two diagrams obtained in i) and ii). Indeed, the following diagram is commutative in $D^+(\mathscr{B})$:

$$\begin{array}{ccccc} K^+(F)(A^\bullet) & \xrightarrow{Q_B(K^+(F)(s))} & K^+(F)(C^\bullet) & \xleftarrow[\cong]{Q_B(K^+(F)(t))} & K^+(F)(R^\bullet) \\ \downarrow{\scriptstyle\gamma(A^\bullet)} & & \downarrow{\scriptstyle\gamma(C^\bullet)} & & \downarrow{\scriptstyle\gamma(R^\bullet)} \\ G(A^\bullet) & \xrightarrow[\cong]{G(Q_A(s))} & G(C^\bullet) & \xleftarrow[\cong]{G(Q_A(t))} & G(R^\bullet) \end{array} \ .$$

Inverting the left pointing isomorphisms on the right side, we get the following solid commutative diagram in $D^+(\mathscr{B})$:

$$\begin{array}{ccc} K^+(F)(A^\bullet) & \xrightarrow{\eta(A^\bullet)} & K^+(F)(R^\bullet) \\ \downarrow{\scriptstyle\gamma(A^\bullet)} & \swarrow{\scriptstyle := \varepsilon(A^\bullet)} & \downarrow{\scriptstyle\gamma(R^\bullet)} \\ G(A^\bullet) & \xrightarrow[\cong]{G(\beta(A^\bullet))} & G(R^\bullet) \end{array} \ .$$

Finally, we define $\varepsilon(A^\bullet) \coloneqq RF(A^\bullet) \to G(A^\bullet)$ as the composition $G(\beta(A^\bullet))^{-1} \circ \gamma(R^\bullet)$, indicated by the dotted arrow above. Naturality of the collection $\{\varepsilon(A^\bullet) \colon A^\bullet \in K^+(\mathscr{A})\}$ follows from the naturality of β. Therefore, by the commutative diagram above, ε clearly fits in diagram (3.4). The claimed uniqueness of ε follows from $G(\beta(A^\bullet))$ being an isomorphism.

3.2.5 Some remarks on the proof

In summary, given a class of F-adapted objects $\mathcal{R}$, one computes RF it as follows: if $A^\bullet \in K^+(\mathscr{A})$, we choose a quasi-resolution of $A^\bullet$ by a bounded above complex $R^\bullet \in K^+(\mathcal{R})$, and apply F term-wise to this complex to get $RF(A^\bullet) = K^+(F)(R^\bullet)$.

The proof of Theorem 3.1.10 sheds light on how the construction of RF does not depend on the choices we made along the way, namely the choice of quasi-resolution $R^\bullet$ for each $A^\bullet \in D^+(\mathscr{A})$ (*i.e.* the choice of quasi-inverse Φ), and the choice of F-adapted class $\mathcal{R}$.

- If $q_1 \colon A^\bullet \to R_1{}^\bullet$ and $q_2 \colon A^\bullet \to R_2{}^\bullet$ are two quasi-resolutions, $K^+(F)(R_1{}^\bullet)$ and $K^+(F)(R_2{}^\bullet)$ are canonically isomorphic in $D^+(\mathscr{B})$, via β. In particular, the value of RF at a bounded below complex whose terms are F-adapted objects can be computed by just applying $K^+(F)$, *i.e.* if $R^\bullet \in K^+(\mathcal{R})$, $RF(R^\bullet)$ is canonically isomorphic to $K^+(F)(R^\bullet)$.
- If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two F-adapted classes in $\mathscr{A}$, the two derived functors RF_1 and RF_2 one constructs are isomorphic, by a unique isomorphism, by Definition 3.1.8.

Lastly, we emphasize the need for the use of right roofs in the proof we constructed. As warned in Remark 2.1.3, to prove the existence of the *right* derived functor of F, we used that the class of quasi-isomorphisms in $K^+(\mathscr{A})$ is a *left* Ore system. Namely, we used that any $A \in \mathrm{Obj}(\mathscr{A})$ can be *embedded* in an object of the class $\mathcal{R}$ (Proposition 3.1.3). Of course, if F is right exact, the dual construction of its *left* derived functor $LF \colon D^-(\mathscr{A}) \to D^-(\mathscr{B})$, requires every $A \in \mathrm{Obj}(\mathscr{A})$ to be a *quotient* object of an object of $\mathcal{R}$, and that the class of quasi-isomorphisms in $K^-(\mathscr{B})$ is a *right* Ore system.

3.2.6 A generalization to functors defined on homotopy categories

Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories. We have seen that if $F \colon \mathscr{A} \to \mathscr{B}$ is an additive functor, it induces an exact functor of triangulated categories $K^*(F) \colon K^*(\mathscr{A}) \to K^*(\mathscr{B})$, for $* = \{\varnothing, +, -, b\}$ (recall Lemma 3.0.1). In addition, if we require F to be left exact, Definition 3.1.8 and Theorem 3.1.10 define, and guarantee the existence, respectively, of the right derived functor $RF \colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ of F. A close inspection of the construction of RF points to the possibility of defining, and guaranteeing the existence of, a right derived version of an exact functor of triangulated categories $K^+(\mathscr{A}) \to K(\mathscr{B})$, which is not necessarily induced from a functor $\mathscr{A} \to \mathscr{B}$ between the underlying abelian categories.

Definition 3.2.1 ([Huy06, pag. 48])**.** Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories, and $V \colon K^+(\mathscr{A}) \to K(\mathscr{B})$ be an exact functor of triangulated categories. A triangulated subcategory $\mathcal{K}_V \subseteq K^+(\mathscr{A})$ is said to be ***adapted*** to V if it satisfies the next two conditions:

a1) if $A^\bullet \in \mathcal{K}_V$ is acyclic, $V(A^\bullet)$ is acyclic;

a2) for any $A^\bullet \in K^+(\mathscr{A})$, there exists $R^\bullet \in \mathcal{K}_V$ and a quis $A^\bullet \to R^\bullet$.

Remark 3.2.2. It is direct to see that, if $F\colon \mathscr{A} \to \mathscr{B}$ is a left exact functor and $\mathcal{R}$ is a class of F-adapted objects in $\mathscr{A}$, the additive subcategory $\mathcal{K}_F \subseteq K^+(\mathscr{A})$ consisting of bounded below complexes with terms in $\mathcal{R}$ is a triangulated subcategory (Definition 2.2.13) by axiom A1 of Definition 3.1.4. Moreover, axiom A2 and Definition/Proposition 3.1.3 shows that $\mathcal{K}_F$ is adapted to $K^+(F)$, in the terms of Definition 3.2.1.

Definition 3.2.3. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories, and $V\colon K^+(\mathscr{A}) \to K(\mathscr{B})$ be exact (as a functor of triangulated categories). The ***right derived functor*** of V is a pair (RV, η) consisting of an exact functor of triangulated categories $RV\colon D^+(\mathscr{A}) \to D(\mathscr{B})$, and a natural transformation $\eta\colon Q_B \circ V \Rightarrow RV \circ Q_A$. This pair is required to be initial with respect to any other such pair.

Let $V\colon K^+(\mathscr{A}) \to K(\mathscr{B})$ be as in the definition above. We will show that, if $K^+(\mathscr{A})$ admits a triangulated subcategory $\mathcal{K}_V$ adapted to V, our reasoning behind the proof of Theorem 3.1.10 holds true to guarantee the existence of RV.

- Following Proposition 3.1.6, the class of quasi-isomorphisms $\mathcal{S}$ in (the underlying additive category of) $\mathcal{K}_V$ is localizing because the cone of a morphism between objects in $\mathcal{K}_V$ is again in $\mathcal{K}_V$, by Definition 2.2.13. Moreover, the exact functor $\mathcal{K}_V[\mathcal{S}^{-1}] \to D^+(\mathscr{A})$ is an equivalence since axiom a2 of Definition 3.2.1 holds.

- Following Section 3.2.1, V maps $\mathcal{S}$ into the class of quasi-isomorphisms in $K(\mathscr{B})$ by axiom a1 of Definition 3.2.1. Therefore, we can define RV using the diagram to the right.

- Using axiom a2 of Definition 3.2.1 when needed, the content of Sections 3.2.2, 3.2.3 and 3.2.4 holds without change.

$$\begin{array}{ccc} D^+(\mathscr{A}) & \xrightarrow{:=RV} & D(\mathscr{B}) \\ \text{quasi}-\text{resolution} \;\; \uparrow & & \| \\ \mathcal{K}_V[\mathcal{S}^{-1}] & \xrightarrow{\overline{V}} & D(\mathscr{B}) \\ \uparrow & & \uparrow \\ \mathcal{K}_V & \xrightarrow{V} & K(\mathscr{B}) \end{array}$$

For future reference, we state this result as a theorem.

Theorem 3.2.4. In the notation of Definition 3.2.3, RV exists if $K^+(\mathscr{A})$ admits a triangulated subcategory adapted to V.

Remark 3.2.5. In the terminology of Remark 3.2.2, the right derived functor of $K^+(F)\colon K^+(\mathscr{A}) \to K^+(\mathscr{B})$ as in Definition 3.2.3 coincides with the right derived functor of $F\colon \mathscr{A} \to \mathscr{B}$ as in Definition 3.1.8, that is $R(K^+(F)) = RF$.

3.3 Classical derived functors

Definition 3.3.1. Let $\mathscr{A}, \mathscr{B}$ be abelian categories and $F\colon \mathscr{A} \to \mathscr{B}$ a left exact functor. Suppose that $RF\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ is everywhere defined. The ***i-th (higher) right derived functor*** of F is the functor $R^iF\colon \mathscr{A} \to \mathscr{B}$ defined as $A \mapsto H^i(RF(A))$, where A is seen as a complex concentrated in degree 0.

Remark 3.3.2. We will also use the notation R^iF to denote the composition $D^+(\mathscr{A}) \xrightarrow{RF} D^+(\mathscr{B}) \xrightarrow{H^i} \mathscr{B}$.

Definition 3.3.3. Let $\mathscr{A}$ be any abelian category. A ***right resolution*** of an object $A \in \mathscr{A}$ is complex $C^\bullet \in \mathrm{Com}_{\mathscr{A}}$ such that

$$0 \longrightarrow A \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \dots \tag{3.5}$$

is an exact sequence. We denote such a resolution by $0 \to A \to C^\bullet$.

Lemma 3.3.4. Let $\mathscr{A}$ be an abelian category and $A \in \mathrm{Obj}(\mathscr{A})$. Viewing A as a complex concentrated in degree 0, giving a right resolution $0 \to A \to C^\bullet$ in $\mathrm{Com}_{\mathscr{A}}$ is the same as giving a quasi-isomorphism $A \to C^\bullet$.

Proof. Suppose the resolution $0 \to A \to C^\bullet$ is given as in diagram (3.5) above, and that the diagram

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & \diamond & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \dots \end{array} \tag{3.6}$$

in $\mathscr{A}$ represents a quis $A \to C^\bullet$. Then, exactness of the sequence (3.5) at C^i, for $i \geq 1$, is the same information as the downward arrow $0 \to C^i$ in diagram (3.6). Moreover, the commutativity of the square marked with $\diamond$ expresses A as the kernel of the differential $C^0 \to C^1$. □

Remark 3.3.5. Recall that, if $F\colon \mathscr{A} \to \mathscr{B}$ is a left exact functor such that $\mathscr{A}$ admits a class of F-adapted objects $\mathcal{R}$, we defined in Definition/Proposition 3.1.3 a quasi-resolution of a complex $A^\bullet \in K^+(\mathscr{A})$ to be a quis $q\colon A^\bullet \to R^\bullet$, where $R^\bullet \in K^+(\mathcal{R})$. This terminology can be justified as follows:

- If $A^\bullet = A \in \mathrm{Obj}(\mathscr{A})$ is concentrated in degree 0, Lemma 3.3.4 shows that a quasi-resolution such as q is just a right resolution of A in $\mathscr{A}$ (as in Definition 3.3.3).
- If $A^\bullet$ is an actual complex, a right resolution $0 \to A^\bullet \to R^{\bullet,\bullet}$ of $A^\bullet$ in $\mathrm{Com}_{\mathscr{A}}$ requires a *double complex* $R^{\bullet,\bullet}$, which will be defined in Section 3.6.

Proposition 3.3.6. Let $F\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor such that $\mathscr{A}$ admits a class of F-adapted objects. Given any $A \in \mathscr{A}$, $R^iF(A) = 0$ for $i < 0$, and $R^0F(A) \cong F(A)$. In particular, R^0F is left exact.

Proof. Choose a quis $A \to R^\bullet$, where $R^\bullet$ is a bounded below complex of F-adapted objects. By Lemma 3.3.4, this is equivalent to giving a right resolution $0 \to A \to R^\bullet$. Then, $RF(A)$ is the complex

$$0 \longrightarrow F(R^0) \longrightarrow F(R^1) \longrightarrow F(R^2) \longrightarrow \dots ,$$

with $F(R^0)$ in degree 0. $R^iF(A)$ is, by definition, the cohomology of this complex at degree i. It is clear that $RF^i(A) = 0$ for $i < 0$. Since F is left exact, applying F to $0 \to A \to R^\bullet$ shows that $F(A) \cong \mathrm{Ker}\,(R^0 \to R^1) = R^0F(A)$. □

Corollary 3.3.7. Under the hypothesis of Proposition 3.3.6, any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$ induces a long exact sequence

$$\begin{array}{c} 0 \to F(A) \longrightarrow F(B) \longrightarrow F(C) \to \\ R^1F(A) \longrightarrow R^1F(B) \longrightarrow R^1F(C) \to \\ R^2F(A) \longrightarrow \cdots \end{array} \tag{3.7}$$

in $\mathscr{B}$.

Proof. The short exact sequence in $\mathscr{A}$ corresponds to a distinguished triangle $A \to B \to C \to A[1]$ in $D^+(\mathscr{A})$ by Proposition 2.4.9. Since RF is an exact functor of triangulated categories (Section 3.2.2), the triangle $RF(A) \to RF(B) \to RF(C) \to RF(A)[1]$ is distinguished in $D^+(\mathscr{B})$. Since the cohomology functors $H^i \colon D^+(\mathscr{B}) \to \mathscr{B}$ are cohomological in the sense of Definition 2.2.5, we get a long exact sequence

$$0 \longrightarrow RF^0(A) \longrightarrow RF^0(B) \longrightarrow RF^0(C) \longrightarrow RF^1(A) \longrightarrow RF^1(B) \longrightarrow \dots$$

in $\mathscr{B}$ by Corollary 2.2.6. Using Proposition 3.3.6, we get the form presented in diagram (3.7). □

Definition 3.3.8. Under the hypothesis of Definition 3.3.1, an object $A \in \mathrm{Obj}(\mathscr{A})$ is called F***-acyclic*** if $R^iF(A) = 0$ for every $i \geq 1$.

Lemma 3.3.9. Under the hypothesis of Definition 3.3.1, any object of an F-adapted class in $\mathscr{A}$ is F-acyclic.

Proof. This is a direct consequence of axiom A2 of Definition 3.1.4. □

3.3.1 Delta functors

In this subsection, we define the concept of cohomological δ-functors. We will only need the broadness of this construction and its properties in Section 4.3. However, these objects generalize the higher derived functors of Definition 3.3.1, and so their definition is easier to understand right after the previous discussion.

Definition 3.3.10 ([Wei94, Defn. 2.1.1])**.** Let $\mathscr{A}, \mathscr{B}$ be abelian categories. A ***(cohomological)*** δ***-functor*** from $\mathscr{A}$ to $\mathscr{B}$ is a family of additive functors $T = \{T^n \colon \mathscr{A} \to \mathscr{B}\}_{n \geq 0}$ indexed in the non-negative integers, together with the following data: for every short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathscr{A}$, a collection of morphisms $\delta = \{\delta^n \colon T^n(A_3) \to T^{n+1}(A_1)\}_{n \geq 0}$ satisfying the following properties:

i) for each short exact sequence as above, there exists a long exact sequence

$$
\begin{array}{l}
0 \longrightarrow T^0(A_1) \longrightarrow T^0(A_2) \longrightarrow T^0(A_3) \xrightarrow{\delta^0} \\
T^1(A_1) \longrightarrow T^1(A_2) \longrightarrow T^1(A_3) \xrightarrow{\delta^1} \\
T^2(A_1) \longrightarrow \cdots
\end{array}
$$

ii) the collection δ is functorial in the short exact sequence, *i.e.* if

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0
\end{array}
$$

is a commutative diagram of short exact sequences in $\mathscr{A}$, then the induced squares

$$
\begin{array}{ccc}
T^n(A_3) & \xrightarrow{\delta^n_A} & T^{n+1}(A_1) \\
\downarrow & & \downarrow \\
T^n(B_3) & \xrightarrow{\delta^n_B} & T^{n+1}(B_1)
\end{array}
$$

are commutative.

Example 3.3.11 ([Wei94, Examp. 2.1.2])**.** If $\mathscr{A}$ is an abelian category, the collection of cohomology functors $\{H^n \colon \mathrm{Com}^+(\mathscr{A}) \to \mathscr{A}\}_{n\geq 0}$ is a δ-functor, where the δ^n are obtained via the Snake Lemma.

Definition 3.3.12. A ***morphism between*** δ***-functors*** $(T,\delta),(S,\overline{\delta}) \colon \mathscr{A} \to \mathscr{B}$ is a collection of natural transformations $F = \{F_n \colon T^n \Rightarrow S^n\}_{n\geq 0}$ such that, for any short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$, the following square is commutative:

$$
\begin{array}{ccc}
T^n(A_3) & \xrightarrow{\delta^n} & T^{n+1}(A_1) \\
{\scriptstyle F_n(A_3)}\Downarrow & & \Downarrow{\scriptstyle F_{n+1}(A_1)} \\
S^n(A_3) & \xrightarrow{\overline{\delta}^n} & S^{n+1}(A_1)
\end{array} .
$$

Definition 3.3.13. We say that a δ-functor $(T,\delta) \colon \mathscr{A} \to \mathscr{B}$ is ***universal*** if it has the following property: given any other δ-functor $(S,\overline{\delta}) \colon \mathscr{A} \to \mathscr{B}$ and a natural transformation $F_0 \colon T^0 \Rightarrow S^0$, there is a unique sequence of natural transformations $\{F_n \colon T^n \Rightarrow S^n\}_{n\geq 1}$ such that, together with F_0, defines a morphism of δ-functors $F \colon (T,\delta) \to (S,\overline{\delta})$.

Remark 3.3.14*.* From the definition above, it is clear that, for every choice of T^0, there can exist at most one (up to unique isomorphism) universal δ-functor $T = \{T^n \colon \mathscr{A} \to \mathscr{B}\}_{n\geq 0}$ with T^0 as the term of degree 0.

There is a procedure to check if a given δ-functor is universal.

Definition 3.3.15. An additive functor $F \colon \mathscr{A} \to \mathscr{B}$ between abelian categories is called ***effaceable*** if, for each $A \in \mathrm{Obj}(\mathscr{A})$, there exists a monomorphism $i \colon A \to A'$ into some $A' \in \mathrm{Obj}(\mathscr{A})$ such that $F(i) = 0$.

Theorem 3.3.16. If $(T,\delta)\colon \mathscr{A} \to \mathscr{B}$ is a δ-functor such that T^n are effaceable for $n \geq 1$, then T is universal.

Proof. Let $(S,\overline{\delta})\colon \mathscr{A} \to \mathscr{B}$ be another δ-functor and $F_0 : T^0 \Rightarrow S^0$ a natural transformation. We construct the extension to a morphism of delta functors by induction on n. Suppose we have already constructed natural transformations $F_i\colon T^i \Rightarrow S^i$ for $i \leq n$ respecting the δ-functors. It suffices to construct a natural transformation $F_{n+1}\colon T^{n+1} \Rightarrow S^{n+1}$ that verifies the commutativity requirement with respect to short exact sequences of the form

$$0 \longrightarrow A \xrightarrow{\ i\ } A' \longrightarrow \operatorname{Coker} i \longrightarrow 0\ ,$$

where $A \in \operatorname{Obj}(\mathscr{A})$ and $i\colon A \to A'$ is a monomorphism such that $T^{n+1}(i) = 0$. Since T is a δ-functor, we have a long exact sequence

$$\ldots \longrightarrow T^n(A) \xrightarrow{T^n(i)} T^n(A') \longrightarrow T^n(\operatorname{Coker} i) \xrightarrow{\delta^n} T^{n+1}(A) \xrightarrow{\ 0\ } T^{n+1}(A') \longrightarrow \ldots\ ,$$

and so δ^n is surjective. Therefore $T^{n+1}(A) \cong T^n(\operatorname{Coker} i)/\operatorname{Ker}\delta^n \cong T^n(\operatorname{Coker} i)/\operatorname{Im}(\operatorname{coker} T^n(i)) =$ $= \operatorname{Coker}(T^n(\operatorname{coker} i))$. We define $F_{n+1}(A)\colon T^{n+1}(A) \to S^{n+1}(A)$ as the map making the bottom square of the diagram

$$\begin{array}{ccc}
T^n(\operatorname{Coker} i) & \xrightarrow{F_n(\operatorname{Coker} i)} & S^n(\operatorname{Coker} i) \\
\downarrow & & \downarrow \\
\operatorname{Coker}(T^n(\operatorname{coker} i)) & \dashrightarrow & \operatorname{Coker}(S^n(\operatorname{coker} i)) \\
\cong\downarrow\delta^n & & \downarrow\overline{\delta}^n \\
T^{n+1}(A) & \xrightarrow{:=F_{n+1}(A)} & S^{n+1}(A)
\end{array}$$

commute. Unicity is then clear. □

Proposition 3.3.17. Let $F\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor between abelian categories, with $\mathscr{A}$ admitting a class of F-adapted objects $\mathcal{R}$, so that $RF\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ is everywhere defined. Then the collection of higher derived functors $\{R^iF\colon \mathscr{A} \to \mathscr{B}\}_{i\geq 0}$ is a universal δ-functor.

Proof. The family $\{R^iF\}_{i\geq 0}$ is a δ-functor by Corollary 3.3.7 and the proof of Theorem 3.1.10. By Theorem 3.3.16, it suffices to show that R^iF is effaceable for each $i \geq 1$. Since $\mathcal{R}$ is a class of F-adapted objects in $\mathscr{A}$, for any $A \in \mathscr{A}$ there is a monomorphism $A \hookrightarrow A'$ with $A' \in \mathcal{R}$ (axiom A3 of Definition 3.1.4). But $R^i(A') = 0$ by Lemma 3.3.9, and so the image of this map under R^i is zero. □

The proposition above proves that, as previously asserted, the higher derived functors are universal δ-functors. We can ask if a converse statement holds. More precisely, do all, universal or not, δ-functors arise as the cohomology of a right derived functor? Under conditions that guarantee the existence of right derived functors, the answer is affirmative for universal δ-functors.

In fact, suppose $T = \{T^i\colon \mathscr{A} \to \mathscr{B}\}_{i\geq 0}$ is any universal δ-functor. Then T^0 is left exact by Definition 3.3.10. If there exists a class of T^0-adapted objects in $\mathscr{A}$, the higher right derived functors R^iT^0 are well defined by Theorem 3.1.10. Since $R^0T^0 \cong T^0$ (Proposition 3.3.6), and we have just seen that the right derived functors are universal in Proposition 3.3.17, there are unique natural isomorphisms $T^i \cong R^iT^0$ for every $i \geq 0$, by Remark 3.3.14.

This discussion also gives an easy example of a δ-functor that does not arise as a higher derived functor: we pick a left exact functor $F\colon \mathscr{A} \to \mathscr{B}$ such that $\mathscr{A}$ admits a class of F-adapted objects, and consider the δ-functor $T = \{T^i\colon \mathscr{A} \to \mathscr{B}\}_{i\geq 0}$ given by $T^0 = 0$ and $T^i = R^{i-1}F$, for $i \geq 1$. T is also necessarily non-universal.

3.4 Classes of adapted objects

3.4.1 On the existence of adapted classes

Throughout this subsection, let $\mathscr{A}, \mathscr{B}$ be abelian categories, and $F\colon \mathscr{A} \to \mathscr{B}$ a left exact functor.

In Theorem 3.1.10, we showed that the right derived functor RF exists if $\mathscr{A}$ has a class of F-adapted objects. Theorem 3.4.2 below gives a partial inversion of this statement: it gives conditions on the existence of F-adapted classes, assuming the existence of RF.

Definition 3.4.1. Suppose that $RF\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ exists. We denote by $\mathrm{acyc}(F)$ the subclass of F-acyclic objects in $\mathscr{A}$. We say that $\mathrm{acyc}(F)$ is ***sufficiently large*** if any object of $\mathscr{A}$ is a sub-object of an object of $\mathrm{acyc}(F)$.

Theorem 3.4.2. Suppose that $RF\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ exists. There exists an F-adapted class $\mathcal{R}$ in $\mathscr{A}$ if and only if $\mathrm{acyc}(F)$ is sufficiently large.

Proof. If $\mathcal{R}$ is an F-adapted class, $\mathcal{R} \subseteq \mathrm{acyc}(F)$ by Lemma 3.3.9, and so $\mathrm{acyc}(F)$ is sufficiently large because $\mathcal{R}$ is (axiom A3 of Definition 3.1.4). Conversely, suppose that $\mathrm{acyc}(F)$ is sufficiently large. Take any sufficiently large subclass $\mathcal{R}$ of $\mathrm{acyc}(F)$ which is stable under direct sum (*e.g.* $\mathrm{acyc}(F)$ itself). To prove that $\mathcal{R}$ is F-adapted, we only need to show that any acyclic complex $R^\bullet \in K^+(\mathcal{R})$ is sent via F to an acyclic complex of $\mathscr{B}$, by Definition 3.1.4. If we write such an $R^\bullet$ as $\ldots \to 0 \to R^0 \xrightarrow{d^0} R^1 \to R^2 \to \ldots$, we can break the complex into short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^0 & \xrightarrow{d^0} & R^1 & \xrightarrow{d^1} & \operatorname{Im} d^1 & \longrightarrow & 0, \\ 0 & \longrightarrow & \operatorname{Im} d^1 & \longrightarrow & R^2 & \xrightarrow{d^2} & \operatorname{Im} d^2 & \longrightarrow & 0, \\ 0 & \longrightarrow & \operatorname{Im} d^2 & \longrightarrow & R^3 & \xrightarrow{d^3} & \operatorname{Im} d^3 & \longrightarrow & 0, \\ & & & & \vdots & & & & \end{array}$$

Each of these short exact sequences induces a long right derived sequence by Corollary 3.3.7. Since all R^i are F-acyclic, $\operatorname{Im} d^i$ are F-acyclic as well. ☐

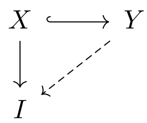
Corollary 3.4.3. Suppose that $RF\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ exists. Any sufficiently large subclass $\mathcal{R}$ of $\mathrm{acyc}(F)$ that is stable under direct sums is F-adapted. If such a $\mathcal{R}$ exists, we can compute $RF(A^\bullet)$ as follows: we find a $R^\bullet \in K^+(\mathcal{R})$ with a quis $A^\bullet \to R^\bullet$, and $RF(A^\bullet) = K^+(F)(R^\bullet)$.

3.4.2 A global adapted class

In this subsection, we will prove that, under certain conditions on the abelian category $\mathscr{A}$, there is a class of objects in $\mathscr{A}$ which is adapted to any left exact functor $F\colon \mathscr{A} \to \mathscr{B}$. Recall the following definition.

Definition/Proposition 3.4.4. Let $\mathscr{A}$ be an abelian category. An object $I \in \mathrm{Obj}(\mathscr{A})$ is said to be ***injective*** if it satisfies two equivalent conditions:

i) Given any morphism $X \to I$ and monomorphism $X \hookrightarrow Y$, there exists a (*not necessarily unique*) morphism $Y \to I$ making the diagram



commute.

ii) The functor $\mathrm{Hom}(-, I)\colon \mathscr{A}^{\mathrm{opp}} \to \mathrm{Ab}$ is exact.

The dual notion is called a ***projective object***.

Definition/Proposition 3.4.5. Let $\mathscr{A}$ be an abelian category. An ***injective resolution*** of an object $A \in \mathrm{Obj}(\mathscr{A})$ is a right resolution of A by injective objects, or, equivalently by Lemma 3.3.4, a quis $A \to I^{\bullet}$, where $I^{\bullet}$ is a bounded below complex of injective objects.

Definition/Proposition 3.4.6. An abelian category $\mathscr{A}$ is said to have ***enough injectives*** if, for any $A \in \mathrm{Obj}(\mathscr{A})$, there exists an injective object I and a monomorphism $A \hookrightarrow I$ (or, equivalently, a short exact sequence $0 \to A \to I \to I/A \to 0$).

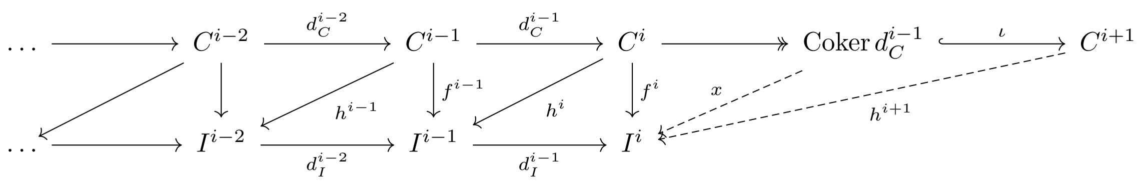
The following two propositions will be crucial for what is to follow.

Proposition 3.4.7 ([Ive86, Pag. 43])**.** Let $\mathscr{A}$ be an abelian category, $C^{\bullet} \in K(\mathscr{A})$ acyclic, and $I^{\bullet} \in K^{+}(\mathscr{A})$ a bounded below complex of injective objects. Then $\mathrm{Hom}_{K(\mathscr{A})}(C^{\bullet}, I^{\bullet}) = 0$.

Proof. We can assume that $I^i = 0$ for all $i < 0$. Let $f\colon C^{\bullet} \to I^{\bullet}$ be a chain map. We will construct a homotopy $\{h^i\colon C^i \to I^{i-1}\}_{i \in \mathbb{Z}}$. Set $h^i = 0$ for all $i \leq 0$. We will built the rest of the collection by induction on i. Assume we have constructed maps h^j, for $1 \leq j \leq i$, such that

$$f^j = h^{j+1} \circ d_C^j + d_I^{j-1} \circ h^j \tag{3.8}$$

for all $0 \leq j \leq (i-1)$. Since $C^{\bullet}$ is acyclic, we can factor $d_C^i\colon C^i \to C^{i+1}$ as the composition $C^i \twoheadrightarrow \mathrm{Coker}\, d_C^{i-1} \overset{\iota}{\hookrightarrow} C^{i+1}$, where ι is a monomorphism induced by d_C^i. Consider the following diagram:



.

Since $d_I^{i-1} \circ \left(h^i \circ d_C^{i-1}\right) = d_I^{i-1} \circ \left(f^{i-1} - d_I^{i-2} \circ h^{i-1}\right) = d_I^{i-1} \circ f^{i-1} = f^i \circ d_C^{i-1}$, we have that $\left(f^i - d_I^{i-1} \circ h^i\right) \circ d_C^{i-1} = 0$. By the the universal property of the cokernel of d_C^{i-1}, there exists an arrow x such that $\left(f^i - d_I^{i-1} \circ h^i\right) = x \circ \operatorname{coker} d_C^{i-1}$. Since ι is a monomorphism and I^i is injective, there exists h^{i+1} as in the diagram, satisfying $x = h^{i+1} \circ \iota$. It is easy to check that this h^{i+1} verifies equation (3.8) when $j = i$. This finishes the induction step, and the proof. □

Proposition 3.4.8. If $\mathscr{A}$ is an abelian category and $I^\bullet \in K^+(\mathscr{A})$ is a bounded below complex of injective objects, for any quis $q\colon A^\bullet \to B^\bullet$ between $A^\bullet, B^\bullet \in K(\mathscr{A})$, the induced homomorphism

$$\operatorname{Hom}_{K(\mathscr{A})}(B^\bullet, I^\bullet) \xrightarrow{(-)\circ q} \operatorname{Hom}_{K(\mathscr{A})}(A^\bullet, I^\bullet)$$

is an isomorphism.

Proof. Consider the standard distinguished triangle $A^\bullet \xrightarrow{q} B^\bullet \xrightarrow{\tau_q} \operatorname{cone}(q) \xrightarrow{\pi_q} A^\bullet[1]$ in $K(\mathscr{A})$. By Corollary 2.2.6 and Example 2.2.7, we have a long exact sequence of abelian groups

$$\cdots \longrightarrow \operatorname{Hom}_{K(\mathscr{A})}(A^\bullet[1], I^\bullet) \xrightarrow{(-)\circ(-\pi_q)} \operatorname{Hom}_{K(\mathscr{A})}(\operatorname{cone}(q), I^\bullet) \xrightarrow{(-)\circ\tau_q} \operatorname{Hom}_{K(\mathscr{A})}(B^\bullet, I^\bullet) \xrightarrow{(-)\circ q} \operatorname{Hom}_{K(\mathscr{A})}(A^\bullet, I^\bullet) \xrightarrow{(-)\circ(-\pi_q[-1])} \operatorname{Hom}_{K(\mathscr{A})}(\operatorname{cone}(q)[-1], I^\bullet) \longrightarrow \cdots.$$

Since q is a quis if and only if $\operatorname{cone}(q)$ is acyclic (Corollary 2.3.21), we get the the desired statement by applying Proposition 3.4.7. □

Corollary 3.4.9. If $\mathscr{A}$ is an abelian category and $q\colon I^\bullet \to C^\bullet$ is a quis from a bounded below complex of injectives $I^\bullet \in K^+(\mathscr{A})$ to $C^\bullet \in K^+(\mathscr{A})$, then there exists a chain map $f\colon C^\bullet \to I^\bullet$ such that $f \circ q \sim \mathrm{id}_{I^\bullet}$. Moreover, f is a quis.

The next theorem is central to the theory of derived functors.

Theorem 3.4.10. Let $\mathscr{A}$ be an abelian category with enough injectives, and $F\colon \mathscr{A} \to \mathscr{B}$ a left exact functor. Then the class $\mathcal{I}$ of injective objects in $\mathscr{A}$ is F-adapted.

Notice that axiom A1 of Definition 3.1.4 follows easily from condition ii) of Definition/Proposition 3.4.4, while A3 follows directly from Definition/Proposition 3.4.6. In this way, we are left to show the following proposition.

Proposition 3.4.11. In the terminology of Theorem 3.4.10, if $I^\bullet \in K^+(\mathcal{I})$ is acyclic, $K^+(F)(I^\bullet)$ is acyclic.

Proof. If $I^\bullet$ is acyclic, the zero map $0\colon I^\bullet \to I^\bullet$ is a quis. By Corollary 3.4.9, this map is homotopy equivalent to the identity $\mathrm{id}_{I^\bullet}$. Since $K^+(F)$ is an additive functor, it follows that the identity map of $K^+(F)(I^\bullet)$ is homotopy equivalent to the zero map $0\colon K^+(F)(I^\bullet) \to K^+(F)(I^\bullet)$. We conclude by Corollary 2.4.4. □

Remark 3.4.12. Proposition 3.4.8 also says something important above quasi-resolutions by complexes of injectives. Using Remark 3.1.7 and Proposition 3.4.11, we know that, given a morphism $f\colon A^\bullet \to B^\bullet$

and quasi-resolutions $A^\bullet \to I_A^\bullet$ and $B^\bullet \to I_B^\bullet$ by complexes of injectives, we can always find a dashed arrow making the square

$$\begin{array}{ccc} A^\bullet & \xrightarrow{\text{quis}} & I_A^\bullet \\ {\scriptstyle f}\downarrow & & \downarrow \\ B^\bullet & \xrightarrow[\text{quis}]{} & I_B^\bullet \end{array}$$

commute. Then, Proposition 3.4.8 asserts that any such arrow is unique up to homotopy. We will use this fact in the proof of Proposition 4.3.11.

Theorems 3.1.10 and 3.4.10 guarantee the existence of right derived functors $RF\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ of left exact functors $F\colon \mathscr{A} \to \mathscr{B}$, under the hypothesis that $\mathscr{A}$ has enough injectives. Dually, the same reasoning shows that, given any right exact functor $G\colon \mathscr{A} \to \mathscr{B}$ from an abelian category with enough projectives, its left derived functor $LG\colon D^-(\mathscr{A}) \to D^-(\mathscr{B})$ exists. Note that, in this case, we can compute RF using F-acyclic quasi-resolutions, according to Corollary 3.4.3.

We finish this section with the following remark: if $q\colon I_1^\bullet \to I_2^\bullet$ is any quis between bounded below complexes of injectives in an abelian category, then q is a homotopy equivalence. This follows trivially from applying Corollary 3.4.9 twice. Consequently, the next proposition is immediate.

Proposition 3.4.13. If $\mathscr{A}$ is an abelian category, $\mathcal{I}$ is its class of injective objects and $\mathcal{S}_{\mathcal{I}}^{-1}$ the localizing class of quis in $K^+(\mathcal{I})$, the natural localization functor $K^+(\mathcal{I}) \to K^+[\mathcal{S}_{\mathcal{I}}^{-1}]$ is an isomorphism.

3.5 Thick subcategories and equivalences

As we will see in Subsection 4.3.4, we are often confronted with the following scenario. We are given a left exact functor $F\colon \mathscr{B} \to \mathscr{C}$ from an abelian category $\mathscr{B}$ with enough injectives, and hence we can speak of its right derived functor $RF\colon D^+(\mathscr{B}) \to D^+(\mathscr{C})$, according to the previous section. We want to consider the restriction of F to a full abelian subcategory $\mathscr{A} \subseteq \mathscr{B}$, $F'\colon \mathscr{A} \to \mathscr{C}$. If $\mathscr{A}$ does not have enough injectives, there is no direct way of defining the right derived functor $RF'\colon D^+(\mathscr{A}) \to D^+(\mathscr{C})$. If we try to use RF applied to objects in $D^+(\mathscr{A})$, there is no way to guarantee that the target of this functor lands in the derived category of bounded below complexes with terms in the essential image of F since, in general, the canonical exact functor $D^+(A) \to D^+(\mathscr{B})$ is not fully faithful.

A way to solve this problem is to find an equivalence φ between $D^+(\mathscr{A})$ and a full triangulated subcategory $\mathscr{N}$ of $D^+(\mathscr{B})$, and use this data to define RF' as the composition

$$\begin{array}{ccc} D^+(\mathscr{A}) & \xrightarrow{\ :=RF'\ } & D^+(\mathscr{C}) \\ & {\scriptstyle \varphi^{-1}}\searrow \quad \nearrow{\scriptstyle RF|_{\mathscr{N}}} & \\ & \mathscr{N} & \end{array} .$$

Any such φ implies that complexes in $\mathscr{N}$ have cohomology groups in the full subcategory $\mathscr{A}$. This motivates the following definition.

Definition 3.5.1. Given a full abelian subcategory $\mathscr{A} \subseteq \mathscr{B}$, we denote by $D^*_{\mathscr{A}}(\mathscr{B})$ the full triangulated subcategory of $D^*(\mathscr{B})$ containing those complexes whose cohomology is in $\mathscr{A}$, where $* = \{\varnothing, +, -, b\}$. A similar notation applies for the homotopy categories.

Motivated by this discussion, we want to find conditions on $\mathscr{A}$ so that the natural functor $D^*(\mathscr{A}) \to D^*(\mathscr{B})$ defines an equivalence of categories between $D^*(\mathscr{A})$ and $D^*_{\mathscr{A}}(\mathscr{B})$.

Definition 3.5.2. Let $\mathscr{A}$ be an abelian category. An ***extension*** of an object Q by an object K is any object X fitting in a short exact sequence of the form $0 \to K \to X \to Q \to 0$.

Definition 3.5.3. A ***thick subcategory*** of an abelian category $\mathscr{B}$ is a full abelian subcategory $\mathscr{A} \subseteq \mathscr{B}$, such that any extension in $\mathscr{B}$ of objects of $\mathscr{A}$ is again in $\mathscr{A}$.

Proposition 3.5.4. Let $* = \{\varnothing, +, -, b\}$. If $\mathscr{A} \subseteq \mathscr{B}$ is a thick subcategory, then the full additive subcategories $K^*(\mathscr{A})$ and $K^*_{\mathscr{A}}(\mathscr{B})$ are full triangulated subcategories of $K^*(\mathscr{B})$.

Proof. According to Definition 2.2.13, we only need to show that the cone of a morphism between objects of $K^*(\mathscr{A})$ (respectively, $K^*_{\mathscr{A}}(\mathscr{B})$) is in $K^*(\mathscr{A})$ (respectively, $K^*_{\mathscr{A}}(\mathscr{B})$). For $K^*(\mathscr{A})$, this is immediate because the cone is an extension object (Proposition 2.3.20). For $K^*_{\mathscr{A}}(\mathscr{B})$, if $f\colon A^\bullet \to B^\bullet$ is a morphism between complexes of $\mathscr{B}$ whose cohomology is in $\mathscr{A}$, consider the long exact sequence of Proposition 2.3.20, associated to the canonical distinguished triangle. We can split the long sequence into short exact sequences of the form $0 \to \operatorname{Im} H^i(f) \to H^i(\operatorname{cone}(f)) \to \operatorname{Ker} H^{i+1}(f) \to 0$ and hence express $H^i(\operatorname{cone}(f))$ as an extension in $\mathscr{B}$ of objects in $\mathscr{A}$, and so $H^i(\operatorname{cone}(f)) \in \mathscr{A}$. □

The next proposition asserts that, under certain conditions, if $\mathscr{A} \subseteq \mathscr{B}$ is a full abelian subcategory, any bounded below complex with terms in $\mathscr{B}$ and cohomology in $\mathscr{A}$ is isomorphic in $D^+(\mathscr{B})$ to a chain complex with terms in $\mathscr{A}$.

Proposition 3.5.5 ([KS90, Prop. 1.7.11])**.** Let $\mathscr{A} \subseteq \mathscr{B}$ be a full abelian subcategory. Suppose the following condition holds:

($\star$) For any monomorphism $A \hookrightarrow B$ with $A \in \mathscr{A}$, there exists a morphism $B \to A'$, where $A' \in \mathscr{A}$, such that the composition $A \hookrightarrow B \to A'$ is still a monomorphism.

Then, for any $B^\bullet \in K^+_{\mathscr{A}}(\mathscr{B})$, there is a complex $A^\bullet \in K^+(\mathscr{A})$ and a quis $B^\bullet \to A^\bullet$.

Theorem 3.5.6. Let $\mathscr{B}$ be an abelian category and $\mathscr{A} \subseteq \mathscr{B}$ a thick subcategory. Suppose that condition ($\star$) of Proposition 3.5.5 holds. Then, the canonical functor $D^*(\mathscr{A}) \to D^*(\mathscr{B})$ is fully faithful and induces an equivalence of categories $D^*(\mathscr{A}) \cong D^*_{\mathscr{A}}(\mathscr{B})$, for $* = \{+, b\}$.

If the dual statement of ($\star$) holds, then we have the same result for $* = \{-, b\}$.

Proof. Since $\mathscr{A}$ is thick, $K^*_{\mathscr{A}}(\mathscr{B})$ is a full triangulated subcategory of $K^*(\mathscr{B})$ by Proposition 3.5.4. If $* = +$, since ($\star$) holds, so does condition ii) of Proposition 2.1.8, by Proposition 3.5.5. We conclude that $D^+(\mathscr{A}) \to D^+(\mathscr{B})$ is fully faithful. It is also essentially surjective by Proposition 3.5.5. For $* = b$, we follow the same reasoning and use truncation functors similar to those defined in Example 2.3.7. □

Corollary 3.5.7. Let $\mathscr{B}$ be an abelian category and $\mathscr{A} \subseteq \mathscr{B}$ a thick subcategory. Suppose that any object $A \in \mathscr{A}$ can be embedded in an object $A' \in \mathscr{A}$ which is injective as an object of $\mathscr{B}$. Then the canonical functor $D^*(\mathscr{A}) \to D^*(\mathscr{B})$ induces an equivalence $D^*(\mathscr{A}) \cong D^*_{\mathscr{A}}(\mathscr{B})$, for $* = \{+, b\}$.

Proof. We show that condition ($\star$) of Proposition 3.5.5 holds. Let $A \hookrightarrow B$ be a monomorphism with $A \in \mathscr{A}$. Then we can find a monomorphism $A \hookrightarrow A'$ where $A' \in \mathscr{A}$ is injective of $\mathscr{B}$. By Definition/Proposition 3.4.4, we can find an arrow $B \to A'$ such that the composition $A \hookrightarrow B \to A'$ is equal to the monomorphism $A \hookrightarrow A'$. □

3.6 Composition of derived functors

As we will see in Chapter 5, composition of derived functors is ubiquitous in the study of derived categories of coherent sheaves, namely in the construction of integral functors (Definition 5.2.4). Under certain conditions, the composition of derived functors is the derived functor of the composition.

Proposition 3.6.1. Let $V_1 \colon K^+(\mathscr{A}) \to K^+(\mathscr{B})$ and $V_2 \colon K^+(\mathscr{B}) \to K(\mathscr{C})$ be two exact functors of triangulated categories. Suppose there exist triangulated subcategories $\mathcal{K}_{V_1} \subseteq K^+(\mathscr{A})$ and $\mathcal{K}_{V_2} \subseteq K^+(\mathscr{A})$ which are adapted to V_1 and V_2, respectively. Then, by Theorem 3.2.4, the right derived functors $RV_1 \colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ and $RV_2 \colon D^+(\mathscr{A}) \to D(\mathscr{C})$ exist. If $V_1(\mathcal{K}_{V_1}) \subseteq \mathcal{K}_{V_2}$, then:

i) $\mathcal{K}_{V_1}$ is adapted to the composition $(V_2 \circ V_1) \colon K^+(\mathscr{A}) \to K(\mathscr{C})$, and hence the right derived functor $R(V_2 \circ V_1) \colon D^+(\mathscr{A}) \to D(\mathscr{C})$ exists.

ii) There is a natural isomorphism of functors $R(V_2 \circ V_1) \xRightarrow{\cong} R(V_2) \circ R(V_1)$.

Proof. Assertion i) is clear. For ii), there is a natural transformation $\gamma \colon Q_C \circ (V_2 \circ V_1) \Rightarrow (RV_2 \circ RV_1) \circ Q_A$ given as follows: if $A^\bullet \in K^+(\mathscr{A})$ and $R^\bullet \in \mathcal{K}_{V_1}$ is such that $A^\bullet \cong R^\bullet$ in $D^+(\mathscr{A})$, represent this isomorphism by a right roof $A^\bullet \xrightarrow{\text{quis}} T^\bullet \xleftarrow{\text{quis}} R^\bullet$, where $T^\bullet \in \mathcal{K}_{V_1}$, as in subsection 3.2.3; applying $V_2 \circ V_1$ we get the roof $V_2(V_1(A^\bullet)) \to V_2(V_1(T^\bullet)) \xleftarrow{\text{quis}} V_2(V_1(R^\bullet))$ in $K(\mathscr{C})$, as $V_1(\mathcal{K}_{V_1}) \subseteq \mathcal{K}_{V_2}$; we define $\gamma(A^\bullet)$ to be the morphism represented by this roof. By the universal property of the pair $(R(V_2 \circ V_1), \eta \colon Q_C \circ (V_2 \circ V_1) \Rightarrow R(V_2 \circ V_1) \circ Q_A)$, there is a unique natural transformation $\varepsilon \colon R(V_2 \circ R_1) \Rightarrow RV_2 \circ RV_1$ such that $(\varepsilon \circ Q_A) \circ \eta = \gamma$. By the construction of subsection 3.2.4, ε is an isomorphism since γ restricts to the identity on objects of $\mathcal{K}_{V_1}$. □

In the case where $\mathscr{A}$ and $\mathscr{B}$ have enough injective objects, and V_1 and V_2 are induced by left exact functors $\mathscr{A} \to \mathscr{B}$ and $\mathscr{B} \to \mathscr{C}$, we can derive the result above via a more hands-on method. This procedure uses a homological tool called a *spectral sequence of a double complex*. We refer the reader to Appendix C, where we introduce this device, and show how it can be used to prove Proposition 3.6.1 in the aforementioned special case. The formalism of spectral sequences will also be required later in Subsection 4.3.5. For future reference, we state right away the result below, which we prove in the appendix.

Corollary 3.6.2. Let $F \colon \mathscr{A} \to \mathscr{B}$ be a left exact functor and $\mathscr{A}$ have enough injectives. If $\mathscr{C} \subseteq \mathscr{B}$ is a thick subcategory and $R^iF(A) \in \mathscr{C}$ for all $A \in \mathscr{A}$, then $RF(A^\bullet) \in D^+_{\mathscr{C}}(\mathscr{B})$ for every $A^\bullet \in D^+(\mathscr{A})$. In other words, RF restricts to $RF \colon D^+(\mathscr{A}) \to D^+_{\mathscr{C}}(\mathscr{B})$.

Proof. We prove this result in page C.11 of Appendix C. □

Chapter 4

Abelian categories of sheaves

We begin by setting the notation for the rest of the text. A ringed space is denoted by a pair $(X, \mathscr{R})$, where X is the topological space, and $\mathscr{R}$ is the accompanying sheaf of rings on X. The category of ringed spaces is denoted $\mathrm{RgSpaces}$. The abelian category of $\mathscr{R}$-modules on a ringed space $(X, \mathscr{R})$ is denoted by $\mathrm{Mod}_{\mathscr{R}}(X)$.

We denote the category of schemes by Sch. If $X \in \mathrm{Sch}$, we always denote its structure sheaf by $\mathscr{O}_X$. In this case, and when there is no chance for confusion, we write the shorthand $\mathrm{Mod}_X \coloneqq \mathrm{Mod}_{\mathscr{O}_X}(X)$.

The structure of a ringed space $(X, \mathscr{R})$ enables one to define several functors out of the abelian category $\mathrm{Mod}_{\mathscr{R}}(X)$. The following table compiles some of the most important ones, along with their type of exactness.

Name	*Required data*	*Symbol*	*Exactness*
Stalk	$p \in X$	$(-)_p \colon \mathrm{Mod}_{\mathscr{R}}(X) \to \mathrm{Mod}_{\mathscr{R}_p}$	Exact
Sections	$U \in \mathrm{Op}(X)$	$\Gamma(U, -) \colon \mathrm{Mod}_{\mathscr{R}}(X) \to \mathrm{Mod}_{\mathscr{R}(U)}$	Left exact
Pushforward	$f \in \mathrm{Mor}_{\mathrm{RgSpaces}}(X, Y)$	$f_* \colon \mathrm{Mod}_{\mathscr{R}}(X) \to \mathrm{Mod}_{\mathscr{S}}(Y)$	Left exact
Tensor product	$\mathscr{F} \in \mathrm{Mod}_{\mathscr{R}}(X)$	$\mathscr{F} \otimes_{\mathscr{R}} (-) \colon \mathrm{Mod}_{\mathscr{R}}(X) \to \mathrm{Mod}_{\mathscr{R}}(X)$	Right exact
Inverse image	$f \in \mathrm{Mor}_{\mathrm{RgSpaces}}(X, Y)$	$f^{-1} \colon \mathrm{Mod}_{\mathscr{S}}(Y) \to \mathrm{Mod}_{f^{-1}(\mathscr{S})}(X)$	Exact
Pullback	$f \in \mathrm{Mor}_{\mathrm{RgSpaces}}(X, Y)$	$f^* \colon \mathrm{Mod}_{\mathscr{S}}(Y) \to \mathrm{Mod}_{\mathscr{R}}(X)$	Right exact

Table 4.1: Functors defined over the ringed spaces $(X, \mathscr{R})$ and $(Y, \mathscr{S})$.

We remind ourselves of the definition of the pullback functor. If $f \colon (X, \mathscr{R}) \to (Y, \mathscr{S})$ is a morphism of ringed spaces, so that $f^\sharp \colon f^{-1}\mathscr{S} \to \mathscr{R}$ is the accompanying map of sheaves of rings on X, for any $U \in \mathrm{Op}(X)$, the maps on sections $f^\sharp(U) \colon f^{-1}\mathscr{S}(U) \to \mathscr{R}(U)$ induce the structure of a $f^{-1}\mathscr{S}$-module on $\mathscr{R}$. Therefore, given any $\mathscr{G} \in \mathrm{Mod}_{f^{-1}\mathscr{S}}(X)$, the tensor product $\mathscr{R} \otimes_{f^{-1}\mathscr{S}} \mathscr{G}$ is a well defined $f^{-1}\mathscr{S}$-module on X. There is a canonical structure of a $\mathscr{R}$-module on this tensor product, induced by the extension of scalars over each $U \in \mathrm{Op}(X)$,

$$\begin{aligned} \mathscr{R}(U) \times \big(\mathscr{R}(U) \otimes_{f^{-1}\mathscr{S}(U)} \mathscr{G}(U)\big) &\longrightarrow \mathscr{R}(U) \otimes_{f^{-1}\mathscr{S}(U)} \mathscr{G}(U) \\ (r', r \otimes g) &\longmapsto (r'r) \otimes g \end{aligned}$$

The functor $f^* \colon \mathrm{Mod}_{\mathscr{S}}(Y) \to \mathrm{Mod}_{\mathscr{R}}(X)$ is then defined as the composition

$$\mathrm{Mod}_{\mathscr{S}}(Y) \xrightarrow{\quad f^{-1} \quad} \mathrm{Mod}_{f^{-1}(\mathscr{S})}(X) \xrightarrow{\mathscr{R} \otimes_{f^{-1}(\mathscr{S})} (-)} \mathrm{Mod}_{\mathscr{R}}(X) \ .$$

To conclude, we recall the adjuntion $f^* \dashv f_*$, [Vak17, 16.3.4].

In order to define the derived analogue of the half-exact functors of Table 4.1 under the framework developed in Chapters 2 and 3, a natural procedure to follow is to survey the abelian category $\mathrm{Mod}_{\mathscr{R}}(X)$ for the existence of injective (respectively, projective) objects, since, according to Theorem 3.4.10, these form a class adapted to any left (respectively, right) exact functor.

4.1 Probing injective and projective objects

4.1.1 Modules over ringed spaces

Proposition 4.1.1. If A is a ring, Mod_A has enough projectives and enough injectives.

Sketch of proof. Let $M \in \mathrm{Mod}_A$. The existence of enough projectives is straightforward: there exists an epimorphism $F \twoheadrightarrow M$ from a free A-module F, [Rei95, 2.4], and free A-modules are projective, [Lan02, III. §4]. Regarding the existence of injectives, Baer's criterion asserts that $Q \in \mathrm{Mod}_A$ is injective if and only if the following condition holds: for every ideal $I \subseteq A$, every A-module homomorphism $I \to Q$ extends to $A \to Q$, [Wei94, 2.3.1]. Using this criterion, we see that an abelian group G is injective as a $\mathbb{Z}$-module if and only if it is divisible (*i.e.* $G = nG$ for every $n \in \mathbb{N}$). Together with the fact that every quotient of a divisible group is again divisible, this shows that the category $\mathrm{Mod}_{\mathbb{Z}} \equiv \mathrm{Ab}$ has enough injectives, since every $\mathbb{Z}$-module is the quotient of a free $\mathbb{Z}$-module F by a submodule K, and hence injects into the divisible group $(F \otimes_{\mathbb{Z}} \mathbb{Q})/K$. Finally, to show that M embeds into an injective module, we consider the group $\mathrm{Hom}_{\mathbb{Z}}(A, M)$ of group homomorphisms from the underlying abelian group of A to the underlying abelian group of M. We can endow $\mathrm{Hom}_{\mathbb{Z}}(A, M)$ with the structure of an A-module via the action $(a, \phi) \mapsto \phi_a$, where $\phi_a \colon A \to M$ is the group homomorphism defined by $\phi_a(a') = \phi(aa')$. There is an injection $M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, M)$ of A-modules defined by $m \mapsto \psi^m$, where $\psi^m(a) = am$. Since $\mathrm{Mod}_{\mathbb{Z}}$ has enough injectives, we can find an injection of $\mathbb{Z}$-modules $M \hookrightarrow Q$, with Q divisible, and so we have $M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, Q)$. One can show that $\mathrm{Hom}_{\mathbb{Z}}(A, Q)$ is an injective A-module, [Wei94, 2.3.11]. □

Proposition 4.1.2. Let $(X, \mathscr{R})$ be a ringed space. Then $\mathrm{Mod}_{\mathscr{R}}(X)$ has enough injectives.

Proof. Let $\mathscr{F} \in \mathrm{Mod}_{\mathscr{R}}(X)$. We want to show that there a monomorphism $\mathscr{F} \hookrightarrow \mathscr{I}$, where $\mathscr{I}$ is an injective $\mathscr{R}$-module on X. Let $x \in X$ be any point and $\iota_x \colon \{x\} \hookrightarrow X$ be the inclusion. Since $\mathscr{F}_x$ is an $\mathscr{R}_x$-module and $\mathrm{Mod}_{\mathscr{R}_x}$ has enough injectives by Proposition 4.1.1, there exists an embedding $h_x \colon \mathscr{F}_x \hookrightarrow I_x$ into an injective $\mathscr{R}_x$-module. Consider the skyscraper sheaf $\iota_{x,*}(I_x)$ with abelian group I_x and supported at x, which is seen an $\mathscr{R}$-module on X via the structure maps of the stalk as a filtered colimit. Define $\mathscr{I} = \prod_{x \in X} \iota_{x,*}(I_x) \in \mathrm{Mod}_{\mathscr{R}}(X)$. Since morphisms of sheaves are determined by their induced maps on stalks [Vak17, 2.4.D],

$$\mathrm{Hom}_{\mathscr{R}}(\mathscr{F}, \mathscr{I}) \cong \prod_{x \in X} \mathrm{Hom}_{\mathscr{R}}(\mathscr{F}, \iota_{x,*}(I_x)) \cong \prod_{x \in X} \mathrm{Hom}_{\mathscr{R}_x}(\mathscr{F}_x, I_x).$$

On one hand, this shows that $\mathscr{I}$ is an injective $\mathscr{R}$-module because $\mathrm{Hom}_{\mathscr{R}}(-, \mathscr{I}) \colon \mathrm{Mod}_{\mathscr{R}}(X)^{\mathrm{opp}} \to \mathrm{Ab}$ is exact (Definition/Proposition 3.4.4) and, on other hand, it determines a monomorphism $\mathscr{F} \hookrightarrow \mathscr{I}$ induced by the collection of embeddings maps $\{h_x\}_{x \in X}$. □

Remark 4.1.3. In particular, this proposition shows that the category of sheaves of abelian groups on X has enough injectives, since $\mathrm{Ab}_X \cong \mathrm{Mod}_{\underline{\mathbb{Z}}}(X)$, [Vak17, 2.2.13].

The next example shows that there exists a ringed space $(X, \mathscr{R})$ such that $\mathrm{Mod}_{\mathscr{R}}(X)$ does not have enough projectives. The discussion is inspired on the blog post [Cla17].

Example 4.1.4. Let X denote a topological space with the following property: X has a closed point x such that, for any connected open neighborhood U of x, there exists a strictly smaller open neighborhood V of x. Suppose there exists an epimorphism $\mathscr{P} \twoheadrightarrow \underline{\mathbb{Z}}$ of sheaves of abelian groups on X, with $\mathscr{P}$ projective. Given any $W \in \mathrm{Op}(X)$, denote by $\iota_W \colon W \hookrightarrow X$ the inclusion, and define the shorthand $\mathscr{Z}_W := \iota_{W,!}(\underline{\mathbb{Z}}_W)$ for the extension by zero of the constant sheaf $\underline{\mathbb{Z}}_W$ associated to $\mathbb{Z}$ on W (with the subspace topology).

Fix some connected open neighborhood U of x. Using the hypothesis on X, there exists $V \in \mathrm{Op}(X)$ with $x \in V \subsetneq U$. Therefore, $X = (X \setminus \{x\}) \cup V$ is an open cover, and there is a surjection $\mathscr{Z}_{X\setminus\{x\}} \oplus \mathscr{Z}_V \twoheadrightarrow \underline{\mathbb{Z}}$. Since $\mathscr{P}$ is projective, there is a commutative diagram

$$\begin{array}{ccc} & & \mathscr{P} \\ & \swarrow & \downarrow \\ \mathscr{Z}_{X\setminus\{x\}} \oplus \mathscr{Z}_V & \twoheadrightarrow & \underline{\mathbb{Z}} \end{array}$$

and, in particular, the map $\mathscr{P}(U) \to \underline{\mathbb{Z}}(U)$ factors through $\mathscr{Z}_{X\setminus\{x\}}(U) \oplus \mathscr{Z}_V(U)$. We have that $\mathscr{Z}_{X\setminus\{x\}}(U) = 0$, as $x \in U$. Moreover, by definition of the extension by zero [Vak17, 2.7.G], $\mathscr{Z}_V(U)$ is in bijection with the set of families of compatible germs over U of the presheaf

$$\mathrm{Op}(X) \ni W \mapsto \begin{cases} \underline{\mathbb{Z}}_V(W) & \text{if } W \subseteq V \\ 0 & \text{otherwise} \end{cases}.$$

Since U is connected, the restriction maps over open subsets contained in V are the identity, and this presheaf just assigns an integer to V, and 0 to any other open subset $W \not\subseteq V$ of X (as is the case of U). We conclude that any family of compatible germs over U is trivial and hence $\mathscr{Z}_V(U) = 0$.

From this discussion, any surjection $\mathscr{P} \twoheadrightarrow \underline{\mathbb{Z}}$ is such that the map on sections $\mathscr{P}(U) \to \underline{\mathbb{Z}}(U)$ over any connected open neighborhood U of x is zero. This implies that the map on stalks $\mathscr{P}_x \to \underline{\mathbb{Z}}_x$ is also zero for every $x \in X$, contracting the surjectivity of $\mathscr{P} \twoheadrightarrow \underline{\mathbb{Z}}$.

Therefore, we have shown that the topological space X has the property that $\mathrm{Mod}_{\underline{\mathbb{Z}}}(X)$ does not have enough projective objects. One example of a topological space satisfying the condition imposed on X is the (underlying topological space of the) projective line over a field k, [Vak17, 23.4.7].

4.1.2 A review of (quasi)coherent modules

As mentioned at the end of the introduction to this chapter, we are a quest to find an abelian category of sheaves of modules that is suitable for the construction of the derived analogue of the half-exact functors of Table 4.1. The previous subsection showed that $\mathrm{Mod}_{\mathscr{R}}(X)$ is not such a category because of lack of projective objects. Since a scheme X is a particular type of ringed space, we may wonder if the additional structure we impose on its definition is sufficient to guarantee the existence of projectives. This is not the case. In fact, Example 4.1.4 can be adapted to show that there exists a scheme X such that Mod_X does not have enough projective objects, [Har77, Exerc. III.6.2.a]. Consequently, we devote our attention to studying the existence of injective and projective objects in abelian subcategories of the category of modules over a scheme.

This subsection is devoted to recalling the definition and main properties of locally free, quasicoherent, and coherent sheaves.

Definition 4.1.5. Let $(X,\mathscr{R}) \in \mathrm{RgSpaces}$ and $\mathscr{F} \in \mathrm{Mod}_{\mathscr{R}}(X)$.

$\mathscr{F}$ is said to be a ***free sheaf*** on X if $\mathscr{F} \cong \mathscr{R}^{\oplus I}$ for some index set I. If I can be taken to be finite, $\mathscr{F}$ is called a ***free sheaf of finite rank***.

$\mathscr{F}$ is said to a ***locally free sheaf*** (respectively, ***locally free sheaf of finite rank***) on X if there exists an open cover $\{U_i\}_{i\in I}$ of X such that, for any $i \in I$, $\mathscr{F}|_{U_i}$ is a free sheaf (respectively, free sheaf of finite rank) on the ringed space $(U_i, \mathscr{R}|_{U_i})$.

We denote the full additive subcategories of locally free sheaves and locally free sheaves of finite rank on a ringed space $(X,\mathscr{R})$ by $\mathrm{LocFree}_{\mathscr{R}}(X)$ and $\mathrm{LocFree}^f_{\mathscr{R}}(X)$, respectively. Again, if $X \in \mathrm{Sch}$ and there is no risk for confusion, we write $\mathrm{LocFree}_X$ (respectively, $\mathrm{LocFree}^f_X$) for the category of locally free $\mathscr{O}_X$-modules (respectively, of finite rank) on X.

Remark 4.1.6. For most schemes X, $\mathrm{LocFree}_X$ is not an abelian category. An example is given in [Vak17, 13.4.1].

Definition/Proposition 4.1.7. If A is a ring, the base of open subsets $\{D(f) \coloneqq \mathrm{Spec}\, A_f \,:\, f \in A\}$ of the affine scheme $\mathrm{Spec}\, A$ is called the ***distinguished affine base***, [Vak17, 3.5.A and 4.1.2].

If $M \in \mathrm{Mod}_A$, $\widetilde{M} \in \mathrm{Mod}_{\mathrm{Spec}\, A}$ is the sheaf of $\mathscr{O}_{\mathrm{Spec}\, A}$-modules on the affine scheme $\mathrm{Spec}\, A$ determined by the sheaf on the distinguished affine base that assigns to each $D(f)$ the localization M_f, and, to each inclusion $D(f) \subseteq D(g)$, the map $\mathrm{res}_{D(g),D(f)} \colon M_g \to M_f$ that arises from the universal property of the localization $M \to M_g$, [Vak17, 4.1.D]. $\widetilde{M}$ is called the ***$\mathscr{O}_{\mathrm{Spec}\, A}$-module induced by*** M.

Since every A-module homomorphism $M \to N$ determines induced maps of A_f-modules $M_f \to N_f$, for any $f \in A$, that fit into commutative diagrams

$$\begin{array}{ccc} M_g & \longrightarrow & N_g \\ {\scriptstyle \mathrm{res}_{D(g),D(f)}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}_{D(g),D(f)}} \\ M_f & \longrightarrow & N_f \end{array}$$

for every $D(f) \subseteq D(g)$, this construction determines a functor $(-)^\sim \colon \mathrm{Mod}_A \to \mathrm{Mod}_{\mathrm{Spec}\, A}$.

Proposition 4.1.8 ([GW10, 7.13])**.** If A is a ring, and $M, N \in \mathrm{Mod}_A$, the maps

$$\mathrm{Hom}_A(M,N) \underset{\Gamma(X,-)}{\overset{(-)^\sim}{\rightleftarrows}} \mathrm{Hom}_{\mathscr{O}_{\mathrm{Spec}\, A}}(\widetilde{M}, \widetilde{N})$$

are mutually inverse. In particular, the functor $(-)^\sim \colon \mathrm{Mod}_A \to \mathrm{Mod}_{\mathrm{Spec}\, A}$ is fully faithful.

Proposition 4.1.9 ([GW10, 7.14]). Let A be a ring.

i) A sequence of A-modules $M \xrightarrow{\phi} N \xrightarrow{\psi} P$ is exact if and only if the sequence of $\mathscr{O}_{\operatorname{Spec} A}$-modules $\widetilde{M} \xrightarrow{\widetilde{\phi}} \widetilde{N} \xrightarrow{\widetilde{\psi}} \widetilde{P}$ is exact. In particular, $(-)^{\sim} \colon \mathrm{Mod}_A \to \mathrm{Mod}_{\operatorname{Spec} A}$ is exact.

ii) If $\phi\colon M \to N$ is an A-module homomorphism, $(\operatorname{Ker}\phi)^{\sim} \cong \operatorname{Ker}\widetilde{\phi}$, $(\operatorname{Coker}\phi)^{\sim} \cong \operatorname{Coker}\widetilde{\phi}$ and $(\operatorname{Im}\phi)^{\sim} \cong \operatorname{Im}\widetilde{\phi}$.

iii) If $(M_i)_{i\in I}$ is a family of A-modules, $(\oplus_{i\in I} M_i)^{\sim} \cong \oplus_{i\in I}\widetilde{M_i}$.

iv) If $M\colon \mathscr{I} \to \mathrm{Mod}_A$ is a filtered diagram of A-modules, then $(\varinjlim_i M_i)^{\sim} \cong \varinjlim_i \widetilde{M_i}$.

Definition 4.1.10 ([Vak17, 13.2.2]). Let $X \in \mathrm{Sch}$ and $\mathscr{F} \in \mathrm{Mod}_X$. $\mathscr{F}$ is said to be ***quasicoherent*** if, for every affine open subset $\operatorname{Spec} A \subseteq X$, $\mathscr{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$, for some $M \in \mathrm{Mod}_A$. The category of quasicoherent $\mathscr{O}_X$-modules on X is denoted QCoh_X.

Remark 4.1.11. Taking $X = \operatorname{Spec} A$ in the definition above expresses $\mathrm{QCoh}_{\operatorname{Spec} A}$ as the essential image of the exact functor of Proposition 4.1.8. We conclude that Mod_A is equivalent to $\mathrm{QCoh}_{\operatorname{Spec} A}$. In particular, $\mathrm{QCoh}_{\operatorname{Spec} A}$ has enough injectives and enough projectives, by Proposition 4.1.1.

Remark 4.1.12. Under the equivalence of categories $\mathrm{Mod}_A \cong \mathrm{QCoh}_{\operatorname{Spec} A}$, the pushforward and the pullback have a useful interpretation as restriction and extension of scalars, respectively. Indeed, if $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of affine schemes, where $A, B \in \mathrm{Rings}$, and $f^\sharp\colon B \to A$ is the corresponding map of rings, then:

i) if $M \in \mathrm{Mod}_A$, we have that $f_*(\widetilde{M}) \cong ({}_B M)^{\sim}$, where ${}_B M$ denotes M viewed as a B-module via $f^\sharp$, [Har77, II.5.2.d];

ii) if $N \in \mathrm{Mod}_B$, $f^*(\widetilde{N}) \cong [{}_A(N \otimes_B A)]^{\sim}$, where A is seen as a B-module via $f^\sharp$ for the tensor product, and ${}_A(N \otimes_B A)$ is notation for the A-module structure on $N \otimes_B A$, given by $A \times (N \otimes_B A) \to N \otimes_B A$, $(a, n \otimes a') \mapsto n \otimes (aa')$, [Har77, II.5.2.e].

Proposition 4.1.13 ([GW10, 7.16]). Let $X \in \mathrm{Sch}$ and $\mathscr{F} \in \mathrm{Mod}_X$. $\mathscr{F} \in \mathrm{QCoh}_X$ if and only if there exists an affine cover $\{\operatorname{Spec} A_i\}_{i\in I}$ of X such that, for all $i \in I$, there exists $M_i \in \mathrm{Mod}_{A_i}$ with $\mathscr{F}|_{\operatorname{Spec} A_i} \cong \widetilde{M_i}$.

Lemma 4.1.14. If $X \in \mathrm{Sch}$ and $\mathscr{F} \in \mathrm{QCoh}_X$, for any $U \in \mathrm{Op}(X)$, $\mathscr{F}|_U \in \mathrm{QCoh}_U$.

Proof. Trivial. □

Proposition 4.1.15 ([GW10, 7.19]). Let $X \in \mathrm{Sch}$.

i) If $\{\mathscr{F}_i\}_{i\in I}$ is a collection of quasicoherent sheaves on X, the $\mathscr{O}_X$-module $\bigoplus_{i\in I} \mathscr{F}_i$ is again quasicoherent.

ii) If $\phi \in \mathrm{Hom}(\mathrm{QCoh}_X)$, then $\operatorname{Ker}\phi, \operatorname{Coker}\phi, \operatorname{Im}\phi \in \mathrm{QCoh}_X$.

iii) If $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}_X$, then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \in \mathrm{QCoh}_X$. Moreover, for any affine open subset $\operatorname{Spec} A \subseteq X$,

$$\Gamma(\operatorname{Spec} A, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) \cong \Gamma(\operatorname{Spec} A, \mathscr{F}) \otimes_{\Gamma(U, \mathscr{O}_X)} \Gamma(\operatorname{Spec} A, \mathscr{G}).$$

If $X \in \mathrm{Sch}$, assertions i) and ii) of Proposition 4.1.15 show, in particular, that the full subcategory $\mathrm{QCoh}_X \subseteq \mathrm{Mod}_X$ is abelian. We have a chain of *strict* inclusions [Vak17, 13.2.A],

$$\begin{array}{ccccc} \mathrm{LocFree}_X & \subsetneq & \mathrm{QCoh}_X & \subsetneq & \mathrm{Mod}_X \\ \text{(not abelian)} & & \text{(abelian)} & & \text{(abelian)} \end{array} .$$

Proposition 4.1.16 ([Har77, II.5.6])**.** Let $X = \operatorname{Spec} A$ be an affine scheme, with A a ring. If $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ is a short exact sequence of $\mathscr{O}_X$-modules and $\mathscr{F}_1 \in \mathrm{QCoh}_X$, then the left exact functor $\Gamma(X, -)\colon \mathrm{Mod}_X \to \mathrm{Mod}_A$ gives rise to an *exact* sequence of A-modules $0 \to \mathscr{F}_1(X) \to \mathscr{F}_2(X) \to \mathscr{F}_3(X) \to 0$.

Corollary 4.1.17 ([Vak17, 13.4.A])**.** If X is a scheme, a sequence $\mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3$ in QCoh_X is exact if and only if, given any affine cover $\{\operatorname{Spec} A_i\}_{i \in I}$ of X, the sequences $\mathscr{F}_1(\operatorname{Spec} A_i) \to \mathscr{F}_2(\operatorname{Spec} A_i) \to \mathscr{F}_3(\operatorname{Spec} A_i)$ are exact in Mod_{A_i}, for every $i \in I$.

Definition 4.1.18. Let A be a ring. $M \in \mathrm{Mod}_A$ is said to be:

i) ***finitely generated*** if there exists a surjection $A^{\oplus p} \twoheadrightarrow M$, for some $p \in \mathbb{Z}^{\geq 0}$.

ii) ***finitely presented*** if there exists a surjection $A^{\oplus p} \twoheadrightarrow M$ whose kernel is also finitely generated, for some $p \in \mathbb{Z}^{\geq 0}$.

iii) ***coherent*** if it is finitely generated and the kernel of *any* morphism $A^{\oplus p} \to M$ is finitely generated, for any $p \in \mathbb{Z}^{\geq 0}$.

Lemma 4.1.19 ([Vak17, 13.6.2])**.** If A is a Noetherian ring, $M \in \mathrm{Mod}_A$ is coherent if and only if it is finitely presented, if and only if it is finitely generated.

Proposition 4.1.20 ([Vak17, 13.6.3])**.** If A is a ring, the category of coherent A-modules is a full abelian subcategory of Mod_A.

Definition 4.1.21. Let $X \in \mathrm{Sch}$ and $\mathscr{F} \in \mathrm{QCoh}_X$. $\mathscr{F}$ is called ***coherent*** (respectively, ***finitely presented***, ***of finite type***) if, for every affine open set $\operatorname{Spec} A \subseteq X$, the A-module $\mathscr{F}(\operatorname{Spec} A)$ is coherent (respectively, finitely presented, finitely generated).

We denote by $\mathrm{Coh}_X \subseteq \mathrm{Mod}_X$ the subcategory of coherent sheaves on a scheme X. According to Proposition 4.1.20, Coh_X is a full abelian subcategory of QCoh_X, [Vak17, 13.6.4]. Therefore, we have a chain of strict inclusions $\mathrm{Coh}_X \subsetneq \mathrm{QCoh}_X \subsetneq \mathrm{Mod}_X$.

Proposition 4.1.13 showed that the property of an $\mathscr{O}_X$-module on a scheme X being quasicoherent can be checked on a specific affine cover of X. The next statement asserts that coherent sheaves also have this behavior.

Proposition 4.1.22 ([Vak17, 5.3.2, 13.6.C and 13.6.D])**.** Let $X \in \mathrm{Sch}$ and $\mathscr{F} \in \mathrm{Mod}_X$. $\mathscr{F} \in \mathrm{Coh}_X$ if and only if there exists an affine cover $\{\operatorname{Spec} A_i\}_{i \in I}$ of X such that, for all $i \in I$, there exists a coherent A_i-module M_i with $\mathscr{F}|_{\operatorname{Spec} A_i} \cong \widetilde{M_i}$.

4.1.3 There is no perfect choice of abelian category

Recall the following definitions.

Definition 4.1.23. Let $X \in \mathrm{Sch}$. X is said to be:

i) ***quasicompact***, if it is quasicompact as a topological space, *i.e.* if it can be covered by a finite number of affine open subsets.

ii) ***quasiseparated***, if the intersection of any two affine open subsets is a finite union of affine open subsets.

iii) ***qcqs***, if it is both quasicompact and quasiseparated.

iv) ***Noetherian***, if it can be covered by a finite number of affine open subsets of the form $\operatorname{Spec} A$, where A is a Noetherian ring.

Remark 4.1.24. We say that a topological space X is Noetherian if it satisfies the descending chain condition for closed subsets, [Vak17, 3.6.14]. In this case, any subset of X (with the subspace topology) is again Noetherian, [Sta21, Tag 0052]. Any Noetherian topological space is quasicompact, [Sta21, Tag 04ZA].

If X is now a scheme, and X is Noetherian as in Definition 4.1.23 iv), its underlying topological space is Noetherian, [Vak17, 5.1.C].

Any open subscheme of a quasiseparated subscheme is again quasiseparated. Any Noetherian scheme is qcqs. We conclude that any open subscheme of a Noetherian scheme is qcqs. In particular, any affine open subscheme of a Noetherian scheme is again Noetherian, [Vak17, 5.3.3, 5.3.4 and 5.3.A].

Definition 4.1.25. Let $\phi\colon X \to Y$ be a morphism of schemes.

ϕ is said to be ***quasicompact*** (respectively, ***quasiseparated***, ***qcqs***) if, for any affine open subscheme $\operatorname{Spec} B \subseteq Y$, $(\phi^{-1}(\operatorname{Spec} B), \mathscr{O}_X|_{\phi^{-1}(\operatorname{Spec} B)})$ is a quasicompact (respectively, ***quasiseparated***, ***qcqs***) scheme.

ϕ is said to be ***of finite type*** if, for any affine open subscheme $\operatorname{Spec} B \subseteq Y$, $\phi^{-1}(\operatorname{Spec} B)$ can be covered by a finite number of affine open subschemes $\{\operatorname{Spec} A_i\}_{i=1}^n$ such that the induced map of rings $B \to A_i$ expresses A_i as a finitely generated B-algebra. In particular, if $\phi^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$ where A is a finite B-algebra (*i.e.* finitely generated as an B-module), then ϕ is said to be ***finite***.

Remark 4.1.26. Any morphism of schemes $X \to Y$ with X Noetherian is qcqs, [Vak17, 7.3.B].

The finiteness conditions imposed on Noetherian schemes bring major simplifications. The next two statements follow directly from Proposition 4.1.22 and Lemma 4.1.19.

Corollary 4.1.27. Let X be a Noetherian scheme. A quasicoherent sheaf $\mathscr{F}$ on X is coherent if and only if it is finitely generated, if and only if it is of finite type.

Corollary 4.1.28. If $X \in \mathrm{Sch}$ is Noetherian, then $\mathrm{LocFree}_X^f \subseteq \mathrm{Coh}_X$. In particular, the structure sheaf $\mathscr{O}_X$ is coherent.

The following proposition concerns extensions in the category of (quasi)coherent sheaves. Again, Noetherianess is useful in the coherent case.

Proposition 4.1.29 ([Har77, II.5.7])**.** If $X \in \mathrm{Sch}$ and $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ is a short exact sequence of $\mathscr{O}_X$-modules, with $\mathscr{F}_1$ and $\mathscr{F}_3$ quasicoherent, then $\mathscr{F}_2$ is also quasicoherent. If X is Noetherian, and $\mathscr{F}_1, \mathscr{F}_3$ are coherent, then $\mathscr{F}_2$ is also coherent.

Proposition 4.1.30 ([Har77, II.5.8 and Exerc. II.5.5.c])**.** Let $f\colon X \to Y$ be a morphism of schemes.

i) If $\mathscr{G} \in \mathrm{QCoh}_Y$, then $f^*\mathscr{G} \in \mathrm{QCoh}_X$.

ii) If X and Y are Noetherian and $\mathscr{G} \in \mathrm{Coh}_Y$, then $f^*\mathscr{G} \in \mathrm{Coh}_X$.

iii) If X is Noetherian (or f is qcqs) and $\mathscr{F} \in \mathrm{QCoh}_X$, then $f_*\mathscr{F} \in \mathrm{QCoh}_Y$.

iv) If X and Y are Noetherian, and f is finite, given $\mathscr{F} \in \mathrm{Coh}_X$, then $f_*\mathscr{F} \in \mathrm{Coh}_Y$.

The next proposition shows that any quasicoherent sheaf on a Noetherian scheme has a resolution by injective quasicoherent sheaves.

Proposition 4.1.31. If X is a Noetherian scheme, the category QCoh_X has enough injectives.

Proof. Since X is Noetherian, we can cover X with by a finite collection of affine schemes $\operatorname{Spec} A_i$, with A_i Noetherian rings, for $i = 1, \dots, n$. If $\mathscr{F}$ is quasicoherent, $\mathscr{F}|_{\operatorname{Spec} A_i} \cong \widetilde{M_i}$, for some A_i-modules M_i, by Proposition 4.1.13. Since Mod_{A_i} has enough injectives by Proposition 4.1.1, we can embed each M_i in an injective A_i-module I_i. Denote by $\iota_{\operatorname{Spec} A_i}\colon \operatorname{Spec} A_i \to X$ the inclusions. Since $\widetilde{I_i}$ is a quasicoherent module on $\operatorname{Spec} A_i$, each pushforward ${\iota_{\operatorname{Spec} A_i}}_*(\widetilde{I_i})$ is also quasicoherent by Proposition 4.1.30 iii). Define

$$\mathscr{G} = \bigoplus_{i=1}^{n} {\iota_{\operatorname{Spec} A_i}}_*(\widetilde{I_i}) \ ,$$

which is again quasicoherent (Proposition 4.1.15 i). The monomorphisms $M_i \to I_i$ of A_i-modules induce monomorphisms (Proposition 4.1.9 ii) of $\mathscr{O}_{\operatorname{Spec} A_i}$-modules $\widetilde{M_i} \to \widetilde{I_i}$ and hence maps $\widetilde{M_i} \to \mathscr{G}$. The induced map $\mathscr{F} \to \mathscr{G}$ is then injective.

The only thing that is left to show is that $\mathscr{G}$ is an injective object in QCoh_X. By Definition/Proposition 3.4.4, this is equivalent to showing that the left-exact functor $\mathrm{Hom}_{\mathrm{QCoh}_X}(-, \mathscr{G})\colon \mathrm{QCoh}_X^{\mathrm{opp}} \to \mathrm{Ab}$ is exact, *i.e.* showing that if $\mathscr{H}_1 \xrightarrow{\phi} \mathscr{H}_2$ is an injective map of quasicoherent sheaves on X, then the induced map $\mathrm{Hom}(\mathscr{H}_2, \mathscr{G}) \to \mathrm{Hom}(\mathscr{H}_1, \mathscr{G})$ is surjective. By the definition of $\mathscr{G}$, this is the same as showing that each map

$$\mathrm{Hom}(\mathscr{H}_2, {\iota_{\operatorname{Spec} A_i}}_*(\widetilde{I_i})) \xrightarrow{-\circ\phi} \mathrm{Hom}(\mathscr{H}_1, {\iota_{\operatorname{Spec} A_i}}_*(\widetilde{I_i}))$$

is surjective. By the adjunction $\iota^* \dashv \iota_*$ and the identification ${\iota_{\operatorname{Spec} A_i}}^*\mathscr{H} \cong \mathscr{H}|_{\operatorname{Spec} A_i}$ this is the same as showing that each map

$$\mathrm{Hom}(\mathscr{H}_2|_{\operatorname{Spec} A_i}, \widetilde{I_i}) \xrightarrow{-\circ\phi|_{\operatorname{Spec} A_i}} \mathrm{Hom}(\mathscr{H}_1|_{\operatorname{Spec} A_i}, \widetilde{I_i})$$

is surjective. Let $\mathscr{H}_1|_{\operatorname{Spec} A_i} \cong \widetilde{N_i}$ and $\mathscr{H}_2|_{\operatorname{Spec} A_i} \cong \widetilde{K_i}$ for A_i-modules N_i and K_i. By the equivalence of categories of Remark 4.1.11, this is the same as showing that each map

$$\operatorname{Hom}(K_i, I_i) \xrightarrow{-\circ\phi(\operatorname{Spec} A_i)} \operatorname{Hom}(N_i, I_i)$$

is surjective. As ϕ is injective, so are the map on sections $\phi(\operatorname{Spec} A_i)\colon N_i \to K_i$. We conclude that the maps above are surjective since I_i is an injective A_i-module. □

The problem with projective objects still holds, as the next example shows.

Example 4.1.32. There exist Noetherian schemes X for which neither QCoh_X or Coh_X have enough projectives. In fact, one can prove that there is no surjection from a projective (quasi)coherent sheaf on the projective line over an infinite field to its structure sheaf, [Har77, Exerc. III.6.2.b].

However, not all is lost. Recall the following definitions.

Definition/Proposition 4.1.33 ([Vak17, Pags. 148-151])**.** Let S be a $\mathbb{Z}^{\geq 0}$-graded ring. The ideal $S_+ := \bigoplus_{n>0} S_n$ is called the irrelevant ideal. The ***homogeneous spectrum*** of S, denoted $\operatorname{Proj} S$, is set $\{\mathfrak{p} \in \operatorname{Spec} S \colon \mathfrak{p} \text{ is homogeneous and } \mathfrak{p} \not\supseteq S_+\}$, endowed with the subspace topology of $\operatorname{Spec} S$. Given a homogeneous element $f \in S_+$, we set $D_+(f) := \{\mathfrak{p} \in \operatorname{Proj} S \colon f \notin \mathfrak{p}\}$. The collection $\{D_+(f)\}$ as f ranges through the homogeneous elements of S_+ is a basis for the topology on $\operatorname{Proj} S$. For any such f, the localization S_f has a canonical structure of a $\mathbb{Z}$-graded ring, and we denote by $S_{[f]}$ its subring of elements of degree 0. Explicitly, $S_{[f]} = \{s/f^k \in S_f \colon \deg(s) = k \deg(f)\}$. Each $D_+(f)$ is homeomorphic to $\operatorname{Spec} S_{[f]}$, compatibly with restrictions $D_+(f) \subseteq D_+(g)$. We define the sheaf $\mathscr{O}_{\operatorname{Proj} S}$ on $\operatorname{Proj} S$ to be the unique (up to unique isomorphism) sheaf such that $\mathscr{O}_{\operatorname{Proj} S}|_{D_+(f)} \cong \mathscr{O}_{\operatorname{Spec} S_{[f]}}$ for any homogeneous element $f \in S_+$. From now on, we write $\operatorname{Proj} S$ for the scheme $(\operatorname{Proj} S, \mathscr{O}_{\operatorname{Proj} S})$.

Definition 4.1.34. If S is a $\mathbb{Z}^{\geq 0}$-graded ring and $S_0 = A$, we say that S is a ***graded ring over*** A.

Definition 4.1.35. We say that $X \in \operatorname{Sch}$ is a ***projective scheme over*** A (or projective A-scheme) if there exists a finitely generated graded ring S over A such that $X \cong \operatorname{Proj} S$.

Example 4.1.36. If A is a ring and $n \geq 0$, $\mathbb{P}^n_A := \operatorname{Proj} A[X_0, \dots, X_n]$ is called the projective n-space over A, [Vak17, 4.5.8].

Remark 4.1.37*.* Any affine scheme $\operatorname{Spec} A$ is a projective scheme over A since $\mathbb{P}^0_A = \operatorname{Proj} A[X] \cong$ $\cong \operatorname{Spec} A$, [Vak17, 4.5.11]. If X is any projective scheme over A, there is a canonical map of schemes $\operatorname{Proj} S \to \operatorname{Spec} A$, [Vak17, 6.3.H].

Proposition 4.1.38 ([Vak17, 5.1.I])**.** All projective A-schemes are qcqs.

Remark 4.1.39*.* Any projective scheme over a Noetherian ring is a Noetherian scheme, [Vak17, 5.3.D].

The following statement due to Serre asserts that, although we do not have enough projective objects in Coh_X over a projective scheme X, we can build resolutions using another type of sheaves.

Proposition 4.1.40 ([Har77, II.5.18])**.** Let X be a projective scheme over a Noetherian ring A. Then any coherent sheaf $\mathscr{F}$ on X is the quotient of a coherent locally free sheaf on X.

Remark 4.1.41. In this situation, we say that Coh_X "has enough locally frees".

On one hand, the passage from the quasicoherent to the coherent setting made possible the existence of enough locally free coherent sheaves on a projective scheme over a Noetherian ring. On other hand, we lose the existence of injective objects. Indeed, since injective modules are usually large (as in not finitely generated) because they should extend morphisms, we cannot hope to find injective resolutions of coherent sheaves by coherent sheaves. We provide a simple example of this behavior below.

Example 4.1.42. We will provide an example of a Noetherian scheme X such that Coh_X does not have enough injectives. Let $X = \operatorname{Spec} A$, where A is a principal ideal domain (hence Noetherian) that is not a field. By Remark 4.1.11, showing that Coh_X does not have enough injectives is equivalent to showing that the category of finitely generated A-modules does not have enough injectives. We do this by proving that any injective finitely generated A-module M is trivial.

If M is as above, by the structure theorem of finitely generated modules over principal ideal domains, we have an isomorphism

$$M \cong A^{\oplus r} \oplus \left(\bigoplus_{i=1}^{n} A/(p_i^{\alpha_i}) \right) ,$$

where $p_i \in A$ are prime elements and $r, n, \alpha_i \in \mathbb{Z}^{\geq 0}$. The sketch of the proof of Proposition 4.1.1 mentions that an abelian group is injective as a $\mathbb{Z}$-module if and only if it is divisible. This assertion can be extended to the case of a module over a principal ideal domain without change. Therefore, $M = aM$ for every $a \in A$. Under this assumption, the A-module homomorphisms $\phi_a \colon M \to M$, $m \mapsto am$, are surjective, for every $a \in A$. If $\alpha_i \neq 0$, the homomorphism $\phi_{p_i^{\alpha_i}}$ sends every element of M to an element which has 0 $(r+i)$-th coordinate, so this map is not surjective. We conclude that $M \cong A^{\oplus r}$. If $a \in A$ is any element, there exists $m \in M$ such that $am = (1, \dots, 0)$. In particular, this implies that every non-zero element of A is invertible. This contradicts the hypothesis that A is not a field.

We compile the results we have so far in Figure 4.1.

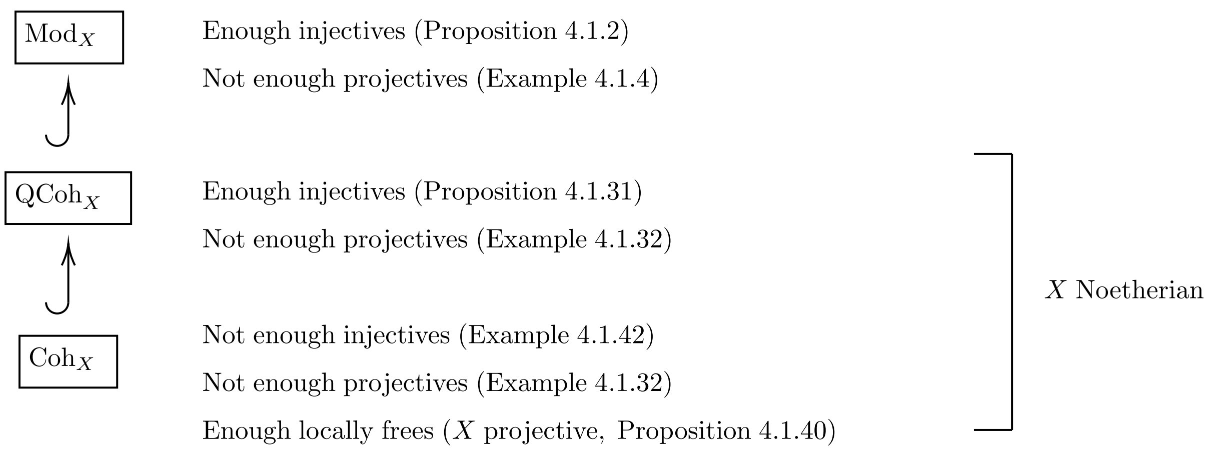


Figure 4.1: Adapted classes of objects in abelian categories of sheaves on schemes.

We conclude that there is no obvious abelian category of sheaves defined on a scheme that is suitable for the *direct* definition of *all* the half exact functors in Table 4.1. By this we mean that none of the abelian categories present in Figure 4.1 has a class of objects adapted to each of the functors in Table 4.1. We are then forced to rely on the chain of inclusions

$$\mathrm{Coh}_X \subsetneq \mathrm{QCoh}_X \subsetneq \mathrm{Mod}_X, \tag{4.1}$$

and study each functor one by one. The next section is devoted to exploiting the chain of inclusions (4.1) in the context of their derived categories. Under certain conditions, we will show that there is a subcategory of $D^+(\mathrm{Mod}_X)$ that is equivalent to $D^+(\mathrm{QCoh}_X)$, and that there is a subcategory of $D^b(\mathrm{QCoh}_X)$ that is equivalent to $D^b(\mathrm{Coh}_X)$. These relationships will be fundamental for defining the derived versions of $\Gamma(X, -)$, f_*, $\mathscr{F} \otimes (-)$, and so on, which will be the subject of Section 4.3.

4.2 Useful equivalences on Noetherian schemes

Throughout this section, let X stand for a Noetherian scheme.

4.2.1 Quasicoherent and coherent sheaves

As we saw in Section 4.1, the abelian category QCoh_X has enough injectives, but it has a full abelian subcategory Coh_X which does not have this property. Therefore, if we have some left exact functor $\xi\colon \mathrm{QCoh}_X \to \mathrm{QCoh}_X$, we can right derive it to obtain $R\xi\colon D^+(\mathrm{QCoh}_X) \to D^+(\mathrm{QCoh}_X)$ yet, as discussed in Section 3.5, a priori there is no reason for the image of a *coherent* sheaf $\mathscr{F} \in \mathrm{Coh}_X$ under $R\xi$ to be a bounded below complex of *coherent* sheaves on X. Despite that, there are special functors ξ such that this happens, and we will study them later in Subsection 4.3.4. Before that, we use the abstract tools developed in Section 3.5 to derive intermediate results.

Proposition 4.2.1. The natural functor $D^b(\mathrm{Coh}_X) \to D^b(\mathrm{QCoh}_X)$ induces an equivalence of categories $D^b(\mathrm{Coh}_X) \cong D^b_{\mathrm{coh}}(\mathrm{QCoh}_X)$ between the bounded derived category of coherent sheaves on X and the bounded derived category of quasicoherent sheaves on X with coherent cohomology, as in Definition 3.5.1.

According to Theorem 3.5.6, it suffices to prove the following two statements.

Proposition 4.2.2. The abelian subcategory $\mathrm{Coh}_X \subseteq \mathrm{QCoh}_X$ is thick.

Proposition 4.2.3. Given $\mathscr{F} \in \mathrm{QCoh}_X$ and $\mathscr{G} \in \mathrm{Coh}_X$ with an epimorphism $\mathscr{F} \twoheadrightarrow \mathscr{G}$, there exists a coherent subsheaf $\mathscr{G}' \subseteq \mathscr{F}$, such that the composition $\mathscr{G}' \hookrightarrow \mathscr{F} \twoheadrightarrow \mathscr{G}$ is still an epimorphism.

Proposition 4.2.2 follows directly from Proposition 4.1.29, while Proposition 4.2.3 requires additional work. The next lemma shows that, on a Noetherian scheme, one can extend coherent sheaves defined on an open subset to the whole scheme.

Lemma 4.2.4. If $U \in \mathrm{Op}(X)$, $\mathscr{G} \in \mathrm{Coh}_U$, $\mathscr{F} \in \mathrm{QCoh}_X$ such that $\mathscr{G} \subseteq \mathscr{F}|_U$, then there exists $\mathscr{G}' \in \mathrm{Coh}_X$ such that $\mathscr{G}' \subseteq \mathscr{F}$ and $\mathscr{G}'|_U \cong \mathscr{G}$.

Proof. Can be found in Appendix D, page D.12. □

Proof of Proposition 4.2.3. Cover X by a finite number of affine open subsets $\operatorname{Spec} A_i$, with A_i Noetherian rings, for $i = 1, \dots, n$. Then, setting $\mathscr{F}|_{\operatorname{Spec} A_i} \cong \widetilde{M_i}$ and $\mathscr{G}|_{\operatorname{Spec} A_i} \cong \widetilde{N_i}$ for A_i-modules M_i, N_i (with N_i finitely generated), we have surjections $M_i \twoheadrightarrow N_i$. Since N_i is projective, there exist finitely generated submodules $M_i' \subseteq M_i$ such that the composition $M_i' \subseteq M_i \twoheadrightarrow N_i$ is surjective. Then because $\widetilde{M_i'}$ is a coherent sheaf on $\operatorname{Spec} A_i$ such that $\widetilde{M_i'} \subseteq \mathscr{F}|_{\operatorname{Spec} A_i}$, there exists a coherent sheaf $\mathscr{G}_i' \subseteq \mathscr{F}$ on X that extends $\widetilde{M_i'}$, by Lemma 4.2.4. We define $\mathscr{G}' = \bigoplus_{i=1}^{n} \mathscr{G}_i'$. □

4.2.2 Sheaves of modules and quasicoherent sheaves

As we will see in Section 4.3, it will also be helpful to deduce an equivalence between the bounded (below) derived category of quasicoherent sheaves on a Noetherian scheme and the bounded (below) derived category of $\mathscr{O}_X$-modules with quasicoherent cohomology. We will use Corollary 3.5.7 for this purpose. The next statement follows directly from Proposition 4.1.29.

Proposition 4.2.5. $\mathrm{QCoh}_X \subseteq \mathrm{Mod}_X$ is a thick subcategory.

As we saw in Proposition 4.1.31, any quasicoherent sheaf on X admits a resolution by injective quasicoherent sheaves. Consider the following result.

Proposition 4.2.6 ([Har66, II.7.18])**.** If $\mathscr{F} \in \mathrm{QCoh}_X$ is an injective object in QCoh_X, there exists $\mathscr{G} \in \mathrm{QCoh}_X$ which is injective in Mod_X, together with a monomorphism $\mathscr{F} \hookrightarrow \mathscr{G}$.

Corollary 4.2.7. Let $\mathscr{F} \in \mathrm{QCoh}_X$. Then, $\mathscr{F}$ is an injective object in Mod_X if and only if $\mathscr{F}$ is an injective object in QCoh_X.

Proof. See Appendix D, page D.12. □

Since QCoh_X has enough injectives, we conclude that any quasicoherent sheaf on X can be embedded in a quasicoherent sheaf which is injective as an object in Mod_X. By Corollary 3.5.7, this result, together with Proposition 4.2.5, yields the statement below.

Proposition 4.2.8. The natural maps $D^*(\mathrm{QCoh}_X) \to D^*(\mathrm{Mod}_X)$ induce equivalences of categories $D^*(\mathrm{QCoh}_X) \cong D^*_{\mathrm{qcoh}}(\mathrm{Mod}_X)$, for $* = \{+, b\}$.

4.3 Derived functors between categories of sheaves

We are now ready to define the derived analogues of the most used functors in Algebraic Geometry, namely the functors in Table 4.1. It will become apparent from our discussion that some of these analogues can be defined over the category of sheaves of modules over a ringed space, while others can only be defined on subcategories, such as that of (quasi)coherent sheaves on a scheme.

The functor we will derive first is the global sections functor. Indeed, its derived functor will be fundamental throughout this text, as it will provide the *definition* of cohomology for sheaves.

4.3.1 Sheaf cohomology

Let $(X, \mathscr{R})$ be a ringed space. Recall that the global sections functor $\Gamma(X, -)\colon \mathrm{Mod}_{\mathscr{R}}(X) \to \mathrm{Mod}_{\mathscr{R}(X)}$ is left exact (Table 4.1). Since $\mathrm{Mod}_{\mathscr{R}}(X)$ has enough injectives (Proposition 4.1.2), the right derived global sections functor

$$R\Gamma(X, -)\colon D^+(\mathrm{Mod}_{\mathscr{R}}(X)) \to D^+(\mathrm{Mod}_{\mathscr{R}(X)})$$

exists by Theorems 3.1.10 and 3.4.10. We use this functor to *define* the cohomology of a sheaf of modules.

Definition 4.3.1. Let $(X, \mathscr{R}) \in \mathrm{RgSpaces}$ and $\mathscr{F} \in \mathrm{Mod}_{\mathscr{R}}(X)$. For $i \geq 0$, the ***i-th cohomology group*** of $\mathscr{F}$, denoted $H^i(X, \mathscr{F})$, is the image of $\mathscr{F}$ under the i-th higher right derived functor of the global sections functor, *i.e.* $H^i(X, \mathscr{F}) := R^i\Gamma(X, \mathscr{F})$, where $\mathscr{F}$ is seen as a complex concentrated in degree 0.

4.3.2 Derived sections of quasicoherent sheaves

If $X \in \mathrm{Sch}$, the global sections functor can be written $\Gamma(X, -)\colon \mathrm{QCoh}_X \to \mathrm{Mod}_{\mathscr{O}_X(X)}$. If X is Noetherian, QCoh_X has enough injectives (Proposition 4.1.31), and the right derived functor

$$R\Gamma(X, -)\colon D^+(\mathrm{QCoh}_X) \to D^+(\mathrm{Mod}_{\mathscr{O}_X(X)}) \tag{4.2}$$

also exists. The following result is due to Grothendieck, and shows that the number of non-trivial cohomology groups of a sheaf of abelian groups on a Noetherian scheme is finite.

Theorem 4.3.2 ([Har77, III.2.7])**.** If $X \in \mathrm{Sch}$ is Noetherian and $\mathscr{F} \in \mathrm{Ab}_X$, then $H^i(X, \mathscr{F}) = 0$ for all $i > \dim(X)$.

In particular, given a Noetherian scheme X, the image of $\mathscr{F} \in \mathrm{QCoh}_X$ (seen as a complex concentrated in degree 0) under the right derived functor (4.2) is a *bounded* complex of $\mathscr{O}_X(X)$-modules, *i.e.* $R\Gamma(X, \mathscr{F}) \in D^b(\mathrm{Mod}_{\mathscr{O}_X(X)})$. The next lemma asserts that this is enough to guarantee that the image under (4.2) of any actual bounded complex $\mathscr{F}^\bullet \in D^b(\mathrm{QCoh}_X)$ also lands in $D^b(\mathrm{Mod}_{\mathscr{O}_X(X)})$. We prove the lemma in full generality since it will be a useful tool for other applications to come.

Lemma 4.3.3. Let $\mathscr{A}, \mathscr{B}$ be abelian categories and $V\colon K^+(\mathscr{A}) \to K^+(\mathscr{B})$ an exact functor of triangulated categories such that $K^+(\mathscr{A})$ admits a triangulated subcategory $\mathcal{K}_V$ adapted to V, as in Definition 3.2.1. If $RV(A) \in D^b(\mathscr{B})$ for all $A \in \mathscr{A}$, then RV induces a functor $RV\colon D^b(\mathscr{A}) \to D^b(\mathscr{B})$.

Proof. Let $A^\bullet \in D^b(\mathscr{A})$ such that $A^i = 0$ for all $i > n$. Let $q\colon A^\bullet \to R^\bullet$ be a quasi-resolution by $R^\bullet \in \mathcal{K}_V \subseteq K^+(\mathscr{A})$. Since $d_R^n \circ q^n = 0$, there exists a canonical arrow $A^n \dashrightarrow \operatorname{Ker} d_R^n$, and the diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & A^{n-1} & \xrightarrow{d_A^{n-1}} & A^n & = & A^n & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle q^{n-1}} & & \downarrow & & \downarrow{\scriptstyle q^n} & & \downarrow & & \\
\cdots & \longrightarrow & R^{n-1} & \xrightarrow[d_R^{n-1}]{} & \operatorname{Ker} d_R^n & \longrightarrow & R^n & \xrightarrow[d_R^n]{} & R^{n+1} & \longrightarrow & \cdots
\end{array}$$

is commutative. Let $T^\bullet$ be the complex $R^n \to R^{n+1} \to \ldots$. Notice that $0 \to \operatorname{Ker} d_R^n \to T^\bullet$ is a resolution of $\operatorname{Ker} d_R^n \in \mathscr{A}$ by a complex $T^\bullet \in \mathcal{K}_V$. Therefore, $RV(\operatorname{Ker} d_R^n) = V(T^\bullet)$. By assumption $\exists\, m \geq 0$ such that $R^iV(\operatorname{Ker} d_R^n) = H^i(V(T^\bullet)) = 0$ for $i > m$, and hence $R^iV(A^\bullet) = H^i(V(R^\bullet)) = 0$ for $i > (n+m)$. By the use of a truncation functor similar to that of Example 2.3.7, we conclude that $RV(A^\bullet) \in D^b(\mathscr{B})$. $\square$

As suggested, Theorem 4.3.2 and Lemma 4.3.3 show that, over a Noetherian scheme X, the right derived functor of the global sections restricts to

$$R\Gamma(X, -) \colon D^b(\mathrm{QCoh}_X) \to D^b(\mathrm{Mod}_{\mathscr{O}_X(X)}).$$

4.3.3 Derived pushforward of quasicoherent sheaves

We restrict ourselves to the study of the derived pushforward for morphisms between Noetherian schemes. If $f \colon X \to Y$ is such a morphism, it takes QCoh_X into QCoh_Y by Proposition 4.1.30 iii). As QCoh_X has enough injectives (Proposition 4.1.31) and $f_* \colon \mathrm{QCoh}_X \to \mathrm{QCoh}_Y$ is left exact (Table 4.1), its right derived functor

$$Rf_* \colon D^+(\mathrm{QCoh}_X) \to D^+(\mathrm{QCoh}_Y)$$

exists.

If we are given two morphisms of Noetherian schemes $X \xrightarrow{f} Y \xrightarrow{g} Z$, it is easy to see that $(g \circ f)_* = g_* \circ f_*$. Therefore, it is legitimate to ask if this behavior extends to the derived functors, namely if there exists a natural isomorphism between $R(g \circ f)_*$ and $Rg_* \circ Rf_*$. This would follow directly from Proposition 3.6.1 if one could show that, given any bounded below complex of injective quasicoherent sheaves $\mathscr{I}^\bullet \in K^+(\mathrm{QCoh}_X)$, $K^+(f_*)(\mathscr{I}^\bullet)$ is a bounded below complex of injective quasicoherent sheaves on Y. It is easy to construct morphisms f such that this fails. We provide the following example.

Example 4.3.4. From Remark 4.1.12, if $f \colon \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of affine schemes and $M \in \mathrm{Mod}_A$, then $f_*(\widetilde{M}) \cong ({}_BM)^\sim$, where ${}_BM$ denotes restriction of scalars via the corresponding map on rings $f^\sharp \colon B \to A$.

Let p be a prime, $A = \mathbb{Z}_p$ and $B = \mathbb{Z}$. Since A is a field and B is a principal ideal domain, $\operatorname{Spec} A$ and $\operatorname{Spec} B$ are Noetherian schemes. Giving an example of an injective quasicoherent sheaf $\mathscr{I} \in \mathrm{QCoh}_{\operatorname{Spec} A}$ such that $f_*(\mathscr{I})$ is not injective in $\mathrm{QCoh}_{\operatorname{Spec} B}$ is equivalent (by Remark 4.1.11) to giving an example of an injective A-module I such that I is not injective when viewed as a B-module. Consider, for example, the A-module $I = \mathbb{Z}_p$. This is injective in Mod_A because all vector spaces are injective. However, I is not an injective abelian group because it is not divisible, as asserted in the proof of Proposition 4.1.2.

This issue can be dealt with by considering a different class of f_*-adapted objects in QCoh_X, that of flasque sheaves.

Definition 4.3.5. A sheaf of $\mathscr{R}$-modules $\mathscr{F}$ on a ringed space $(X, \mathscr{R})$ is called ***flasque*** if, for every inclusion of open sets $V \subseteq U$, the restriction maps $\mathrm{res}_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ are surjective.

Lemma 4.3.6 ([Har77, II.1.16])**.** Let $(X, \mathscr{R}) \in \mathrm{RgSpaces}$ and $0 \to \mathscr{F}_1 \xrightarrow{g} \mathscr{F}_2 \xrightarrow{h} \mathscr{F}_3 \to 0$ be an exact sequence in $\mathrm{Mod}_{\mathscr{R}}(X)$. Then:

i) If $\mathscr{F}_1$ is flasque, the sequence $0 \to \mathscr{F}_1(U) \to \mathscr{F}_2(U) \to \mathscr{F}_3(U) \to 0$ is exact in $\mathrm{Mod}_{\mathscr{R}(U)}$, for all $U \in \mathrm{Op}(X)$.

ii) If $\mathscr{F}_1$ and $\mathscr{F}_2$ are flasque, so is $\mathscr{F}_3$.

Lemma 4.3.7. If $f\colon (X, \mathscr{R}) \to (Y, \mathscr{S})$ is a map of ringed spaces and $\mathscr{F} \in \mathrm{Mod}_{\mathscr{R}}(X)$ is flasque, the pushforward $f_*\mathscr{F}$ is also flasque.

Proof. Trivial. □

Proposition 4.3.8. If $(X, \mathscr{R})$ is a ringed space, any injective $\mathscr{R}$-module on X is flasque.

Proof. We refer to Appendix D, page D.13. □

After this short detour, we are ready to show that flasque quasicoherent sheaves are f_*-adapted. As an intermediate step, we show that this class is $\Gamma(X, -)$-adapted.

Proposition 4.3.9. The class of quasicoherent flasque sheaves on a Noetherian scheme is adapted to the left exact functor $\Gamma(X, -)\colon \mathrm{QCoh}_X \to \mathrm{Mod}_{\mathscr{O}_X(X)}$.

According to Corollary 3.4.3, since $R\Gamma(X, -)\colon D^+(\mathrm{QCoh}_X) \to D^+(\mathrm{Mod}_{\mathscr{O}_X(X)})$ exists, it suffices to prove the following lemma.

Lemma 4.3.10. Let $X \in \mathrm{Sch}$ be Noetherian. Then:

i) If $\{\mathscr{F}_i\}_{i \in I}$ is a collection of flasque quasicoherent sheaves on X, $\bigoplus_{i \in I} \mathscr{F}_i$ is a flasque quasicoherent sheaf on X.

ii) If $\mathscr{F} \in \mathrm{QCoh}_X$ is flasque, $H^i(X, \mathscr{F}) = R^i\Gamma(X, \mathscr{F}) = 0$ for $i \geq 1$.

iii) If $\mathscr{F} \in \mathrm{QCoh}_X$, there exists a flasque sheaf $\mathscr{G} \in \mathrm{QCoh}_X$ and a monomorphism $\mathscr{F} \hookrightarrow \mathscr{G}$.

Proof. Again, we refer to Appendix D, page D.13. □

The next proposition gives a useful characterisation of the higher right derived functors of the pushforward. Its proof uses the formalism of δ-functors, introduced in Subsection 3.3.1.

Proposition 4.3.11. Let $f\colon X \to Y$ be continuous map of topological spaces, $\mathscr{F} \in \mathrm{Ab}_X$ and $i \geq 0$. Define the presheaf $\mathscr{H}^i_{\mathrm{pre}}(\mathscr{F})$ of abelian groups on Y by assigning, to each $U \in \mathrm{Op}(Y)$, the abelian group $\mathscr{H}^i_{\mathrm{pre}}(\mathscr{F})(U) \coloneqq H^i(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)})$. If $\mathscr{H}^i(\mathscr{F})$ is the sheafification of $\mathscr{H}^i_{\mathrm{pre}}(\mathscr{F})$, there is an isomorphism $R^i f_*(\mathscr{F}) \cong \mathscr{H}^i(\mathscr{F})$.

Proof. See page D.14 in Appendix D. □

As promised, the next proposition asserts that the class of quasicoherent flasque sheaves on a Noetherian scheme is f_*-adapted.

Proposition 4.3.12. Given a morphism of Noetherian schemes $f\colon X \to Y$, the class of quasicoherent flasque sheaves on X is adapted to the left exact functor $f_*\colon \mathrm{QCoh}_X \to \mathrm{QCoh}_Y$.

Proof. Since $Rf_*\colon D^+(\mathrm{QCoh}_X) \to D^+(\mathrm{QCoh}_Y)$ exists, it suffices to prove the hypothesis of Corollary 3.4.3. Two out of three of the hypothesis have already been proven in Lemma 4.3.10. Therefore, we only need to show that, if $\mathscr{F} \in \mathrm{QCoh}_X$ is flasque, $R^i f_*(\mathscr{F}) = 0$ for $i \geq 1$. For this, it suffices to show that $R^i f_*(\mathscr{F})(V) = 0$ for every $V \in \mathrm{Op}(Y)$ and $i \geq 1$. From Proposition 4.3.11, we see that $R^i f_*(\mathscr{F})|_V = R^i g_* \left(\mathscr{F}|_{f^{-1}(V)}\right)$, where $g\colon f^{-1}(V) \to Y$ is f restricted to $f^{-1}(V)$. Since the restriction of a flasque sheaf is flasque, the result follows from the previously mentioned proposition and from Lemma 4.3.10 ii). □

After settling that quasicoherent flasque sheaves on a Noetherian scheme are adapted to pushforwards, we arrive at the desired equality stated at the beginning of this subsection.

Proposition 4.3.13. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms of Noetherian schemes, then there is a natural isomorphism $R(g \circ f)_* \cong Rg_* \circ Rf_*\colon D^+(\mathrm{QCoh}_X) \to D^+(\mathrm{QCoh}_Z)$.

Proof. As hinted before, we use Proposition 3.6.1. The classes of flasque quasicoherent sheaves on X and on Y are adapted to f_* and g_*, respectively. Moreover, given any flasque sheaf $\mathscr{F} \in \mathrm{QCoh}_X$, $f_*(\mathscr{F})$ is flasque by Lemma 4.3.7. □

In addition, since quasicoherent flasque sheaves on a Noetherian scheme are also adapted to the global sections functor (Proposition 4.3.9), the next statement follows effortlessly.

Proposition 4.3.14. If $f\colon X \to Y$ is a morphism of Noetherian schemes, there is a natural isomorphism $R\Gamma(X, -) \cong R\Gamma(Y, -) \circ Rf_*\colon D^+(\mathrm{Ab}_X) \to D^+(\mathrm{Ab})$.

Proof. Follows directly from Propositions 3.6.1 and 4.3.9, together with the equality $\Gamma(Y, f_*(\mathscr{F})) = \Gamma(X, \mathscr{F})$, for every Ab_X. □

As we did for the global sections functor, we now try to restrict to the bounded derived category $D^b(\mathrm{QCoh}_X)$ on a Noetherian scheme X. This is straightforward with the tools developed so far.

Proposition 4.3.15. If $f\colon X \to Y$ is a morphism of Noetherian schemes, given any $\mathscr{F} \in \mathrm{QCoh}_X$, $R^i f_*(\mathscr{F}) = 0$ for every $i > \dim(X)$.

Proof. Follows directly from Grothendieck's Theorem 4.3.2 and from Proposition 4.3.11. □

Proposition 4.3.16. If $f\colon X \to Y$ is a morphism of Noetherian schemes, the right derived functor $Rf_*\colon D^+(\mathrm{QCoh}_X) \to D^+(\mathrm{QCoh}_Y)$ restricts to the right derived functor

$$Rf_*\colon D^b(\mathrm{QCoh}_X) \to D^b(\mathrm{QCoh}_Y).$$

Proof. Given any $\mathscr{F} \in \mathrm{QCoh}_X$, $Rf_*(\mathscr{F}) \in D^b(\mathrm{QCoh}_Y)$ by Proposition 4.3.15. Lemma 4.3.3 finishes the proof. $\square$

4.3.4 Restricting to the coherent setting

Up until this point, we have defined the right derived functors

$$R\Gamma(X, -)\colon D^b(\mathrm{QCoh}_X) \to D^b(\mathrm{Mod}_{\mathscr{O}_X(X)}), \tag{4.3}$$

$$Rf_*\colon D^b(\mathrm{QCoh}_X) \to D^b(\mathrm{QCoh}_Y) \tag{4.4}$$

on Noetherian schemes X, Y. In this subsection, we will restrict this functors to the bounded derived category of coherent sheaves $D^b(\mathrm{Coh}_X)$. As mentioned at the end of subsection 4.1.3, there is no direct way of constructing the right derived analogue of these functors because Coh_X does not have enough injectives, even if X is Noetherian. However, as we will shortly see, the results derived in Section 4.2 will enable us to define the coherent analogue of the functors (4.3) and (4.4) indirectly. A downside is that we need to restrict ourselves to even more specific schemes.

Recall that a ***scheme over a ring*** A is a pair (X, ϕ), consisting of $X \in \mathrm{Sch}$ and a *structure morphism* of schemes $\phi\colon X \to \operatorname{Spec} A$, [Vak17, 6.3.7]. Such a morphism induces a structure of an A-algebra on the rings of sections of the structure sheaf $\mathscr{O}_X$, over all open sets, and all restriction maps are maps of A-algebras, [Vak17, 5.3.6]. We denote the category of schemes over A by Sch_A. Since $\operatorname{Spec} \mathbb{Z}$ is the terminal object in Sch, [Vak17, 6.3.I], any scheme is (canonically) a scheme over $\mathbb{Z}$. Moreover, according to Remark 4.1.37, any projective A-scheme is a scheme over A.

If we take $A = k$, a field, any quasicoherent sheaf on $X \in \mathrm{Sch}_k$ takes values in the category of k-vector spaces, denoted Vec_k. Suppose further that X is Noetherian (*e.g.* any projective scheme over k, Remark 4.1.39), then:

- any coherent sheaf on X takes values in the category of finitely generated k-vector spaces, denoted Vec_k^f;
- the global sections functor $\Gamma(X, -)\colon \mathrm{QCoh}_X \to \mathrm{Vec}_k$ is a special case of the pushforward. Indeed, $\Gamma(X, \mathscr{F}) = f_*(\mathscr{F})$ if $\mathscr{F} \in \mathrm{QCoh}_X$ and $f\colon X \to \operatorname{Spec} k$ is the structure morphism.

We conclude that, in order to restrict the functors (4.3) and (4.4) over Noetherian k-schemes, it suffices to define the right derived functor of the pushforward, $Rf_*\colon D^b(\mathrm{Coh}_X) \to D^b(\mathrm{Coh}_Y)$.

Recall the following results.

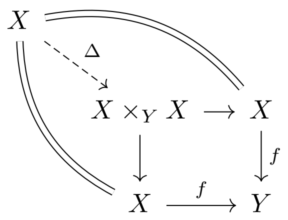
Theorem 4.3.17 ([Vak17, 9.1.1])**.** Sch has all fibered products. In order words, given any two morphisms of schemes $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$, the limit of the diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ exists. We denote it by $X \times_Z Y$.

Definition 4.3.18. A morphism of schemes $f\colon X \to Y$ is called:

i) a ***closed embedding*** if, for any affine subscheme $\operatorname{Spec} B \subseteq Y$, $f^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$ and the corresponding ring homomorphism $B \to A$ is surjective;

ii) ***universally closed*** if, for any scheme Z together with a morphism $Z \to Y$, the projection $X \times_Y Z \to Z$ from the fibered product is a closed map of the underlying topological spaces;

iii) ***separated*** if the ***diagonal morphism*** $\Delta\colon X \to X \times_Y X$, *i.e.* the unique map arising from the universal property of the fibered product



,

is a closed embedding; in this case, we also say that X is separated over Y (or over A if $Y = \operatorname{Spec} A$);

iv) ***proper*** if f is universally closed, separated and of finite type; in this case, we also say that X is proper over Y (or over A if $Y = \operatorname{Spec} A$).

Example 4.3.19. Any closed embedding is proper, [Vak17, 10.3.2].

Lemma 4.3.20 ([Sta21, Tag 02WG])**.** If $f\colon X \to Y$ and $g\colon Y \to Z$ are two closed embeddings (respectively, universally closed morphisms, separated morphisms, proper morphisms), the composition $g \circ f$ is also a closed embedding (respectively, universally closed morphism, separated morphism, proper morphism).

Proposition 4.3.21 ([Vak17, 10.3.5])**.** Given any projective A-scheme X, the structure morphism $X \to \operatorname{Spec} A$ of Remark 4.1.37 is proper.

The following theorem is a deep result that was firstly introduced in *Éléments de géométrie algébrique: III*. It is now known as *Grothendieck's Coherence Theorem*, [Vak17, 18.9.1].

Theorem 4.3.22 ([Gro61, 3.2.1])**.** Let $f\colon X \to Y$ be a proper morphism of Noetherian schemes. Then, for any $\mathscr{F} \in \mathrm{Coh}_X$, the higher direct images $R^i f_*(\mathscr{F})$ are again coherent, for all $i \geq 0$.

Corollary 4.3.23. Let X be a projective scheme over a field k. Then, given any $\mathscr{F} \in \mathrm{Coh}_X$, $H^i(X, \mathscr{F})$ is a finitely generated A-module, for each $i \geq 0$.

Proof. Follows directly from Theorem 4.3.22 by choosing f to be the structure morphism $X \to \operatorname{Spec} k$, which is proper by Proposition 4.3.21. □

By Theorem 4.3.22, if $f\colon X \to Y$ is a proper morphism of Noetherian schemes, given any $\mathscr{F} \in \mathrm{Coh}_X$, its image under the right derived functor $Rf_*\colon D^b(\mathrm{QCoh}_X) \to D^b(\mathrm{QCoh}_Y)$ lands in the full triangulated subcategory $D^b_{\mathrm{coh}}(\mathrm{QCoh}_Y)$ of bounded complexes of quasicoherent sheaves on Y that have coherent cohomology. Since $\mathrm{Coh}_X \subseteq \mathrm{QCoh}_X$ is a thick subcategory, Corollary 3.6.2 asserts that this property is sufficient to guarantee that, for any *complex* $\mathscr{F}^\bullet \in D^b(\mathrm{Coh}_X)$, $Rf_*(\mathscr{F}^\bullet) \in D^b_{\mathrm{coh}}(\mathrm{QCoh}_Y)$. Consequently, we can restrict Rf_* to $Rf_*\colon D^b(\mathrm{Coh}_X) \to D^b_{\mathrm{coh}}(\mathrm{QCoh}_Y)$.

By Proposition 4.2.1, the inclusion $D^b(\mathrm{Coh}_Y) \to D^b(\mathrm{QCoh}_Y)$ induces an equivalence $D^b(\mathrm{Coh}_Y) \cong D^b_{\mathrm{coh}}(\mathrm{QCoh}_Y)$, so we have defined $Rf_*\colon D^b(\mathrm{Coh}_X) \to D^b(\mathrm{Coh}_Y)$ in the case of proper morphisms of Noetherian schemes. We summarize our journey until here in the following diagram:

$$
\begin{array}{ccc}
D^+(\mathrm{QCoh}_X) & \xrightarrow{\quad Rf_* \quad} & D^+(\mathrm{QCoh}_Y) \\
\big\uparrow & & \big\uparrow \\
D^b(\mathrm{QCoh}_X) & \xrightarrow{\text{Proposition 4.3.16}} & D^b(\mathrm{QCoh}_Y) \\
\big\uparrow & & \big\uparrow \\
D^b(\mathrm{Coh}_X) & \xrightarrow[\text{and Corollary 3.6.2}]{\text{Theorem 4.3.22}} & D^b_{\mathrm{coh}}(\mathrm{QCoh}_Y) \\
\Big\| & \text{Proper morphisms} & \cong \big\uparrow \text{Proposition 4.2.1} \\
D^b(\mathrm{Coh}_X) & \xrightarrow{\qquad\qquad} & D^b(\mathrm{Coh}_Y)
\end{array} .
$$

As discussed before, if X is a Noetherian scheme over a field k, this procedure also defined the right derived functor of global sections, $R\Gamma(X,-)\colon D^b(\mathrm{Coh}_X) \to D^b\left(\mathrm{Vec}_k^f\right)$.

4.3.5 Derived tensor product

Given a Noetherian scheme X and $\mathscr{F} \in \mathrm{Coh}_X$, the tensor product introduced in Table 4.1 restricts to a right exact functor $\mathscr{F} \otimes_{\mathscr{O}_X} (-)\colon \mathrm{Coh}_X \to \mathrm{Coh}_X$, by Proposition 4.1.15 and the fact that the tensor product of two finitely generated modules is again finitely generated, [Sta21, Tag 05BS].

Referencing Figure 4.1, Coh_X does not have enough projectives, but it has enough locally free sheaves if X is projective, by Proposition 4.1.40. Consequently, and to simplify the arguments, we focus our attention on projective schemes over a field k.

Proposition 4.3.24. If $X \in \mathrm{Sch}_k$ is projective, the subclass of locally free coherent sheaves on X is adapted to the right exact functor $\mathscr{F} \otimes_{\mathscr{O}_X} (-)\colon \mathrm{Coh}_X \to \mathrm{Coh}_X$, where $\mathscr{F} \in \mathrm{Coh}_X$.

Proof. We verify the dual axioms of Definition 3.1.4. Axiom A1 is clear and the dual to Axiom A3 is just Proposition 4.1.40. Therefore, we just need to prove that, given any bounded above acyclic complex $\mathscr{G}^\bullet$ of coherent locally free sheaves on X, the complex $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}^\bullet$ is acyclic. Since checking exactness of a complex of sheaves can be preformed at the level of stalks, it suffices to prove acyclicity over a small enough open subset U of X such that each $\mathscr{G}^i$ is free. Then, it is clear that tensoring with any coherent sheaf over U maintains exactness. ☐

Thus, in the notation of the previous proposition, we have defined the left derived functor

$$L(\mathscr{F} \otimes (-))\colon D^-(\mathrm{Coh}_X) \to D^-(\mathrm{Coh}_X). \tag{4.5}$$

Definition 4.3.25. A local Noetherian ring $(A, \mathfrak{m})$ is called ***regular*** if $\mathfrak{m}$ can be generated by $\dim(A)$ elements. A Noetherian ring is called ***regular*** if, for every $\mathfrak{p} \in \operatorname{Spec} A$, the local ring $A_{\mathfrak{p}}$ is regular. Finally, a scheme X is called ***regular*** if, for every $x \in X$, there exists an affine open neighbourhood $\operatorname{Spec} A \subseteq X$ of x such that $\mathscr{O}_X(U)$ is Noetherian and regular.

The next result is crucial in order to restrict the derived tensor product to the bounded derived category of coherent sheaves.

Theorem 4.3.26 ([KK21, Exerc. 5.4.22])**.** If X is a regular Noetherian scheme, any $\mathscr{F} \in \mathrm{Coh}_X$ admits a resolution by coherent locally free sheaves of finite length.

In particular, Theorem 4.3.26 implies that, given two coherent sheaves $\mathscr{F}$ and $\mathscr{G}$ on a regular projective scheme over a field, $L(\mathscr{F} \otimes_{\mathscr{O}_X} (-))(\mathscr{G}) \in D^b(\mathrm{Coh}_X)$. The dual statement of Lemma 4.3.3 shows that (4.5) restricts to a left derived functor between the bounded derived categories. In summary,

$$\begin{array}{ccc} D^-(\mathrm{Coh}_X) & \xrightarrow{\quad L(\mathscr{F}\otimes(-)) \quad} & D^-(\mathrm{Coh}_X) \\ \uparrow & & \uparrow \\ D^b(\mathrm{Coh}_X) & \xrightarrow[\text{Theorem 4.3.26 and Lemma 4.3.3}]{X \text{ regular}} & D^b(\mathrm{Coh}_X) \end{array} .$$

We extend this construction to the derived tensor product of two complexes, using the formalism of double complexes and spectral sequences (Appendix C). If X is any scheme and $\mathscr{F}^\bullet, \mathscr{G}^\bullet \in K^-(\mathrm{Coh}_X)$, we build the third quadrant double complex $E^{\bullet,\bullet}$ as follows.

$$\begin{aligned} E^{p,q} &:= \mathscr{F}^p \otimes \mathscr{G}^q \\ d_h^{p,q} &:= d^p_{\mathscr{F}} \otimes 1 \\ d_v^{p,q} &:= (-1)^{p+q} 1 \otimes d^q_{\mathscr{G}} \end{aligned} \qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & \mathscr{F}^{-2} \otimes \mathscr{G}^0 & \longrightarrow & \mathscr{F}^{-1} \otimes \mathscr{G}^0 & \longrightarrow & \mathscr{F}^0 \otimes \mathscr{G}^0 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow {\scriptstyle d_v^{-1,-1}} & & \uparrow & & \\ \dots & \longrightarrow & \mathscr{F}^{-2} \otimes \mathscr{G}^{-1} & \longrightarrow & \mathscr{F}^{-1} \otimes \mathscr{G}^{-1} & \xrightarrow[d_h^{-1,-1}]{} & \mathscr{F}^0 \otimes \mathscr{G}^{-1} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & \mathscr{F}^{-2} \otimes \mathscr{G}^{-2} & \longrightarrow & \mathscr{F}^{-1} \otimes \mathscr{G}^{-2} & \longrightarrow & \mathscr{F}^0 \otimes \mathscr{G}^{-2} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

We define $\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet \in K^-(\mathrm{Coh}_X)$ to be the total complex of $E^{\bullet,\bullet}$, $\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet := (\mathrm{Tot}\, E)^\bullet$, as in Definition C.1. This defines a functor

$$\begin{aligned} \mathscr{F}^\bullet \otimes (-) \colon K^-(\mathrm{Coh}_X) &\to K^-(\mathrm{Coh}_X) \\ \mathscr{G}^\bullet &\mapsto \mathscr{F}^\bullet \otimes \mathscr{G}^\bullet. \end{aligned} \tag{4.6}$$

Lemma 4.3.27. If $X \in \mathrm{Sch}$ and $\mathscr{F}^\bullet, \mathscr{G}^\bullet, \mathscr{H}^\bullet \in K^-(\mathrm{Coh}_X)$, then:

i) $\mathscr{F}^\bullet \otimes (\mathscr{G}^\bullet[1]) = (\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet)[1]$.

ii) $\mathscr{F}^\bullet \otimes (\mathscr{G}^\bullet \oplus \mathscr{H}^\bullet) = (\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet) \oplus (\mathscr{F}^\bullet \otimes \mathscr{H}^\bullet)$.

Proof. Straightforward. □

Using the lemma above, one can see that (4.6) is an exact functor of triangulated categories. Since this is not a functor induced from a functor defined over the underlying abelian categories, in order to construct its left derived counterpart we need to use the generalizations developed in Subsection 3.2.6.

Proposition 4.3.28. Let $X \in \mathrm{Sch}_k$ be projective.

i) The additive subcategory $\mathscr{L}$ of bounded above complexes of coherent locally free sheaves on X is a full triangulated subcategory of $K^-(\mathrm{Coh}_X)$.

ii) $\mathscr{L}$ is adapted to the exact functor (4.6), in the sense of Definition 3.2.1.

Proof. According to Definition 2.2.13, proving i) is a matter of checking that, given a morphism $f\colon \mathscr{G}^\bullet \to \mathscr{H}^\bullet$ of bounded above complexes of coherent locally free sheaves, the terms of $\mathrm{cone}\,(f)$ are also locally free. This is a direct consequence of the trivial fact that $\mathrm{LocFree}_X$ is stable under direct sums.

To prove assertion ii), we check the axioms of Definition 3.2.1. Axiom a2 follows from the fact that Coh_X "has enough locally frees" (Proposition 4.1.40) and the fact that this implies the existence of quasi-resolutions (Definition/Proposition 3.1.3). For axiom a1, let $\mathscr{G}^\bullet \in K^-(\mathrm{Coh}_X)$ be an acyclic bounded above complex of coherent locally free sheaves. We want to show that $\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet$ is also acyclic. For this, we use the spectral sequences of Theorem C.3, applied to the double complex $E^{p,q} = \mathscr{F}^p \otimes \mathscr{G}^q$. Consider the spectral sequence whose zero-th differential is $d_v^{p,q} = (-1)^{p+q} 1 \otimes d_{\mathscr{G}}^q$, which abuts to a quotient of a filtration of $H^{p+q}(\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet)$. Its second page is ${}_vE_2^{p,q} = H_h^q H_v^p(\mathscr{F}^p \otimes \mathscr{G}^q)$. For fixed p, the complex $\mathscr{F}^p \otimes \mathscr{G}^\bullet$ is exact, as in the proof of Proposition 4.3.24, and hence ${}_vE_2^{p,q} = 0$ for all p, q, and $\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet$ is acyclic, as desired. □

Having settled the existence of a full triangulated subcategory adapted to $\mathscr{F}^\bullet \otimes (-)\colon K^-(\mathrm{Coh}_X) \to K^-(\mathrm{Coh}_X)$, Theorem 3.2.4 then guarantees that the left derived functor

$$L(\mathscr{F}^\bullet \otimes (-))\colon D^-(\mathrm{Coh}_X) \to D^-(\mathrm{Coh}_X) \tag{4.7}$$

exists, under the assumption that X is a projective k-scheme.

Finally, consider the bifunctor

$$\begin{aligned} K^-(\mathrm{Coh}_X) \times D^-(\mathrm{Coh}_X) &\to D^-(\mathrm{Coh}_X) \\ (\mathscr{F}^\bullet, \mathscr{G}^\bullet) &\mapsto L(\mathscr{F}^\bullet \otimes (-))(\mathscr{G}^\bullet). \end{aligned} \tag{4.8}$$

Proposition 4.3.29. The bifunctor (4.8) is exact in the first argument and descends to a well-defined left derived bifunctor $(-) \otimes^L (-)\colon D^-(\mathrm{Coh}_X) \times D^-(\mathrm{Coh}_X) \to D^-(\mathrm{Coh}_X)$.

Proof. Exactness in the first argument can be proven easily by choosing quasi-resolutions by complexes of locally free sheaves and relying on the analogous properties of Lemma 4.3.27 for the first argument.

Regarding the descent to a bifunctor on the derived categories, the claim follows from the universal property of localisation (Proposition 2.1.5) if we show that, given two quasi-isomorphic $\mathscr{F}_1^\bullet, \mathscr{F}_2^\bullet \in K^-(\mathrm{Coh}_X)$, $L(\mathscr{F}_1^\bullet \otimes (-))(\mathscr{G}^\bullet) \cong L(\mathscr{F}_2^\bullet \otimes (-))(\mathscr{G}^\bullet)$ for any $\mathscr{G}^\bullet \in D^-(\mathrm{Coh}_X)$. Equivalently by Corollary 2.3.21, one has to show that, if $\mathscr{F}^\bullet \in K^-(\mathrm{Coh}_X)$ is acyclic, $L(\mathscr{F}^\bullet \otimes (-))(\mathscr{G}^\bullet)$ is acyclic for any $\mathscr{G}^\bullet \in D^-(\mathrm{Coh}_X)$. Let $\mathscr{E}^\bullet \to \mathscr{G}^\bullet$ be a quasi-resolution of $\mathscr{G}^\bullet$ by a bounded above complex $\mathscr{E}^\bullet$ of coherent locally free sheaves. Then, $L(\mathscr{F}^\bullet \otimes (-))(\mathscr{G}^\bullet) = \mathscr{F}^\bullet \otimes \mathscr{E}^\bullet$. Again, we use a spectral sequence associated to the double complex

$E^{p,q} = \mathscr{F}^p \otimes \mathscr{E}^q$, but starting with the horizontal differential $d_h^{p,q} = d_{\mathscr{F}}^p \otimes 1$. The second page of the sequence ${}_hE_2^{p,q} = H_v^p H_h^q(\mathscr{F}^p \otimes \mathscr{E}^q)$. For fixed q, tensoring with a locally free sheaf $\mathscr{F}^\bullet \otimes \mathscr{E}^q$ maintains exactness (we can reduce to the case where $\mathscr{E}^q$ is free), and therefore ${}_hE_2^{p,q} = 0$ for all p, q. By Theorem C.3, this implies that $H^i(\mathscr{F}^\bullet \otimes \mathscr{E}^\bullet) = 0$ for all i. □

In order to restrict to the coherent setting, we rely on the same method that was used to define $L(\mathscr{F} \otimes (-)) \colon D^b(\mathrm{Coh}_X) \to D^b(\mathrm{Coh}_X)$ for a single sheaf $\mathscr{F} \in D^b(\mathrm{Coh}_X)$. Given a pair of coherent sheaves over a regular projective scheme over a field $(\mathscr{F}, \mathscr{G})$, by Theorem 4.3.26, we pick quasi-resolutions $\mathscr{E}_1^\bullet \to \mathscr{F}$ and $\mathscr{E}_2^\bullet \to \mathscr{G}$ for $\mathscr{E}_1^\bullet, \mathscr{E}_2^\bullet \in K^b(\mathrm{Coh}_X)$ bounded complexes of coherent locally free sheaves, and $\mathscr{F} \otimes^L \mathscr{G} = \mathscr{E}_1^\bullet \otimes \mathscr{E}_2^\bullet$ is again a bounded complex. Lemma 4.3.3 is used once again to guarantee that this is sufficient for the bifunctor to restrict to the bounded derived categories. In summary,

$$\begin{array}{ccc} D^-(\mathrm{Coh}_X) \times D^-(\mathrm{Coh}_X) & \xrightarrow{(-)\otimes^L(-)} & D^-(\mathrm{Coh}_X) \times D^-(\mathrm{Coh}_X) \\ \uparrow & & \uparrow \\ D^b(\mathrm{Coh}_X) \times D^b(\mathrm{Coh}_X) & \xrightarrow[\text{Theorem 4.3.26 and Lemma 4.3.3}]{X \text{ regular}} & D^b(\mathrm{Coh}_X) \times D^b(\mathrm{Coh}_X) \end{array}.$$

We remark the following natural isomorphisms: for $\mathscr{F}^\bullet, \mathscr{G}^\bullet, \mathscr{H}^\bullet \in D^b(\mathrm{Coh}_X)$,

$$\mathscr{F}^\bullet \otimes^L \mathscr{G}^\bullet \cong \mathscr{G}^\bullet \otimes^L \mathscr{F}^\bullet,$$
$$\mathscr{F}^\bullet \otimes^L \left(\mathscr{G}^\bullet \otimes^L \mathscr{H}^\bullet\right) \cong \left(\mathscr{F}^\bullet \otimes^L \mathscr{G}^\bullet\right) \otimes^L \mathscr{H}^\bullet.$$

Proving the existence of these maps reduces to giving the isomorphisms for tensor products of complexes of locally free coherent sheaves, which reduces further to the well-known isomorphisms for tensor products of single locally free coherent sheaves.

4.3.6 Derived pullback

Recall from the introduction of this chapter that, if $f \colon (X, \mathscr{R}) \to (Y, \mathscr{S})$ is a morphism of ringed spaces, the pullback under f is the right exact functor

$$\mathrm{Mod}_{\mathscr{S}}(Y) \xrightarrow{f^{-1}} \mathrm{Mod}_{f^{-1}(\mathscr{S})}(X) \xrightarrow{\mathscr{R} \otimes_{f^{-1}(\mathscr{S})}(-)} \mathrm{Mod}_{\mathscr{R}}(X) .$$

Therefore, since f^{-1} is exact, if we want to define the left derived functor Lf^* we only need to derive the functor $\mathscr{R} \otimes_{f^{-1}(\mathscr{S})} (-)$. If $X, Y \in \mathrm{Sch}$, $\mathscr{R} = \mathscr{O}_X$ and $\mathscr{S} = \mathscr{O}_Y$, given $\mathscr{F} \in \mathrm{Mod}_Y$, its image under f^{-1} is not an $\mathscr{O}_X$-module, but a $f^{-1}(\mathscr{O}_Y)$-module on X. In particular, $\mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} (-)$ is not a functor $\mathrm{Mod}_X \to \mathrm{Mod}_X$, but a functor $\mathrm{Mod}_{f^{-1}(\mathscr{O}_Y)}(X) \to \mathrm{Mod}_X$. In Subsection 4.3.5, and specifically in equation (4.7), we only defined a derived tensor product for coherent modules over the structure sheaf of a scheme. Therefore, we need to generalize our arguments in order define the left derived analogue of $\mathrm{Mod}_{f^{-1}(\mathscr{O}_Y)}(X) \to \mathrm{Mod}_X$. Recall the following definition.

Definition 4.3.30. Given a ring A, $M \in \mathrm{Mod}_A$ is said to be ***flat*** if the functor $(-) \otimes_A M \colon \mathrm{Mod}_A \to \mathrm{Mod}_A$ is exact.

Definition 4.3.31 ([GW10, 7.37])**.** Let $f \colon (X, \mathscr{R}) \to (Y, \mathscr{S})$ be a morphism of ringed spaces and $\mathscr{F} \in \mathrm{Mod}_{\mathscr{R}}(X)$. For any $p \in X$, we can view the $\mathscr{R}_p$-module $\mathscr{F}_p$ as a $\mathscr{S}_{f(p)}$-module via (the induced map on stalks of) the accompanying map of sheaves $f^\sharp \colon \mathscr{S} \to f_*(\mathscr{R})$. Then, $\mathscr{F}$ is:

i) f***-flat over*** $p \in X$ if $\mathscr{F}_p$ is flat as a $\mathscr{S}_{f(p)}$-module.

ii) f***-flat*** if $\mathscr{F}$ is f-flat over every $p \in X$.

We say that f is ***flat*** if $\mathscr{R}$ is f-flat. Taking $Y = X$, $\mathscr{S} = \mathscr{R}$ and $f = \mathrm{id}_X$, we say that $\mathscr{F}$ is ***flat*** (respectively, over $p \in X$) if $\mathscr{F}$ is id_X-flat (respectively, over $p \in X$), *i.e.* if the stalk $\mathscr{F}_p$ is a flat $\mathscr{R}_p$-module for every $p \in X$.

Proposition 4.3.32 ([Sta21, Tag 05NE])**.** If $(X, \mathscr{R}) \in \mathrm{RgSpaces}$, $\mathscr{F} \in \mathrm{Mod}_{\mathscr{R}}(X)$ is flat if and only if the functor $\mathrm{Mod}_{\mathscr{R}}(X) \to \mathrm{Mod}_{\mathscr{R}}(X)$, $\mathscr{G} \mapsto \mathscr{G} \otimes_{\mathscr{R}} \mathscr{F}$ is exact.

Proposition 4.3.33 ([Sta21, Tag 02N2])**.** Let $f \colon (X, \mathscr{R}) \to (Y, \mathscr{S})$ be a morphism of ringed spaces.

i) f is flat if and only if the functor $\mathrm{Mod}_{f^{-1}(\mathscr{S})}(X) \to \mathrm{Mod}_{f^{-1}(\mathscr{S})}(X)$, $\mathscr{G} \mapsto \mathscr{G} \otimes_{f^{-1}(\mathscr{S})} \mathscr{R}$ is exact.

ii) If f is flat, the change of rings functor

$$\begin{aligned} \mathrm{Mod}_{f^{-1}(\mathscr{S})}(X) &\to \mathrm{Mod}_{\mathscr{R}}(X) \\ \mathscr{G} &\mapsto \mathscr{R} \otimes_{f^{-1}(\mathscr{S})} \mathscr{G} \end{aligned}$$

is exact. In particular, the pullback $f^* \colon \mathrm{Mod}_{\mathscr{S}}(Y) \to \mathrm{Mod}_{\mathscr{R}}(X)$ is exact.

It can be shown that, given a morphism of schemes $f \colon X \to Y$, the class of flat $\mathscr{O}_X$-modules is adapted to the pullback functor $f^* \colon \mathrm{Mod}_Y \to \mathrm{Mod}_X$, [Har66, Pag. 99], allowing us to define its left derived functor $Lf^* \colon D^-(\mathrm{Mod}_Y) \to D^-(\mathrm{Mod}_X)$. Moreover, Lf^* can be proven to take $D^-(\mathrm{QCoh}_X)$ into $D^-(\mathrm{QCoh}_X)$, and, in the case where X and Y are Noetherian, $D^-(\mathrm{Coh}_X)$ into $D^-(\mathrm{Coh}_X)$, [Har66, II.4.4]. We will not get into the details of this construction because, in the rest of this text, we will only deal with pullbacks of flat morphisms between schemes, which, according to Proposition 4.3.33 ii), are already exact. We state this fact as a proposition for future reference.

Proposition 4.3.34. If $f \colon X \to Y$ is a flat morphism between Noetherian schemes, the pullback $f^* \colon \mathrm{Coh}_Y \to \mathrm{Coh}_X$ extends to a well defined functor $D^b(\mathrm{Coh}_Y) \to D^b(\mathrm{Coh}_X)$ by applying f^* term-wise, which we also denote by f^*.

Chapter 5

Integral functors

We finish the text with a chapter devoted to applications. Namely, we will define the notion of integral functors and Fourier-Mukai transforms, which are functors between the bounded derived categories of coherent sheaves on projective schemes. We will present no more than the basic definitions and immediate properties. Lastly, we state one of the most important results in the theory of integral functors, due to Orlov.

5.1 Compatibilities between derived functors

Before defining integral transforms, we state two useful relationships between the derived functors constructed in Chapter 4. First and foremost, we recall some properties of fibered products of schemes and of the pullback functor.

Proposition 5.1.1 ([Sta21, Tag 02WF]). Let P stand for the following properties of morphisms of schemes: being an open embedding, being a closed embedding, being qcqs, being proper, being flat. Then, if

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{u} & Y \end{array}$$

is a fibered product square, if f (respectively, u) has property P, then so does g (respectively, v).

Proposition 5.1.2 ([Vak17, 16.3.7]). Let $\phi\colon X \to Y$ be a morphism of schemes and $\mathscr{F} \in \mathrm{QCoh}_Y$.

i) There is a canonical isomorphism $\phi^*\mathscr{O}_Y \cong \mathscr{O}_X$.

ii) If $\varphi\colon Z \to X$ is another morphism of schemes, there is a canonical isomorphism $\varphi^*\phi^*\mathscr{F} \cong (\phi \circ \varphi)^*\mathscr{F}$.

iii) If $\mathscr{G} \in \mathrm{QCoh}_Y$, $\phi^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) \cong \phi^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \phi^*(\mathscr{G})$.

If $\phi\colon X \to Y$ is a qcqs morphism between schemes, the pushforward $\phi_*\colon \mathrm{QCoh}_X \to \mathrm{QCoh}_Y$ and the pullback $\phi^*\colon \mathrm{QCoh}_Y \to \mathrm{QCoh}_X$ are an adjoint pair, $\phi^* \dashv \phi_*$, [Vak17, 16.3.6]. Explicitly, this means that, for each pair $\mathscr{F} \in \mathrm{QCoh}_X$ and $\mathscr{G} \in \mathrm{QCoh}_Y$, there is an isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}_X}(\phi^*\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathrm{Mod}_Y}(\mathscr{G}, \phi_*\mathscr{F}),$$

which is functorial in both $\mathscr{F}$ and $\mathscr{G}$. Consequently, there are natural maps:

i) $\phi^*\phi_*\mathscr{F} \to \mathscr{F}$, induced by the identity in $\mathrm{Hom}_{\mathrm{Mod}_Y}(\phi_*\mathscr{F}, \phi_*\mathscr{F})$;

ii) $\mathscr{G} \to \phi_*\phi^*\mathscr{G}$, induced by the identity in $\mathrm{Hom}_{\mathrm{Mod}_X}(\phi^*\mathscr{G}, \phi^*\mathscr{G})$.

We are now set to state a compatibility result between the pullback and pushforward functors.

Proposition 5.1.3. If

$$\begin{array}{ccc} W & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{u} & Y \end{array}$$

is a commutative diagram of schemes, f and g are qcqs, and $\mathscr{F} \in \mathrm{QCoh}_X$, there is a natural morphism $u^* f_* \mathscr{F} \to g_* v^* \mathscr{F}$ of quasicoherent sheaves on Z.

Proof. Note the following sequence of functorial isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Mod}_Z}(u^* f_* \mathscr{F}, g_* v^* \mathscr{F}) &\cong \mathrm{Hom}_{\mathrm{Mod}_Y}(f_* \mathscr{F}, (g \circ u)_* v^* \mathscr{F}) \\ &= \mathrm{Hom}_{\mathrm{Mod}_Y}(f_* \mathscr{F}, (f \circ v)_* v^* \mathscr{F}) = \mathrm{Hom}_{\mathrm{Mod}_Y}(f_* \mathscr{F}, f_* v_* v^* \mathscr{F}). \end{aligned}$$

The claimed morphism is the image of the natural map $\mathscr{F} \to v_* v^* \mathscr{F}$ under the pushforward $f_* \colon \mathrm{QCoh}_X \to \mathrm{QCoh}_Y$. □

The following result states that, under certain conditions, not only is the natural map we defined in the above proposition an isomorphism, but we also get an isomorphism of derived functors $u^* \circ Rf_* \cong Rg_* \circ v^*$.

Proposition 5.1.4 ([Sta21, Tag 02KH])**.** Let $f \colon X \to Y$ be a qcqs morphism and $u \colon Z \to Y$ be a flat morphism. Consider the fibered product square

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{u} & Y \end{array} .$$

Then, for any $i \geq 0$ and $\mathscr{F} \in \mathrm{QCoh}_X$, there is a natural isomorphism

$$u^* R^i f_* \mathscr{F} \xrightarrow{\cong} R^i g_* v^* \mathscr{F}.$$

The statement of Proposition 5.1.4 is usually called ***flat base change***. We can also derive a compatibility result between the pushforward and the tensor product functors. Indeed, consider the following propositions.

Proposition 5.1.5. If $\phi \colon X \to Y$ is a qcqs morphism of schemes, $\mathscr{F} \in \mathrm{QCoh}_X$ and $\mathscr{G} \in \mathrm{QCoh}_Y$, there is a natural morphism $(\phi_* \mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G} \to \phi_*(\mathscr{F} \otimes_{\mathscr{O}_X} \phi^* \mathscr{G})$. Moreover, if $\mathscr{G}$ is locally free, this morphism is an isomorphism.

Proof. Note the following natural isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Mod}_Y}((\phi_* \mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G}, \phi_*(\mathscr{F} \otimes_{\mathscr{O}_X} \phi^* \mathscr{G})) &\cong \mathrm{Hom}_{\mathrm{Mod}_{\mathscr{O}_X}}(\phi^*((\phi_* \mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G}), \mathscr{F} \otimes_{\mathscr{O}_X} \phi^* \mathscr{G}) \\ &\cong \mathrm{Hom}_{\mathrm{Mod}_{\mathscr{O}_X}}((\phi^* \phi_* \mathscr{F}) \otimes_{\mathscr{O}_X} \phi^* \mathscr{G}, \mathscr{F} \otimes_{\mathscr{O}_X} \phi^* \mathscr{G}). \end{aligned}$$

The mentioned morphism is therefore the image of the canonical map $\phi^*\phi_*\mathscr{F} \to \mathscr{F}$ under the functor $(-) \otimes_{\mathscr{O}_X} \phi^*\mathscr{G} \colon \mathrm{QCoh}_X \to \mathrm{QCoh}_X$. If $\mathscr{G}$ is locally free, in order to prove that the morphism is an isomorphism, we replace Y by a sufficiently small open subset so that $\mathscr{G}$ is free, *i.e.* $\mathscr{G} = \mathscr{O}_Y^{\oplus I}$. The result then follows from the fact that tensor product and pullback commute with direct sums. □

Proposition 5.1.6. Let $f \colon X \to Y$ be a flat, proper morphism between regular projective schemes over a field. Then, given any $\mathscr{F}^\bullet \in D^b(\mathrm{Coh}_X)$ and $\mathscr{G}^\bullet \in D^b(\mathrm{Coh}_Y)$, there exists a natural isomorphism

$$Rf_*(\mathscr{F}^\bullet) \otimes^L \mathscr{G}^\bullet \xrightarrow{\cong} Rf_*(\mathscr{F}^\bullet \otimes^L f^*\mathscr{G}^\bullet).$$

Proof. Note that we do not need to derive f^* since f is flat (Proposition 4.3.34). Pick a quasi-resolution $\mathscr{E}^\bullet \to \mathscr{G}^\bullet$ by a bounded complex of locally free coherent sheaves $\mathscr{E}^\bullet$ by Proposition 4.3.28, and a quasi-resolution $\mathscr{F}^\bullet \to \mathscr{I}^\bullet$ by, say a bounded complex of injective quasicoherent sheaves $\mathscr{I}^\bullet$, by Proposition 4.3.16. Then, by construction, we want to give an isomorphism $f_*(\mathscr{I}^\bullet) \otimes \mathscr{E}^\bullet \xrightarrow{\cong} f_*(\mathscr{I}^\bullet \otimes f^*\mathscr{E}^\bullet)$. Using the definition of the tensor product of complexes, one sees that this reduces to showing Proposition 5.1.5 in the locally free case. □

Comparing Propositions 5.1.5 and 5.1.6, one sees a clear advantage of working over the derived category $D^b(\mathrm{Coh}_X)$, as it enables us to draw more compatibilities than we could in its underlying abelian category Coh_X. Proposition 5.1.6 is usually called the ***projection formula***, [Huy06, Pag. 83].

5.2 Introduction to integral functors

As mentioned at the introduction of this chapter, integral functors are functors between the bounded derived categories of coherent sheaves on projective schemes. Having said that, not all projective schemes carry enough structure to define such a functor. We restrict ourselves to smooth projective varieties, which we now define.

Definition/Proposition 5.2.1 ([Vak17, 5.2.4 and 5.2.F])**.** A scheme X is said to be ***integral*** if it is non-empty and for every non-empty open subset $U \subseteq X$ the ring $\mathscr{O}_X(U)$ is an integral domain. Equivalently, X is integral if it is both reduced (*i.e.* for every $x \in X$, the local ring $\mathscr{O}_{X,x}$ has no non-zero nilpotents, [Vak17, 5.2.A]) and irreducible.

Definition 5.2.2 ([Har77])**.** Let k be a field.

A ***variety*** over k is a k-scheme X that is integral and such that the structure morphism $X \to \operatorname{Spec} k$ is separated and of finite type.

A ***projective variety*** over a field k is an integral projective k-scheme.

Remark 5.2.3. Some sources (*e.g.* [Vak17]) define a variety over k simply as a reduced, separated scheme of finite type over k, hence not requiring irreducibility. We stick to the more classical definition, as is the case in [Har77] or [Sta21].

The notion of regularity on schemes of finite type over a field k can be generalized to a concept known as *smoothness*. In fact, smoothness is a relative definition, *i.e.* we say that a morphism of finite type k-schemes $X \to Y$ is smooth, and say that X is smooth if its structure morphism $X \to \operatorname{Spec} k$ is smooth. Although a more robust definition that regularity [Vak17, 12.2.5], smoothness is more cumbersome to define. We bypass the need to define this concept by working over algebraically closed fields $k = \overline{k}$. Indeed, a $\overline{k}$-scheme of finite type is smooth if and only if it is regular, as in Definition 4.3.25, [Har77, III.10.0.3]. Note, however, that the arguments made in this section can be generalized to smooth k-schemes of finite type. A standard text that deals with this more general case is [Huy06], on which the following discussion is based on.

Working over algebraically closed fields also has the upside that the fibered product $X \times_{\operatorname{Spec} k} Y$ of two (projective) varieties is again a (projective) variety, [Sta21, Tag 05P3] and [Vak17, 9.6].

Until the end of this chapter, let k stand for an algebraically closed field and $D(X) \coloneqq D^b(\mathrm{Coh}_X)$ for a scheme X. If $X, Y \in \mathrm{Sch}_k$, we also write the shorthand $X \times Y$ for the fibered product $X \times_{\operatorname{Spec} k} Y$.

Definition 5.2.4. Let X and Y be regular projective varieties over k, and

$$\begin{array}{ccc} & X \times Y & \\ {}^{q}\swarrow & & \searrow^{p} \\ X & & Y \end{array}$$

be the projections. The ***integral functor*** $\Phi^{X \to Y}_{\mathscr{P}^\bullet}$ with ***kernel*** $\mathscr{P}^\bullet \in D(X \times Y)$ is the functor

$$\begin{aligned} \Phi^{X \to Y}_{\mathscr{P}^\bullet} \colon D(X) &\to D(Y) \\ \mathscr{F}^\bullet &\mapsto Rp_*(q^*\mathscr{F}^\bullet \otimes^L \mathscr{P}^\bullet). \end{aligned}$$

We say that $\Phi^{X \to Y}_{\mathscr{P}^\bullet}$ is a ***Fourier-Mukai transform*** if it is an equivalence of categories. In this case, we say that X and Y are ***Fourier-Mukai partners***.

Remark 5.2.5. We do not need to derive the pullback q^* because q is flat. Indeed, by Proposition 5.1.1, the projections q and p are flat if the structure morphisms $X \to \operatorname{Spec} k$ and $Y \to \operatorname{Spec} k$, respectively, are flat. By Definition 4.3.31, $X \to \operatorname{Spec} k$ is flat if $(-) \otimes_k \mathscr{O}_{X,x} \colon \mathrm{Vec}_k \to \mathrm{Vec}_k$ is exact for every $x \in X$. This is the case because, for example, $(-) \otimes_k \mathscr{O}_{X,x}$ is left adjoint to $\mathrm{Hom}_{\mathrm{Vec}_k}(\mathscr{O}_{X,x}, -)$, and all vector spaces are projective.

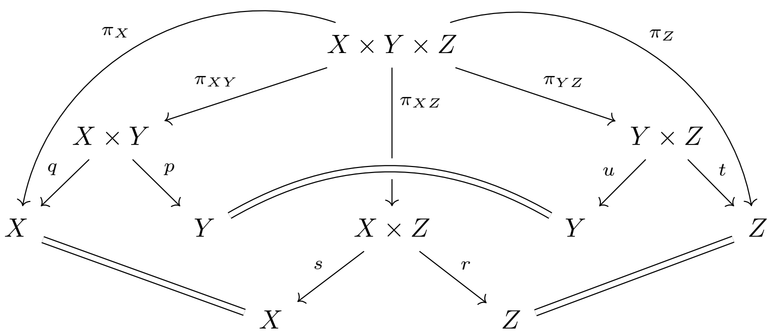
Notation. Whenever suitable, we drop the label $(\cdot)^{X \to Y}$ from the notation of the integral functor $\Phi^{X \to Y}_{\mathscr{P}^\bullet}$.

An immediate observation is that any integral functor is an exact functor of triangulated categories since it is the composition of the three exact functors q^*, $(-) \otimes^L \mathscr{P}^\bullet$ and Rp_*. We provide a simple example of a Fourier-Mukai transform.

Example 5.2.6. If X is an in Definition 5.2.4, the identity functor $\mathrm{id} \colon D(X) \to D(X)$ is a Fourier-Mukai transform. Indeed, let $\Delta \colon X \to X \times X$ be the diagonal morphism, which is a closed embedding by Propositions 4.3.21 and 5.1.1 (and hence proper by Example 4.3.19), and also flat by the discussion of Remark 5.2.5. Write $\delta = \Delta(X) \subseteq X \times X$ and let $\mathscr{O}_\delta \coloneqq \Delta_*\mathscr{O}_X$ be its structure sheaf. Then, $\mathrm{id} \cong \Phi_{\mathscr{O}_\delta}$. Indeed, if $\mathscr{F}^\bullet \in D(X)$,

$$
\begin{aligned}
\Phi_{\mathscr{O}_\delta}(\mathscr{F}^\bullet) &= Rp_*(q^*\mathscr{F}^\bullet \otimes^L \mathscr{O}_\delta) \\
&= Rp_*(q^*\mathscr{F}^\bullet \otimes^L \Delta_*\mathscr{O}_X) \\
&\cong Rp_*\Big(R\Delta_*\big(\Delta^*q^*\mathscr{F}^\bullet \otimes^L \mathscr{O}_X\big)\Big) && \text{by Proposition 5.1.6,} \\
&\cong R(p_* \circ \Delta_*)(\Delta^*q^*\mathscr{F}^\bullet \otimes^L \mathscr{O}_X) && \text{by Proposition 4.3.13,} \\
&\cong R(p_* \circ \Delta_*)(\Delta^*q^*\mathscr{F}^\bullet) && \text{since } \mathscr{O}_X \text{ is free,} \\
&\cong R(p_* \circ \Delta_*)((q \circ \Delta)^*\mathscr{F}^\bullet) && \text{by Proposition 5.1.2,} \\
&\cong \mathscr{F}^\bullet && \text{as } p \circ \Delta = \mathrm{id}_X = q \circ \Delta.
\end{aligned}
$$

Proposition 5.2.7. Let X, Y, Z be regular projective varieties over k. Let $\mathscr{P}^\bullet \in D(X \times Y)$ and $\mathscr{Q}^\bullet \in D(Y \times Z)$. Consider the diagram of projections to the right. Define



$$\mathscr{R}^\bullet := R\pi_{XZ,*}(\pi_{XY}^*\mathscr{P}^\bullet \otimes^L \pi_{YZ}^*\mathscr{Q}^\bullet).$$

Then the composition of integral functors

$$D(X) \xrightarrow{\Phi_{\mathscr{P}^\bullet}} D(Y) \xrightarrow{\Phi_{\mathscr{Q}^\bullet}} D(Z)$$

is naturally isomorphic to the integral functor $\Phi_{\mathscr{R}^\bullet}\colon D(X) \to D(Z)$.

Proof. As in Example 5.2.6, we prove the statement by direct calculation. Note the fibered product square

$$
\begin{array}{ccc}
X \times Y \times Z & \xrightarrow{\pi_{YZ}} & Y \times Z \\
{\scriptstyle \pi_{XY}}\big\downarrow & & \big\downarrow{\scriptstyle u} \\
X \times Y & \xrightarrow{\quad p \quad} & Y
\end{array} .
$$

Given some $\mathscr{F}^\bullet \in D(X)$,

$$
\begin{aligned}
\Phi_{\mathscr{R}^\bullet}(\mathscr{F}^\bullet) &\cong Rr_*(s^*\mathscr{F}^\bullet \otimes^L \mathscr{R}^\bullet) \\
&= Rr_*\Big(s^*\mathscr{F}^\bullet \otimes^L R\pi_{XZ,*}\big(\pi_{XY}^*\mathscr{P}^\bullet \otimes^L \pi_{YZ}^*\mathscr{Q}^\bullet\big)\Big) \\
&\cong Rr_*\Big(R\pi_{XZ,*}\big(\pi_{XZ}^*s^*\mathscr{F}^\bullet \otimes^L \pi_{XY}^*\mathscr{P}^\bullet \otimes^L \pi_{YZ}^*\mathscr{Q}^\bullet\big)\Big) && \text{by Proposition 5.1.6,} \\
&\cong Rr_*\Big(R\pi_{XZ,*}\big(\pi_X^*\mathscr{F}^\bullet \otimes^L \pi_{XY}^*\mathscr{P}^\bullet \otimes^L \pi_{YZ}^*\mathscr{Q}^\bullet\big)\Big) && \text{since } s \circ \pi_{XZ} = \pi_X, \\
&\cong R\pi_{Z_*}\Big(\pi_X^*\mathscr{F}^\bullet \otimes^L \pi_{XY}^*\mathscr{P}^\bullet \otimes^L \pi_{YZ}^*\mathscr{Q}^\bullet\Big) && \text{since } r \circ \pi_{XZ} = \pi_Z, \\
&\cong R\pi_{Z,*}\Big(\pi_{XY}^*\big(q^*\mathscr{F}^\bullet \otimes^L \mathscr{P}^\bullet\big) \otimes^L \pi_{YZ}^*\mathscr{Q}^\bullet\Big) && \text{since } q \circ \pi_{XZ} = \pi_X,
\end{aligned}
$$

$$\begin{aligned}
\Phi_{\mathscr{R}^\bullet}(\mathscr{F}^\bullet) &\cong Rt_*\Big[R\pi_{YZ,*}\Big(\pi_{XY}^*\big(q^*\mathscr{F}^\bullet \otimes^L \mathscr{P}^\bullet\big) \otimes^L \pi_{YZ}^*\mathscr{Q}^\bullet\Big)\Big] && \text{since } t\circ\pi_{YZ} = \pi_Z,\\
&\cong Rt_*\Big[R\pi_{YZ,*}\pi_{XY}^*\big(q^*\mathscr{F}^\bullet \otimes^L \mathscr{P}^\bullet\big) \otimes^L \mathscr{Q}^\bullet\Big] && \text{by Proposition 5.1.6,}\\
&\cong Rt_*\Big[u^*Rp_*\big(q^*\mathscr{F}^\bullet \otimes^L \mathscr{P}^\bullet\big) \otimes^L \mathscr{Q}^\bullet\Big] && \text{by Proposition 5.1.4,}\\
&= Rt_*\Big[u^*\Phi_{\mathscr{P}^\bullet}(\mathscr{F}^\bullet) \otimes^L \mathscr{Q}^\bullet\Big]\\
&= \Phi_{\mathscr{Q}^\bullet}\Big(\Phi_{\mathscr{P}^\bullet}(\mathscr{F}^\bullet)\Big).
\end{aligned}$$

□

The previous proposition asserts that the composition of integral functors is again an integral functor. Taking $Z = X$ in the statement above,

$$D(X) \underset{\Phi_{\mathscr{Q}^\bullet}}{\overset{\Phi_{\mathscr{P}^\bullet}}{\rightleftarrows}} D(Y)\ , \qquad \begin{array}{ccccc} \mathscr{P}^\bullet, \mathscr{Q}^\bullet & & X\times Y\times X & & \mathscr{O}_\delta^X \\ \big| & \overset{\pi_{XY}}{\swarrow} & & \overset{\pi_{XX}}{\searrow} & \big| \\ X\times Y & & & & X\times X \end{array} ,$$

we conclude that $\Phi_{\mathscr{Q}^\bullet}\circ\Phi_{\mathscr{P}^\bullet} \cong \mathrm{id}_{D(X)}$ if and only if $\Phi_{\mathscr{R}^\bullet} \cong \Phi_{\mathscr{O}_\delta^X}$, where $\mathscr{O}_\delta^X$ is the structure sheaf of the diagonal $\Delta\colon X \to X\times X$, as in Example 5.2.6. In particular, setting $\mathscr{S}^\bullet := \big(\mathscr{P}^\bullet \otimes^L \mathscr{Q}^\bullet\big) \in D(X\times Y)$, X and Y are Fourier-Mukai partners if one can find an isomorphism $R\pi_{XX,*}(\pi_{XY}^*\mathscr{S}^\bullet) \cong \mathscr{O}_\delta^X$ on $D(X\times X)$, and similarly for the other direction, $R\pi_{YY,*}(\pi_{YX}^*\mathscr{S}^\bullet) \cong \mathscr{O}_\delta^Y$ on $D(Y\times Y)$, where $\pi_{YX}\colon Y\times X\times Y \to X\times Y$. An immediate corollary of the following theorem reveals the existence of such isomorphisms is, not only sufficient, but necessary in order for $\Phi_{\mathscr{P}^\bullet}$ to be an equivalence.

Theorem 5.2.8. Let $F\colon D(X)\to D(Y)$ be a fully faithful exact functor. Then there exists an object $\mathscr{P}^\bullet \in D(X\times Y)$, unique up to unique isomorphism, such that F is naturally isomorphic to the integral transform $\Phi_{\mathscr{P}^\bullet}^{X\to Y}$ with kernel $\mathscr{P}^\bullet$.

Corollary 5.2.9. Any exact equivalence of categories $D(X)\to D(Y)$ is given by a Fourier-Mukai transform $\Phi_{\mathscr{P}^\bullet}^{X\to Y}$, with uniquely defined kernel $\mathscr{P}^\bullet$.

The content of Theorem 5.2.8 is known as Orlov's Theorem. The statement we present is not the original one appearing in Orlov's articles [Orl97], and later in [Orl03], where he required F to admit right and left adjoints. This assumption has since been seen to always be satisfied, [Huy06, 5.14].

We conclude this text by claiming that Corollary 5.2.9 can be used to give a simpler proof of Gabriel's Theorem, that dates back to the sixties, [Huy06].

Theorem 5.2.10 ([Gab62])**.** Suppose X and Y are smooth projective varieties over k. If there exists an equivalence $\mathrm{Coh}_X \cong \mathrm{Coh}_Y$, then X and Y are isomorphic.

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Appendix A

Abelian triangulated categories

In this short appendix, we explore if the structure of a triangulated category is compatible with the structure of an abelian category. Our goal is then to prove Proposition 2.2.11. For convenience, we repeat Definition 2.2.10.

Definition 2.2.10. Let $\mathscr{A}$ be an abelian category. We say that $\mathscr{A}$ is ***semisimple*** if, for any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the following three equivalent conditions hold:

i) There exists a right-inverse for the epimorphism g (which we call a *section*).

ii) There exists a left-inverse for the monomorphism f (which we call a *retraction*).

iii) The short exact sequence *splits*, *i.e.* is isomorphic to the short exact sequence

$$0 \longrightarrow A \xrightarrow{\binom{1}{0}} A \oplus C \xrightarrow{(0\ 1)} C \longrightarrow 0 \ .$$

Proposition A.1. Let $(\mathscr{D}, T)$ be a triangulated category.

i) If

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$$
$$A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} T(A')$$

are two distinguished triangles, then so is the triangle

$$A \oplus A' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} B \oplus B' \xrightarrow{\begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}} C \oplus C' \xrightarrow{\begin{pmatrix} h & 0 \\ 0 & h' \end{pmatrix}} T(A) \oplus T(A') \ .$$

ii) Given any $A, C \in \mathscr{D}$, the following triangle is distinguished:

$$A \xrightarrow{\binom{1}{0}} A \oplus C \xrightarrow{(0\ 1)} C \xrightarrow{0} T(A).$$

Proof. For i), consider the map

$$A \oplus A' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} B \oplus B'$$

and complete it to a distinguished triangle by axiom A2 of Definition 2.2.2:

$$A \oplus A' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} B \oplus B' \xrightarrow{w_1} X \xrightarrow{w_2} T(A) \oplus T(A').$$

Consider the commutative diagrams induced by A3:

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} & B \oplus B' & \xrightarrow{w_1} & X & \xrightarrow{w_2} & T(A) \oplus T(A') \\
\downarrow{\scriptstyle p_A} & & \downarrow{\scriptstyle p_B} & & \downarrow{\scriptstyle w_3} & & \downarrow{\scriptstyle p_{T(A)}} \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & T(A)
\end{array}$$

and

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} & B \oplus B' & \xrightarrow{w_1} & X & \xrightarrow{w_2} & T(A) \oplus T(A') \\
\downarrow{\scriptstyle p_{A'}} & & \downarrow{\scriptstyle p_{B'}} & & \downarrow{\scriptstyle w_4} & & \downarrow{\scriptstyle p_{T(A')}} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & T(A')
\end{array}$$

Therefore, we get a commutative diagram

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} & B \oplus B' & \xrightarrow{w_1} & X & \xrightarrow{w_2} & T(A) \oplus T(A') \\
\| & & \| & & \downarrow{\scriptstyle \begin{pmatrix} w_3 & 0 \\ 0 & w_4 \end{pmatrix}} & & \| \\
A \oplus A' & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} & B \oplus B' & \xrightarrow{w_1} & C \oplus C' & \xrightarrow{w_2} & T(A) \oplus T(A')
\end{array}$$

and we are done by Proposition 2.2.8 i).

For ii), start with the distinguished triangle $0 \to C \xrightarrow{\mathrm{id}_C} C \to 0$. By the previous proposition, the "direct sum" of this distinguished triangle with the distinguished triangle $A \xrightarrow{\mathrm{id}_A} A \to 0 \to T(A)$ is distinguished. ☐

Motivated by Definition 2.2.10, we say that a triangle in a triangulated category $(\mathscr{D}, T)$ ***splits*** (or is split) if it is isomorphic to a triangle of the form of Proposition A.1 ii).

Lemma A.2. Let $(\mathscr{D}, T)$ be a triangulated category. If

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$$

is a distinguished triangle, then the following are equivalent:

i) The triangle is split.

ii) $f = 0$ or $g = 0$ or $h = 0$.

Proof. If i) holds then g is surjective. Since $h \circ g = 0$ this implies that $h = 0$.

Conversely, if ii) holds we can assume that $h = 0$ (otherwise we rotate the triangle). Then we have a commutative diagram of distinguished (Proposition A.1 ii) triangles

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & T(A) \\
\| & & \downarrow & & \| & & \| \\
A & \xrightarrow{\binom{1}{0}} & A \oplus C & \xrightarrow{(0\ 1)} & C & \xrightarrow{0} & T(A)
\end{array}$$

which we can complete by an isomorphism (Proposition 2.2.8 i). □

Proposition A.3. Let $(\mathscr{D}, T)$ be a triangulated category and $f\colon A \to B$ be a monomorphism in $\mathscr{D}$. Then f splits, *i.e.* any distinguished triangle induced by f as in axiom A1 i) of Definition 2.2.2 splits.

Proof. If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$ is such an extension, $T^{-1}(C) \xrightarrow{T^{-1}(h)} A \xrightarrow{f} B \xrightarrow{g} C$ is also distinguished by axiom A2 of Definition 2.2.2. Then $f \circ T^{-1}(h) = 0 \implies T^{-1}(h) = 0 \Leftrightarrow h = 0$. Lemma A.2 finishes the proof. □

Remark A.4*.* By a similar argument, it is also true that any epimorphism in a triangulated category splits.

The next result first appeared in Verdier's PhD thesis, [Ver96, Prop. 1.2.9, pgs. 102/103], where he introduced the notions of triangulated and derived categories, under the supervisor of Grothendieck.

Proposition A.5. Let $(\mathscr{D}, T)$ be a triangulated category. Then $\mathscr{D}$ is abelian if and only if every morphism $f\colon A \to B$ in $\mathscr{D}$ is isomorphic to a morphism of the form

$$A' \oplus I \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} I \oplus B' \ .$$

Proof. Let us prove the converse direction first. The kernel of a morphism of the form above is A', and its cokernel is B'. It is then easy to see that any monomorphism (resp. epimorphism) of the form above is its image (resp. coimage), and so $\mathscr{D}$ is abelian.

For the direct implication, let $f\colon A \to B$ be any morphism in $\mathscr{D}$. Factor f as

$$\begin{array}{ccccc} A & & \xrightarrow{\quad f \quad} & & B \\ & \overset{e}{\searrow\!\!\!\searrow} & & \overset{m}{\nearrow} & \\ & & \operatorname{Im} f & & \end{array}$$

Then, by Proposition A.3, there exist subobjects $A' \hookrightarrow A$ and $B' \hookrightarrow B$ such that e is isomorphic to the natural projection map $A' \oplus \operatorname{Im} f \to \operatorname{Im} f$, and m is isomorphic to the natural inclusion map $\operatorname{Im} f \to \operatorname{Im} f \oplus B$. This concludes the proof. □

Finally, we prove Proposition 2.2.11.

Proposition 2.2.11. Let $\mathscr{C}$ be an additive category.

i) If $\mathscr{C}$ is triangulated (with automorphism T) and abelian, then $\mathscr{C}$ is semisimple. Moreover any distinguished triangle is isomorphic to a triangle of the form

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } T(\operatorname{Ker} f) \oplus \operatorname{Coker} f \xrightarrow{\ h\ } T(A)$$

for natural maps g, h.

ii) Conversely, if $\mathscr{C}$ is abelian and semisimple, $\mathscr{C}$ has the structure of a triangulated category, by picking any automorphism T and setting a triangle to be distinguished if it is isomorphic to a triangle of the form above.

Proof. For i), Proposition A.3 provides a retraction for every monomorphism in $\mathscr{C}$, and hence we prove semisimplicity by Definition 2.2.10. Now, by Proposition A.5, any morphism $f\colon A \to B$ in $\mathscr{C}$ is isomorphic to the map

$$\operatorname{Ker} f \oplus \operatorname{Im} f \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} \operatorname{Im} f \oplus \operatorname{Coker} f \ \ .$$

By application of axioms A1 i) and A2 of Definition 2.2.2, the following are distinguished triangles:

$$\begin{aligned} &\operatorname{Ker} f \longrightarrow 0 \longrightarrow T(\operatorname{Ker} f) \xrightarrow{-\mathrm{id}} T(\operatorname{Ker} f), \\ &\operatorname{Im} f \xrightarrow{\mathrm{id}} \operatorname{Im} f \longrightarrow 0 \longrightarrow T(\operatorname{im} f), \\ &0 \longrightarrow \operatorname{Coker} f \xrightarrow{\mathrm{id}} \operatorname{Coker} f \longrightarrow 0. \end{aligned}$$

It follows that the triangle

$$\operatorname{Ker} f \oplus \operatorname{Im} f \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)} \operatorname{Im} f \oplus \operatorname{Coker} f \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)} T(\operatorname{Ker} f) \oplus \operatorname{Coker} f \xrightarrow{\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} T(\operatorname{Ker} f) \oplus T(\operatorname{Im} f)$$

is also distinguished by Proposition A.1 i). Proposition 2.2.8 iii) finishes the proof.

Regarding ii), it is easy to show that axioms A1-A3 of Definition 2.2.2 hold. For axiom A4, we refer to a sketch of the proof in [Fri14, pgs. 38/39]. ☐

Remark A.6. The proof of the previous proposition shows that any distinguished triangle in an abelian semisimple category $\mathscr{A}$ can be obtained as the direct sum of triangles of the form

$$\begin{aligned} &\bullet \xrightarrow{\mathrm{id}} \bullet \longrightarrow 0 \longrightarrow \bullet \\ &0 \longrightarrow \bullet \xrightarrow{\mathrm{id}} \bullet \longrightarrow 0 \ . \\ &\bullet \longrightarrow 0 \longrightarrow \bullet \xrightarrow{-\mathrm{id}} \bullet \end{aligned}$$

In particular, since additive functors preserve direct sums, any additive functor $F\colon \mathscr{A} \to \mathrm{Ab}$ is cohomological (Definition 2.2.5). These remarks show that abelian semisimple categories can be taken as prototypical examples of triangulated categories.

Appendix B

Proof of Theorem 2.4.6

As the name suggests, this appendix is devoted to proving Theorem 2.4.6. For convenience, we repeat the definition of a localizing class compatible with triangulation.

Definition 2.4.5. Let $(\mathscr{D}, T)$ be a triangulated category and $\mathcal{S}$ be a localizing class in $\mathscr{D}$. We say that $\mathcal{S}$ is ***compatible with the triangulation*** if the following conditions hold:

a1) If s is a morphism of $\mathscr{D}$, $s \in \mathcal{S}$ if and only if $T(s) \in \mathcal{S}$.

a2) Axiom A3 of Definition 2.2.2 is "well-behaved with respect to localization"; more precisely, every solid diagram with rows consisting of distinguished triangles

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & T(A) \\ \downarrow{\scriptstyle \alpha\in\mathcal{S}} & & \downarrow{\scriptstyle \beta\in\mathcal{S}} & & \downarrow{\scriptstyle \gamma\in\mathcal{S}} & & \downarrow{\scriptstyle T(\alpha)\in\mathcal{S}} \\ D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & T(D) \end{array}$$

and with $\alpha, \beta \in \mathcal{S}$ can be completed *(not necessarily uniquely)* by $\gamma \in \mathcal{S}$ to a morphism of triangles.

Theorem 2.4.6. Let $(\mathscr{D}, T)$ be a triangulated category and $\mathcal{S}$ a localizing class in $\mathscr{D}$ compatible with the triangulation. Denote by $Q\colon \mathscr{D} \to \mathscr{D}[\mathcal{S}^{-1}]$ the natural functor of the localization.

The functor $T_{\mathcal{S}}\colon \mathscr{D}[\mathcal{S}^{-1}] \to \mathscr{D}[\mathcal{S}^{-1}]$ defined by $T_{\mathcal{S}}(A) = T(A)$ for $A \in \mathrm{Obj}(\mathscr{D}[\mathcal{S}^{-1}]) = \mathrm{Obj}(\mathscr{D})$, and $T_{\mathcal{S}}(fs^{-1}) = T(f)(T(s))^{-1}$, is well-defined with respect to equivalence of roofs and is an automorphism.

Then, the pair $(\mathscr{D}[\mathcal{S}^{-1}], T_{\mathcal{S}})$ has the structure of a triangulated category if we define a triangle in $\mathscr{D}[\mathcal{S}^{-1}]$ to be distinguished if it is isomorphic to the image under Q of a distinguished triangle of $\mathscr{D}$.

Proof. The fact that $T_{\mathcal{S}}$ is well defined on roofs follows directly from functoriality of T and axiom a1 of Definition 2.4.5. It is easy to check that $T_{\mathcal{S}}$ is an automorphism. We check the axioms of Definition 2.2.2:

A1) Assertion ii) follows from definition of distinguished triangles in $\mathscr{D}[\mathcal{S}^{-1}]$, and assertion i) from the definition of Q (Proposition 2.1.5). For iii), let $A \stackrel{s}{\leftarrow} X \stackrel{f}{\to} B$ be a representative of a morphism $A \to B$ in $\mathscr{D}[\mathcal{S}^{-1}]$. Complete the morphism $X \stackrel{f}{\to} B$ to a distinguished triangle $X \stackrel{f}{\to} B \stackrel{g}{\to} C \stackrel{h}{\to} T(X)$ in $\mathscr{D}$. Consider the triangle $A \xrightarrow{fs^{-1}} B \xrightarrow{g\mathrm{id}_B^{-1}} C \xrightarrow{(T(s)\circ h)\mathrm{id}_C^{-1}} T(A)$ of $\mathscr{D}[\mathcal{S}^{-1}]$. Then the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f\mathrm{id}_X^{-1}} & B & \xrightarrow{g\mathrm{id}_B^{-1}} & C & \xrightarrow{h\mathrm{id}_C^{-1}} & T(X) \\ \downarrow{\scriptstyle s\mathrm{id}_X^{-1}} & & \| & & \| & & \downarrow{\scriptstyle T(s)\mathrm{id}_{T(X)}^{-1}} \\ A & \xrightarrow{fs^{-1}} & B & \xrightarrow{g\mathrm{id}_B^{-1}} & C & \xrightarrow{(T(s)\circ h)\mathrm{id}_C^{-1}} & T(A) \end{array}$$

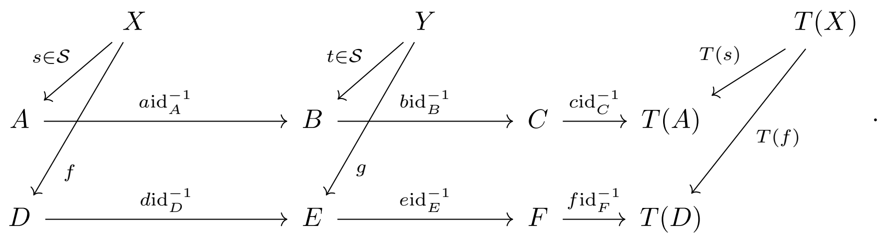
provides an isomorphism from a distinguished triangle because $s\mathrm{id}_X^{-1}$ is an isomorphism in $\mathscr{D}[\mathcal{S}^{-1}]$, as s is a quis.

A2) Follows directly from the definition of $T_{\mathcal{S}}$ on morphisms and the fact that $(\mathscr{D}, T)$ is triangulated.

A3) This is the axiom that is most cumbersome to show. Given a solid commutative diagram in $\mathscr{D}[\mathcal{S}^{-1}]$, with rows consisting of distinguished triangles,

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow & & \downarrow & & \vdots & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & D[1] \end{array},$$

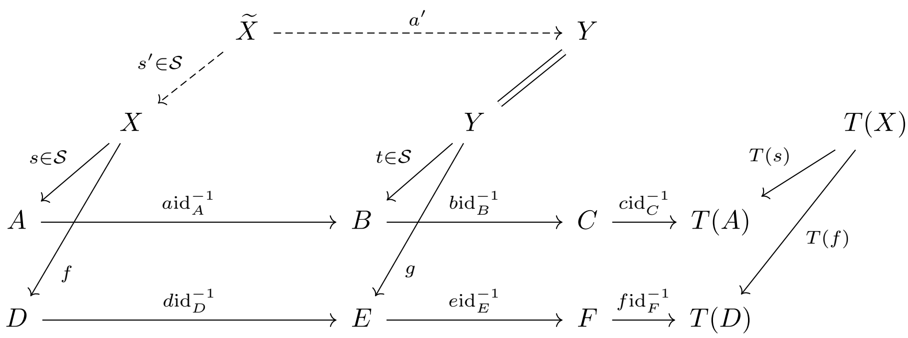
we want to fill in the dashed arrow. First of all, by the definition of distinguished triangles in $\mathscr{D}[\mathcal{S}^{-1}]$, we can assume that the horizontal arrows above are just (images under Q of) morphisms in $\mathscr{D}$. Choosing representatives for the vertical arrows, we have the following diagram:



We want to construct a roof from C to F that makes everything commute.

Claim. We can assume the existence of a morphism $X \to Y$ in $\mathscr{D}$ making the "back" and "front" squares commute (by changing, if necessary, the roof representing the equivalence class fs^{-1}).

Proof. Using axiom A2 of Definition 2.1.1, complete the arrows $X \xrightarrow{a\circ s} B \xleftarrow{t} Y$ to a square by the arrows $s' \in \mathcal{S}$ and a', as shown below.

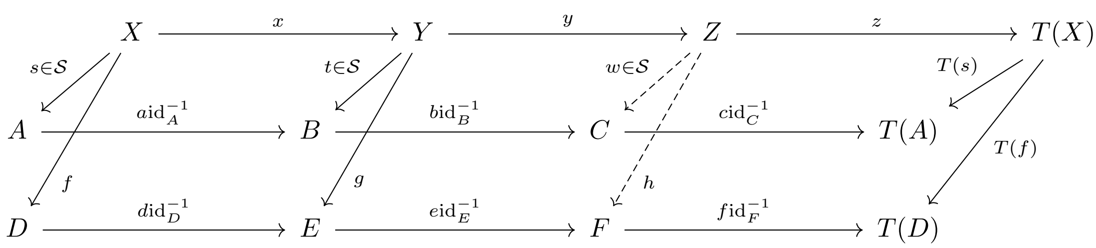


This makes the "back" square commute. Moreover, by Proposition 2.1.4 a) i), the roof $(f \circ s', s \circ s')$ is equivalent to the roof (f, s). Now, the commutativity condition $(d\mathrm{id}_D^{-1}) \circ (fs^{-1}) = (gt^{-1}) \circ (a\mathrm{id}_A^{-1})$ is saying that $Q(d \circ f) \circ Q(s)^{-1} = Q(g) \circ Q(t)^{-1} \circ Q(a)$ (again by Proposition 2.1.4 a). Now,

$$\begin{aligned} Q(d \circ f \circ s') &= Q(d \circ f) \circ Q(s)^{-1} \circ Q(s \circ s') = Q(g) \circ Q(t)^{-1} \circ Q(a) \circ Q(s \circ s') \\ &= Q(g) \circ Q(a') = Q(g \circ a'), \end{aligned}$$

that is, commutativity of "back" square in $\mathscr{D}$ makes the "front" square commute in $\mathscr{D}[\mathcal{S}^{-1}]$. But then, Proposition 2.1.4 c) says that there exists $\widetilde{\widetilde{X}}$ and a morphism $s'' \colon \widetilde{\widetilde{X}} \to \widetilde{X}$ in $\mathcal{S}$ such that $d \circ f \circ s' \circ s'' = g \circ a' \circ s''$ in $\mathscr{D}$. $\square$

With the claim proved, we can fit an arrow $x \colon X \to Y$ in the diagram below, that makes the "back" and "front" squares of the left side commute. Now, complete x by the arrows y and z to a distinguished triangle in $\mathscr{D}$:



.

Finally, we take advantage of axiom a2 of Definition 2.4.5 to construct the dashed arrows above the objects C and F, so that the "back" and "front" lattices commute. The morphism $C \to F$ represented by the class hw^{-1} gives what we want.

A4) Once more, we refer to the literature [GM03, IV.2.6] for the octahedron axiom.

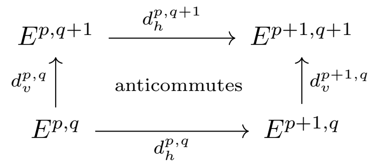
$\square$

Appendix C

Spectral sequences of double complexes

Definition C.1. Let $\mathscr{A}$ be an abelian category. A ***double complex*** (with terms in $\mathscr{A}$) is a collection of objects $E^{\bullet,\bullet} = \{E^{p,q}\}_{p,q\in\mathbb{Z}}$ of $\mathscr{A}$, together with morphisms

$$d_h^{p,q} \colon E^{p,q} \to E^{p+1,q}$$
$$d_v^{p,q} \colon E^{p,q} \to E^{p,q+1}$$



such that, for every (p,q), $d_h^{p+1,q} \circ d_h^{p,q} = 0$, $d_v^{p,q+1} \circ d_v^{p,q} = 0$ and $d_v^{p+1,q} \circ d_h^{p,q} = -\, d_h^{p,q+1} \circ d_v^{p,q}$. This means that, for every p, the data $(E^{p,\bullet}, d_v^{p,\bullet})$ is an "upward" directed "column" complex, and, for every q, the data $(E^{\bullet,q}, d_h^{\bullet,q})$ is an "right-ward" directed "row" complex.

The ***total complex*** associated to $E^{\bullet,\bullet}$ is the complex $((\operatorname{Tot} E)^\bullet, D)$, where

$$(\operatorname{Tot} E)^n = \bigoplus_{p+q=n} E^{p,q}$$

for every $n \in \mathbb{Z}$, and with differential $D^n \colon (\operatorname{Tot} E)^n \to (\operatorname{Tot} E)^{n+1}$ induced by the maps $E^{p,q} \to (\operatorname{Tot} E)^{n+1}$, $x \mapsto d_h^{p,q}(x) + d_v^{p,q}(x)$, where $p + q = n$. The n-th term of the total complex corresponds to the "anti-diagonal" $q = -p + n$.

We say that $E^{\bullet,\bullet}$ is in the ***first*** (respectively, ***third***) ***quadrant*** if $E^{p,q} = 0$ for every $p, q < 0$ (respectively, $p, q > 0$).

We can picture a double complex $E^{\bullet,\bullet}$ and its total complex $(\operatorname{Tot} E)^\bullet$ as in the figure below.

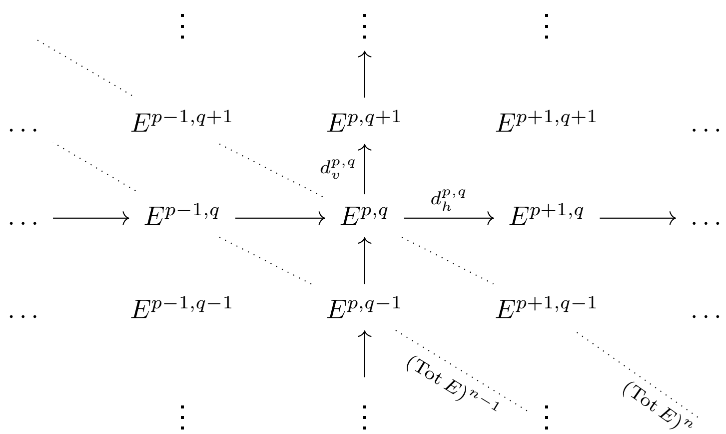


Figure C.1: A double complex $E^{\bullet,\bullet}$, together with its total complex $(\operatorname{Tot} E)^\bullet$.

Definition C.2. A ***spectral sequence*** E in an abelian category $\mathscr{A}$ consists of the following data.

i) A family of ***pages*** $\{E_r\}_{r\in\mathbb{Z}^{\geq 0}}$, where each page is a bigraded object $E_r = \{E_r^{p,q}\}_{p,q\in\mathbb{Z}}$, with each $E_r^{p,q} \in \mathrm{Obj}(\mathscr{A})$.

ii) For each $p, q \in \mathbb{Z}$ and $r \in \mathbb{Z}^{\geq 0}$, a morphism

$$d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

These morphisms should satisfy $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ for any $p, q \in \mathbb{Z}$ and $r \in \mathbb{Z}^{\geq 0}$. Consequently, we call $d_r^{p,q}$ ***differentials***.

iii) Isomorphisms $\alpha_r^{p,q} \colon H^{p,q}(E_r) \to E_{r+1}$, where $H^{p,q}(E_r)$ is the chain cohomology

$$H^{p,q}(E_r) = \mathrm{Ker}\,(d_r^{p,q})/\mathrm{Im}\,\big(d_r^{p-r,q+r-1}\big).$$

If there exists $r_0 \geq 0$ such that $E_r^{p,q} \cong E_{r_0}^{p,q}$ for every p, q and $r \geq r_0$, we set $E_\infty^{p,q} := E_{r_0}^{p,q}$ and say that E_∞ is the ***limit page*** of E, or that the spectral sequence ***abuts*** to E_∞. In this case, we write $E_r^{p,q} \Rightarrow E_\infty^{p,q}$.

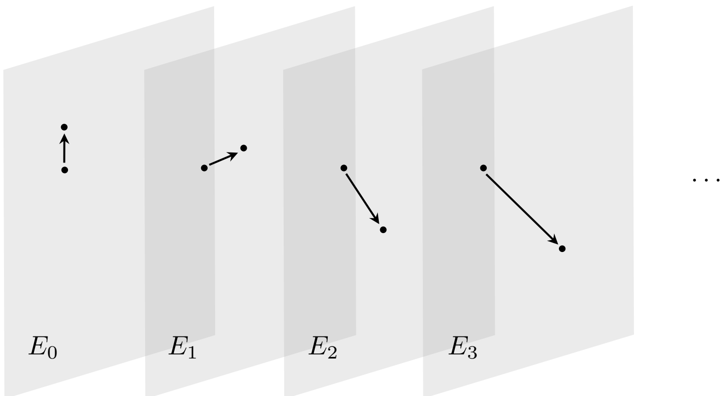


Figure C.2: The pages and differentials of a spectral sequence E_r.

Theorem C.3 ([Lan02, XX §9])**.** Let $\{E^{\bullet,\bullet}, d_h^{\bullet,\bullet}, d_v^{\bullet,\bullet}\}$ be a first quadrant double complex with terms in an abelian category $\mathscr{A}$. Then, there exist two spectral sequences $\{{}_vE_r, {}_vd_r\}$ and $\{{}_hE_r, {}_hd_r\}$ respecting the following properties.

i) ${}_vE_0^{p,q} = {}_hE_0^{p,q} = E^{p,q}$, *i.e.* their 0-th page is just the bigraded object underlying the double complex $E^{\bullet,\bullet}$.

ii) ${}_vd_0^{p,q} = d_v^{p,q}$, and hence the first page of ${}_vE$ is the bigraded object $H_v^{p,\bullet} := H^p(E^{\bullet,q})$. Analogously, ${}_hd_0^{p,q} = d_h^{p,q}$, and hence ${}_hE_1$ is the bigraded object $H_h^{\bullet,q} := H^q(E^{p,\bullet})$. We say that ${}_vE$ starts with the vertical differential, and that ${}_hE$ starts with the horizontal differential.

iii) ${}_vd_1^{p,q}$ is the differential $H_v^{p,q} \to H_v^{p+1,q}$ induced by $d_h^{p,q}$. Therefore, the second page of ${}_vE$ is the bigraded object one obtains from taking cohomology again (with respect to this differential), which we denote by $H_h^q H_v^p(E^{\bullet,\bullet})$. Similarly, ${}_h d_1^{p,q}$ is the differential $H_h^{p,q} \to H_h^{p,q+1}$ induced by $d_v^{p,q}$, and the second page of ${}_vE$ is the bigraded object $H_v^p H_h^q(E^{\bullet,\bullet})$.

iv) There exist two decreasing filtrations on $(\operatorname{Tot} E)^\bullet$, *i.e.* sequences of subcomplexes

$$\ldots \subseteq {}_vF^{p+1}(\operatorname{Tot} E)^\bullet \subseteq {}_vF^{p}(\operatorname{Tot} E)^\bullet \subseteq \ldots \subseteq {}_vF^{1}(\operatorname{Tot} E)^\bullet \subseteq {}_vF^{0}(\operatorname{Tot} E)^\bullet = (\operatorname{Tot} E)^\bullet$$
$$\ldots \subseteq {}_hF^{q+1}(\operatorname{Tot} E)^\bullet \subseteq {}_hF^{q}(\operatorname{Tot} E)^\bullet \subseteq \ldots \subseteq {}_hF^{1}(\operatorname{Tot} E)^\bullet \subseteq {}_hF^{0}(\operatorname{Tot} E)^\bullet = (\operatorname{Tot} E)^\bullet$$

with $D({}_vF^p(\operatorname{Tot} E)^\bullet) \subseteq {}_vF^p(\operatorname{Tot} E)^\bullet$ (and similarly for ${}_hF^\bullet$). For fixed n, these satisfy ${}_vF^i(\operatorname{Tot} E)^n = {}_hF^i(\operatorname{Tot} E)^n = 0$ for $i > n$. There are two induced decreasing filtrations ${}_vF^\bullet H^\bullet(\operatorname{Tot} E)$ and ${}_hF^\bullet H^\bullet(\operatorname{Tot} E)$ on the cohomology of the total complex, with the same properties.

v) Both spectral sequences abut. Despite ${}_vE_\infty^{p,q}$ and ${}_hE_\infty^{p,q}$ not being isomorphic in general, both limiting pages are related to the cohomology of the total complex. More precisely, the consecutive quotients satisfy

$$
\begin{aligned}
{}_vE_\infty^{p,q} &\cong {}_vF^p H^{p+q}(\operatorname{Tot} E) / {}_vF^{p+1} H^{p+q}(\operatorname{Tot} E),\\
{}_hE_\infty^{p,q} &\cong {}_hF^q H^{p+q}(\operatorname{Tot} E) / {}_hF^{q+1} H^{p+q}(\operatorname{Tot} E).
\end{aligned}
$$

Remark C.4. This result can be easily adapted to a third quadrant double complex.

Note that what is deep in the result above is that we can start with either the vertical or horizontal differential and both algorithms provide information above the *same* object, the total complex. Although a filtration only provides partial information about the total complex, there are situations where $H^n(\operatorname{Tot} E)$ is completely determined. For example, introducing the shorthand $\mathcal{H}^n := H^n(\operatorname{Tot} E)$, we can write

$$ {}_vE_\infty^{n,0} \cong {}_vF^n\mathcal{H}^n \xhookrightarrow{{}_vE_\infty^{n-1,1}} {}_vF^{n-1}\mathcal{H}^n \xhookrightarrow{{}_vE_\infty^{n-1,1}} \ldots \xhookrightarrow{{}_vE_\infty^{1,n-1}} {}_vF^1\mathcal{H}^n \xhookrightarrow{{}_vE_\infty^{0,n}} \mathcal{H}^n \ , $$

where the quotients are represented above the hooked arrows. Therefore, if ${}_vE_\infty^{n-i,i} = 0$ for all i, we have that $\mathcal{H}^n = 0$. This type of argument will be useful to prove Proposition 4.3.28.

The concept of spectral sequences is more easily digested by understating spectral sequences of filtered complexes, and then relating to the case presented above of a double complex. We recommend the references [Lan02, XX §9] and [Vak17, 1.7]. The gadget of spectral sequences is "*a powerful book-keeping tool for proving things involving complicated commutative diagrams*", [Vak17, 1.7]. We highlight two useful results that can be proven with the help of this machinery.

Theorem C.5 (***Grothendieck's composition of functors***, [Lan02, XX §9.6])**.** Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories with enough injective objects, and $F\colon \mathscr{A} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{C}$ be left exact functors. Suppose F maps injective objects of $\mathscr{A}$ into G-acyclic objects of $\mathscr{B}$. Then, for each $A \in \mathscr{A}$, there exists a spectral sequence E such that $E_2^{p,q} = R^qG(R^pF(A))$ and that abuts to $E_\infty^{p,q} = R^{p+q}(G \circ F)(A)$.

Note the similarity of the theorem above and Proposition 3.6.1, which we state below for convenience.

Proposition 3.6.1. Let $V_1\colon K^+(\mathscr{A}) \to K^+(\mathscr{B})$ and $V_2\colon K^+(\mathscr{B}) \to K(\mathscr{C})$ be two exact functors of triangulated categories. Suppose there exist triangulated subcategories $\mathcal{K}_{V_1} \subseteq K^+(\mathscr{A})$ and $\mathcal{K}_{V_2} \subseteq K^+(\mathscr{A})$ which are adapted to V_1 and V_2, respectively. Then, by Theorem 3.2.4, the right derived functors $RV_1\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ and $RV_2\colon D^+(\mathscr{A}) \to D(\mathscr{C})$ exist. If $V_1(\mathcal{K}_{V_1}) \subseteq \mathcal{K}_{V_2}$, then:

i) $\mathcal{K}_{V_1}$ is adapted to the composition $(V_2 \circ V_1)\colon K^+(\mathscr{A}) \to K(\mathscr{C})$, and hence the right derived functor $R(V_2 \circ V_1)\colon D^+(\mathscr{A}) \to D(\mathscr{C})$ exists.

ii) There is a natural isomorphism of functors $R(V_2 \circ V_1) \overset{\cong}{\Longrightarrow} R(V_2) \circ R(V_1)$.

Under the assumption that $\mathscr{A}, \mathscr{B}$ have enough injectives and picking $V_1 = K^+(F)$ and $V_2 = K^+(G)$ in the aforementioned proposition, the results are equivalent. Theorem C.5 was first introduced by Grothendieck in [Gro57]. By Hartshorne's wording, Proposition 3.6.1 "*shows the convenience of derived functors in the context of derived categories. What used to be a spectral sequence becomes now simply a composition of functors. (And of course one can recover the old spectral sequence from this proposition by taking cohomology and using the spectral sequence of a double complex)*", [Har66, Pag. 60].

Despite the bird's-eye perspective that the general theory of derived functors provides, spectral sequences are still useful for explicit computations. The proof of Grothendieck's composition of functors spectral sequence can be adapted to give the following result.

Proposition C.6 ([Huy06, 2.66])**.** Let $\mathscr{A}, \mathscr{B}$ be abelian categories and assume that $\mathscr{A}$ has enough injectives. If $F\colon \mathscr{A} \to \mathscr{B}$ is a left exact functor, for each $A^\bullet \in D^+(\mathscr{A})$, there exists a spectral sequence E such that $E_2^{p,q} = R^pF(H^q(A^\bullet))$ and that abuts to $R^{p+q}F(A^\bullet)$.

As asserted in Section 3.6, the proposition above enables us to prove the result below, which is used in Subsection 4.3.4.

Corollary 3.6.2. Let $\mathscr{A}, \mathscr{B}$ and F be as in the previous proposition. If $\mathscr{C} \subseteq \mathscr{B}$ is a thick subcategory and $R^iF(A) \in \mathscr{C}$ for all $A \in \mathscr{A}$, then $RF(A^\bullet) \in D^+_{\mathscr{C}}(\mathscr{B})$ for every $A^\bullet \in D^+(\mathscr{A})$. In other words, RF restricts to

$$RF\colon D^+(\mathscr{A}) \to D^+_{\mathscr{C}}(\mathscr{B}).$$

Proof. Let $A^\bullet \in D^+(\mathscr{A})$. By Proposition C.6, there exits a spectral sequence E such that $E_2^{p,q} = R^pF(H^p(A^\bullet)) \Rightarrow R^{p+q}F(A^\bullet)$. Since $H^p(A^\bullet) \in \mathscr{A}$, $R^qF(H^p(A^\bullet)) \in \mathscr{C}$ for all q by hypothesis. In other words, all the objects in the second page of the spectral sequence are in $\mathscr{C}$. Since subsequent pages are obtained by taking cohomology over and over again, and $\mathscr{C}$ is closed under extensions, all subsequent pages have terms in $\mathscr{C}$. In particular, the limiting page $E_\infty^{p,q} = R^{p+q}F(A^\bullet) \in \mathscr{C}$. By the definition of the higher derived functors, we conclude that $RF(A^\bullet)$ lands in $D^+_{\mathscr{C}}(\mathscr{B})$. □

Appendix D

Supplement to Chapter 4

This appendix is devoted to presenting proofs of certain statements made in Chapter 4.

Lemma 4.2.4. If $U \in \mathrm{Op}(X)$, $\mathscr{G} \in \mathrm{Coh}_U$, $\mathscr{F} \in \mathrm{QCoh}_X$ such that $\mathscr{G} \subseteq \mathscr{F}|_U$, then there exists $\mathscr{G}' \in \mathrm{Coh}_X$ such that $\mathscr{G}' \subseteq \mathscr{F}$ and $\mathscr{G}'|_U \cong \mathscr{G}$.

Proof. We split the proof into two steps.

1) Firstly, we prove the claim for the affine case. We set $X = \operatorname{Spec} A$, where A is a Noetherian ring. Let $\iota\colon U \to X$ denote the inclusion. By Lemma 4.1.14, $\mathscr{F}|_U \in \mathrm{QCoh}_U$. Since U is qcqs by Remark 4.1.24, $\iota_*(\mathscr{F}|_U)$ and $\iota_*(\mathscr{G})$ are quasicoherent sheaves on X by Proposition 4.1.30 iii). Consider the morphism $\phi\colon \mathscr{F} \to \iota_*(\mathscr{F}|_U)$ of quasicoherent sheaves on X, defined over each $V \in \mathrm{Op}(X)$, by $\phi(V)(f) = f|_{U\cap V}$, where $f \in \mathscr{F}(V)$. Since QCoh_X is abelian, we have an induced map of quasicoherent shaves on X, $\phi\colon \mathscr{F} \to \iota_*(\mathscr{F}|_U)/\iota_*(\mathscr{G})$, and $\mathscr{K} := \operatorname{Ker}\phi \in \mathrm{QCoh}_X$. Note that, if $V \in \mathrm{Op}(X)$,

$$\mathscr{K}(V) = \{f \in \mathscr{F}(V)\colon f|_{U\cap V} \in \mathscr{G}(U\cap V)\},$$

$\mathscr{K} \subseteq \mathscr{F}$ and $\mathscr{K}|_U = \mathscr{G}$. Since X is affine, we can write $\mathscr{K} \cong \widetilde{K}$ for $K \in \mathrm{Mod}_A$, by Remark 4.1.11. Since U is quasicompact, we can cover it by a finite number of distinguished affine open subsets of the form $\operatorname{Spec} A_{f_i}$, for $f_i \in A$, $i = 1, \ldots, n$. Then K_{f_i} is generated by finitely many elements, say $x_{i,1}/f_i^N, \ldots, x_{i,r_i}/f_i^N$ for $x_{1,1}, \ldots, x_{1,r_i} \in K$. Let $K' \subseteq K$ be the submodule generated by $\{x_{i,j}\}_{i=1,j=1}^{n,r_i}$. Then $\mathscr{G}' := \widetilde{K'} \in \mathrm{Coh}_X$, $\mathscr{G}' \subseteq \mathscr{K} \subseteq \mathscr{F}$ and $\mathscr{G}'|_U \cong \mathscr{K}|_U = \mathscr{G}$.

2) For the general case, cover X by affine opens $\operatorname{Spec} A_i$, for A_i Noetherian rings, $i = 1, \ldots, n$. Then, by step 1), there exist $\mathscr{G}_i' \in \mathrm{Coh}_{\operatorname{Spec} A_i}$ such that $\mathscr{G}_i' \subseteq \mathscr{F}|_{\operatorname{Spec} A_i}$ and $\mathscr{G}_i'|_{U\cap \operatorname{Spec} A_i} \cong \mathscr{G}|_{U\cap\operatorname{Spec} A_i}$. Notice that $\mathscr{G}_i'|_{\operatorname{Spec} A_i \cap \operatorname{Spec} A_j} \cong \mathscr{G}_j'|_{\operatorname{Spec} A_i\cap\operatorname{Spec} A_j}$ since each $\mathscr{G}_i'$ is a subobject of $\mathscr{F}|_{\operatorname{Spec} A_i}$, and $\mathscr{F}$ is a sheaf. Let $\mathscr{G}'$ be the (unique up to unique isomorphism) gluing of the $\mathscr{G}_i'$ along the intersections $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$, [Vak17, 2.5.1]. This sheaf of $\mathscr{O}_X$-modules is coherent by Proposition 4.1.22. Moreover, it is easy to see that $\mathscr{G}' \subseteq \mathscr{F}$ and $\mathscr{G}'|_U \cong \mathscr{G}$.

□

Corollary 4.2.7. Let $\mathscr{F} \in \mathrm{QCoh}_X$. Then, $\mathscr{F}$ is an injective object in Mod_X if and only if $\mathscr{F}$ is an injective object in QCoh_X.

Proof. We prove the sufficient condition. By Proposition 4.2.6, there exists $\mathscr{G} \in \mathrm{QCoh}_X$, injective in Mod_X, and a monomorphism $\mathscr{F} \hookrightarrow \mathscr{G}$. Since $\mathscr{F}$ is injective in QCoh_X, this monomorphism splits:

$$\begin{array}{ccc} \mathscr{F} & \hookrightarrow & \mathscr{G} \\ \| & \swarrow & \\ \mathscr{F} & & \end{array} .$$

Let $\mathscr{H}_1 \hookrightarrow \mathscr{H}_2$ is a monomorphism in $\mathrm{Mod}_{\mathscr{O}_X}$, and consider the following diagram:

$$\begin{array}{ccc} \mathscr{H}_1 & \hookrightarrow & \mathscr{H}_2 \\ \downarrow & & \\ \mathscr{F} & \swarrow & \\ \uparrow\!\downarrow & & \\ \mathscr{G} & & \end{array} .$$

The arrow $\mathscr{H}_2 \to \mathscr{G}$ exists since $\mathscr{G}$ is injective in $\mathrm{Mod}_{\mathscr{O}_X}$, and so we can define the arrow $\mathscr{H}_2 \to \mathscr{F}$ as the composition with the splitting found previously. □

Proposition 4.3.8. If $(X, \mathscr{R})$ is a ringed space, any injective $\mathscr{R}$-module on X is flasque.

Proof. Let $\mathscr{I}$ be an injective $\mathscr{R}$-module and $\iota_U : U \to X$ the inclusion of an open set $U \subseteq X$. If $V \subseteq U$, consider the natural injection $(\iota_V)_!(\mathscr{R}) \hookrightarrow (\iota_U)_!(\mathscr{R})$. Since $\mathscr{I}$ is injective, we can complete any diagram

$$\begin{array}{ccc} (\iota_V)_!(\mathscr{R}) & \hookrightarrow & (\iota_U)_!(\mathscr{R}) \\ \downarrow & \swarrow & \\ \mathscr{I} & & \end{array}$$

This means that the natural map $\mathrm{Hom}_{\mathscr{O}_X}((\iota_U)_!(\mathscr{R}), \mathscr{I}) \to \mathrm{Hom}_{\mathscr{R}}((\iota_V)_!(\mathscr{R}), \mathscr{I})$ is surjective. Since any $\phi \in \mathrm{Hom}_{\mathscr{R}}((\iota_U)_!(\mathscr{R}), \mathscr{I})$ is determined by a map $\mathscr{R}(U) \to \mathscr{I}(U)$, *i.e.* an element of $\mathscr{I}(U)$, and the restriction maps of $\mathscr{I}$, and similarly for V, we get that

$$\begin{array}{ccc} \mathscr{R}(U) & \longrightarrow & \mathscr{I}(U) \\ \downarrow & & \downarrow\!\!\downarrow \mathrm{res}_{U,V} \\ \mathscr{R}(V) & \longrightarrow & \mathscr{I}(V) \end{array} ,$$

as desired. □

Lemma 4.3.10. Let $X \in \mathrm{Sch}$ be Noetherian. Then:

i) If $\{\mathscr{F}_i\}_{i \in I}$ is a collection of flasque quasicoherent sheaves on X, $\bigoplus_{i \in I} \mathscr{F}_i$ is a flasque quasicoherent sheaf on X.

ii) If $\mathscr{F} \in \mathrm{QCoh}_X$ is flasque, $H^i(X, \mathscr{F}) = R^i\Gamma(X, \mathscr{F}) = 0$ for $i \geq 1$.

iii) If $\mathscr{F} \in \mathrm{QCoh}_X$, there exists a flasque sheaf $\mathscr{G} \in \mathrm{QCoh}_X$ and a monomorphism $\mathscr{F} \hookrightarrow \mathscr{G}$.

Proof. The first statement holds even without the Noetherian hypothesis and follows easily from Proposition 4.1.15 i) and the definition of flasque sheaves.

The second statement holds for general $\mathscr{R}$-modules on arbitrary ringed spaces $(X, \mathscr{R})$. Since $\mathrm{Mod}_{\mathscr{R}}(X)$ has enough injectives (Proposition 4.1.2), there is a short exact sequence of $\mathscr{R}$-modules

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I} \longrightarrow \mathscr{I}/\mathscr{F} \longrightarrow 0 \ ,$$

where $\mathscr{I}$ is injective. We prove the result by induction on i: if all cohomology groups of a flasque sheaf are trivial up to the i-th one, so is its $(i+1)$-th cohomology group. For the induction base $i = 1$, since $\mathscr{F}$ is flasque, applying $\Gamma(X, -)$ to the short exact sequence above gets us an exact sequence of $\mathscr{R}(X)$-modules by Lemma 4.3.6 i), and so $H^1(X, \mathscr{F}) = 0$. For the induction step, consider the induced long exact sequence of Corollary 3.3.7:

$$\ldots \longrightarrow H^i(X,\mathscr{F}) \longrightarrow H^i(X,\mathscr{I}) \longrightarrow H^i(X,\mathscr{F}/\mathscr{I}) \longrightarrow H^{i+1}(X,\mathscr{F}) \longrightarrow H^{i+1}(X,\mathscr{I}) \longrightarrow \ldots \ .$$

Since the class of injective sheaves is adapted to any functor (Theorem 3.4.10), $H^i(X, \mathscr{I}) = 0$ for every $i \geq 1$ by Lemma 3.3.9, and hence $H^{i+1}(X, \mathscr{F}) \cong H^i(X, \mathscr{F}/\mathscr{I})$. But since $\mathscr{F}, \mathscr{I}$ are flasque, so is $\mathscr{F}/\mathscr{I}$ by Lemma 4.3.6 ii). By induction hypothesis, $H^i(X, \mathscr{F}/\mathscr{I}) = 0$ and we are done.

The last statement is the only assertion that relies on all the hypothesis on X and $\mathscr{F}$. It follows from Proposition 4.1.31, Corollary 4.2.7 and Proposition 4.3.8. □

Proposition 4.3.11. Let $f\colon X \to Y$ be continuous map of topological spaces, $\mathscr{F} \in \mathrm{Ab}_X$ and $i \geq 0$. Define the presheaf $\mathscr{H}^i_{\mathrm{pre}}(\mathscr{F})$ of abelian groups on Y by assigning, to each $U \in \mathrm{Op}(Y)$, the abelian group

$$\mathscr{H}^i_{\mathrm{pre}}(\mathscr{F})(U) \coloneqq H^i(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)}).$$

If $\mathscr{H}^i(\mathscr{F})$ is the sheafification of $\mathscr{H}^i_{\mathrm{pre}}(\mathscr{F})$, there is an isomorphism $R^i f_*(\mathscr{F}) \cong \mathscr{H}^i(\mathscr{F})$.

Proof. We start by specifying the restriction maps of $\mathscr{H}^i_{\mathrm{pre}}(\mathscr{F})$. Recall from Remark 4.1.3 that Ab_X has enough injectives. If $V \subseteq U$, let $\mathscr{F}|_{f^{-1}(U)} \xrightarrow{\text{quis}} \mathscr{I}_U{}^\bullet$ and $\mathscr{F}|_{f^{-1}(V)} \xrightarrow{\text{quis}} \mathscr{I}_V{}^\bullet$ be a resolution by injective sheaves of abelian groups on $f^{-1}(U)$ and on $f^{-1}(V)$, respectively. Let $i\colon f^{-1}(V) \hookrightarrow f^{-1}(U)$ be the inclusion. Since $i_! \dashv i^{-1}$, $i_!(\mathscr{I}_V{}^\bullet)$ is a complex of injective sheaves of abelian groups on $f^{-1}(U)$ and, since $i_!$ is exact, $i_!\left(\mathscr{F}|_{f^{-1}(V)}\right) \to i_!(\mathscr{I}_V{}^\bullet)$ is still a quis. Consider the square

$$\begin{array}{ccc} \mathscr{F}|_{f^{-1}(U)} & \xrightarrow{\text{quis}} & \mathscr{I}_U{}^\bullet \\ \downarrow & & \downarrow{\scriptstyle \exists\,!} \\ i_!\left(\mathscr{F}|_{f^{-1}(V)}\right) & \xrightarrow{\text{quis}} & i_!(\mathscr{I}_V{}^\bullet) \end{array} \ ,$$

where the left downward map is the canonical map arising from the adjunction. By Remark 3.4.12, there exists a unique (up to homotopy) dashed arrow making the square commute in $K^+(\mathrm{QCoh}_{f^{-1}(U)})$. We define the restriction map $\mathscr{H}^i_{\mathrm{pre}}(\mathscr{F})(U) \to \mathscr{H}^i_{\mathrm{pre}}(\mathscr{F})(V)$ to be the map on cohomology determined by applying f_* to the downward arrow $\mathscr{I}_U{}^\bullet \to i_!(\mathscr{I}_V{}^\bullet)$.

Consider the functor $\mathscr{H}^i(-)\colon \mathrm{Ab}_X \to \mathrm{Ab}_Y$, taking $\mathscr{G} \mapsto \mathscr{H}^i(\mathscr{G})$. Note that, for fixed $U \in \mathrm{Op}(X)$, $\mathscr{H}^i_{\mathrm{pre}}(-)(U)\colon \mathrm{Ab}_X \to \mathrm{Ab}$ is just the right derived functor of the left exact "sections over U" functor,

$\Gamma(U,-)\colon \mathrm{Ab}_X \to \mathrm{Ab}$. Therefore, by Proposition 3.3.17, the collection $\{\mathscr{H}^i_{\mathrm{pre}}(-)(U)\}_{i\geq 0}$ is a (universal) δ-functor $\mathrm{Ab}_X \to \mathrm{Ab}$. Since sheafification is exact, this implies that the collection $\{\mathscr{H}^i(-)\}_{i\geq 0}$ is a δ-functor $\mathrm{Ab}_X \to \mathrm{Ab}_Y$. Given any $\mathscr{G} \in \mathrm{Ab}_X$ and $U \in \mathrm{Op}(X)$,

$$\mathscr{H}^0_{\mathrm{pre}}(\mathscr{G})(U) = H^0(f^{-1}(U), \mathscr{G}|_{f^{-1}(U)}) = \Gamma(f^{-1}(U), \mathscr{G}) = f_*(\mathscr{G})(U),$$

by Proposition 3.3.6 and the definition of the pushforward. According to Remark 3.3.14 and Theorem 3.3.16, we prove the claim if we show that $\mathscr{H}^i(-)$ is effaceable for any $i \geq 1$. If $\mathscr{G}$ is any sheaf of abelian groups on X, again since Ab_X has enough injectives, there exists an embedding $\mathscr{G} \hookrightarrow \mathscr{I}$ into an injective sheaf of abelian groups. By Proposition 4.3.8, $\mathscr{I}$ is flasque. It is clear that $\mathscr{I}|_V$ is also flasque, for any $V \in \mathrm{Op}(X)$. Therefore, $\mathscr{H}^i(\mathscr{F}) = 0$ by Lemma 4.3.10 ii). □