Representations of Thompson groups from Cuntz algebras

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Abstract

We associate a representation of the Thompson group V (and thus of F and T by considering the restriction) to every representation of the Cuntz algebra $\mathcal{O}_2$. The well-developed theory of representations of the Cuntz algebras leads us to exhibit an uncountable family of unitary representations of V which are pairwise non equivalent, and two others for F and T.

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1. Introduction

In this paper we investigate the unitary equivalence and he irreducibility of representations of Thompson groups arising from representations of the Cuntz algebra $\mathcal{O}_2$, paying a particular attention to the permutative representations of $\mathcal{O}_2$.

A representation of the Cuntz algebra $\mathcal{O}_n$ on a Hilbert space H is a family of isometries $S_1, ..., S_n$ acting on H with orthogonal ranges such that the underlying space H is subdivided into the these ranges. Since the celebrated work of Bratteli and Jorgensen in [5] on the permutative representations of $\mathcal{O}_n$ (where

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the isometries $S_1, ..., S_n$ permute the vectors of a fixed orthonormal basis of H) have a host of applications, for example to fractals, wavelets, dynamical systems see e.g. [5, 11, 7], and quantum field theory in [1]. For example, it is known that these representations serve as a computational tool for wavelets analysts, see [11]. For this one uses the subdivision of H into orthogonal subspaces that arises from the Cuntz algebra representations. Then the problem in wavelet theory is to build orthonormal bases in $L^2(\mathbb{R})$ from these data. Indeed this can be done and these wavelet bases have advantages over the earlier known basis constructions (one advantage is the efficiency of computation), see [11].

On the other hand, the Thompson groups F, T and V, introduced in the 1960's by Richard Thompson, are finitely generated and finitely presented groups such that $F \subseteq T \subseteq V$. These are countable and discrete groups and fairly easy to define as certain piecewise linear maps from the interval $[0, 1]$ onto itself (see [6]). Almost every question related to these groups is a challenge, typically harder for the smaller groups, as for example it is still an open problem whether F is an amenable group whereas the other two contain copies of the free group and thus are non-amenable [9]. Several approximation properties for groups are based on suitable asymptotic behaviour of matrix coefficients of unitary representations. For this reason, it is of interest to determine as much as possible about the representation theory of the Thompson groups (especially useful in solving analytical problems where of course amenability springs to mind). For that we first need to construct families of representations of Thompson groups. Recently, V.F.R. Jones has recently developed [10] a machinery to produce unitary representations of F, which we may call *Jones representations*. Besides this, the coefficients $\langle \tau(g)\Omega, \Omega \rangle$ (of a specific "vacuum" vector Ω and $g \in F$) of a certain Jones representation τ of F together with the geometric description of F was used in [10, 2] to fabricate (unoriented and oriented) knot and link invariants.

Then a relation between two these two subjects, representations of Cuntz algebras and Thompson groups, was unveiled by Nekrashevych in [12], where a *canonical* realization of the Thompson groups as subgroups of the unitary group of the Cuntz algebras was discovered. It is therefore expected that tools from the well developed representation theory of Cuntz algebras can be used in the study of the representation theory of Thompson groups, and the interplay serves to enrich both subjects.

The above mentioned interplay is what we plan to investigate in this paper. Given the wealthy source of Cuntz algebras representations, our results indicate that it is important to study the related Thompson groups representations. Indeed, we clarify why every representation of the Cuntz algebra $\mathcal{O}_2$ gives rise to a representation of the Thompson group V, and when we consider its restriction we get representations of the smaller groups T and F. Then we identify a class of representations of these groups using those representations $\{\pi_x\}_{x\in[0,1]}$ of the Cuntz algebra $\mathcal{O}_2$ fabricated in [7], and carry on with the study of the corresponding questions of unitary equivalence and irreducibility of these Thompson groups representations. As remarked in [10], it is typically difficult to distinguish equivalence classes or irreducible representations for representations of Thompson groups. We however succeeded in characterizing the inequivalent and irreducible unitary representations of V in our aforementioned class, and two other inequivalent classes for T and F. For V, this is done by borrowing the related study of the Cuntz algebra representations previously done in [7] together with the use of the natural action of V on the interval $[0, 1]$ and proving that range of every such representation of V generated the whole Cuntz algebra. For T and F, we identify the new inequivalent representations by a direct inspection, where we use the geometric description of F (and T) and use the fact that one of the generators has only two fixed points. We remark that the image of the canonical realization of F in not the whole Cuntz algebra $\mathcal{O}_2$, see [9, Prop. 4.3], thus we cannot apply [7] as we did for V which is why we have a different proof for F (and T).

More precisely, we first show in Theorem 3.1 – using a more group theoretic approach, namely, working with generators and relations of the underlying groups – that indeed every representation π of the Cuntz algebra $\mathcal{O}_2$ leads to a (unitary) representation ρ_π of the Thompson group V,

$$\pi \in \mathrm{Rep}(\mathcal{O}_2, H) \ \mapsto \ \rho_\pi \in \mathrm{Rep}(V, H)$$

thus producing unitary representations of the smaller groups T and F by considering the restriction of ρ_π.

Then we consider a family of $\mathcal{O}_2$ representations $\{\pi_x\}_{x\in[0,1]}$ from [7] that were built on the Hilbert spaces $\ell^2(\mathrm{orb}(x))$ ($x \in [0,1]$) with $\mathrm{orb}(x)$ being the orbit of x under the interval map defined by $f(x) = 2x$ (mod 1). Each of these representations π_x thus leads to a representation $\rho_x := \rho_{\pi_x}$ of V. In this setting, the representation $\rho_{\frac{1}{2}}$ is in fact the canonical representation of V considered in [12, 9]. We prove in Theorem 3.1 that the $g \cdot y = g(y)$ (with $g \in V$ and $y \in \mathrm{orb}(x)$) is a well-defined action of V on $\mathrm{orb}(x)$ and furthermore it coincides with the representation ρ_x.

The main result of this paper is Theorem 4.3 where we show that the representations ρ_x and ρ_y are unitarily equivalent if and only if x and y live in the same orbit

$$\rho_x \sim \rho_y \text{ if and only if } \ x \sim y.$$

This is a consequence of Theorem 4.2 where we establish that the C*-algebra generated by $\rho_x(V)$ equals $\pi_x(\mathcal{O})$. This latter result is also used when we prove in Theorem 4.4 that ρ_x is irreducible. The unitary equivalence of the restrictions τ_x and σ_x of π_x to the smaller Thompson groups F and T (respectively) are harder as usual. In Theorem 5.1 we show that if x does not belong to the orbit of $\frac{1}{2}$, then τ_x and $\tau_{\frac{1}{2}}$ are not unitarily equivalent (the same for σ_x and $\sigma_{\frac{1}{2}}$):

$$\tau_x \sim \tau_{\frac{1}{2}} \text{ if and only if } x \sim \frac{1}{2} \text{ and } \ \sigma_x \sim \sigma_{\frac{1}{2}} \text{ if and only if } x \sim \frac{1}{2}.$$

2. Representations of C*-algebras and of discrete groups

In this section we provide some background on representation theory of C*-algebras and discrete groups. Besides this, we also review the definitions of the Thompson groups and Cuntz algebras.

Let H be a (complex) Hilbert space and $B(H)$ the C^*-algebra of all bounded linear operators on H, and denote by $\mathbf{1}$ the identity operator. Let A be a C^*-algebra. A representation of A on the Hilbert space H is a *-preserving homomorphism $\pi : A \to B(H)$. Such map π is automatically continuous [13] with respect to the norm topologies of A and $B(H)$.

Then π is said to be an irreducible representation of A if the only invariant subspaces of π are the trivial ones, i.e., 0 and H. For $S \subseteq B(H)$ consider its commutant $S' = \{t \in B(H) : ts = st, \forall s \in S\}$. Then the representation π of A is irreducible iff $\pi(A)' = \mathbb{C}\mathbf{1}$, see [13].

Another important aspect is the equivalence of two representations by a unitary operator. Given two representations $\pi_1 : A \to B(H_1)$ and $\pi_2 : A \to B(H_2)$ of a C*-algebra A, we say π_1 and π_2 are unitarily equivalent representations (and we write $\pi_1 \sim \pi_2$) if there exists a unitary operator $U : H_1 \to H_2$ such that

$$\pi_2(a) = U\pi_1(a)U^*$$

for all $a \in A$, where U^* is the adjoint operator of U, so that $\langle U\xi, \eta\rangle = \langle \xi, U^*\eta\rangle$ for all $\xi, \eta \in H$.

We now turn our attention to discrete group representations. Most of the definitions we need are direct translations of the definitions given above by replacing the algebra A by a discrete group G. We recall that a discrete group G is a group G equipped with the discrete topology. A group representation of G on a Hilbert space H is a group homomorphism $\rho : G \to B(H)$. If $\rho(g^{-1}) = \rho(g)^*$, then we say that π is a unitary group representation of G (thus $\rho(g)$ is a unitary operator on H). Since G is assumed to be discrete, ρ is automatically continuous.

The notions of irreducible and unitary equivalent representations for representations of groups are similar to those for algebras. We remark that a discrete group representation ρ is irreducible iff $\rho(G)' = \mathbb{C}\mathbf{1}$, see [3]. This will be useful in the sequel

2.1. The Thompson groups

The Thompson groups are discrete groups, F, T and V, introduced by Richard Thompson in 1965. A good reference for the Thompson groups is [6].

The smallest of these groups, the Thompson group F, is the set of piecewise linear bijections of $[0, 1]$ which:

- are homeomorphisms of $[0, 1[$;
- have only a finite number of non-differentiability points in the set of dyadic rationals in $[0, 1[$;
- at the points of differentiability, the derivative is a power of 2;
- map $\mathbb{Z}[1/2] \cap [0, 1]$ bijectively onto itself

where $\mathbb{Z}[1/2]$ denotes the dyadic numbers. As an example of elements of F, consider the functions A and B, whose analytic expressions are given by

$$A(x) = \begin{cases} \frac{x}{2}, & \text{if } 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{4}, & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ 2x - 1, & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases}, \qquad B(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ x + \frac{1}{8}, & \text{if } \frac{3}{4} \leq x \leq \frac{7}{8} \\ 2x - 1, & \text{if } \frac{7}{8} \leq x \leq 1 \end{cases}, \quad (1)$$

whose graphs are in Fig. 1.

The Thompson group T is also a subset of the set of piecewise linear bijections of $[0, 1]$, but in this case it contains the functions that:

- are homeomorphisms of $[0, 1[$ when given the circle topology (this is equivalent to saying that it is a piecewise linear bijection of the unit interval with one discontinuity, at most);
- have only a finite number of non-differentiability points in the set of dyadic rationals in $[0, 1[$;
- at the points of differentiability, the derivative is a power of 2;

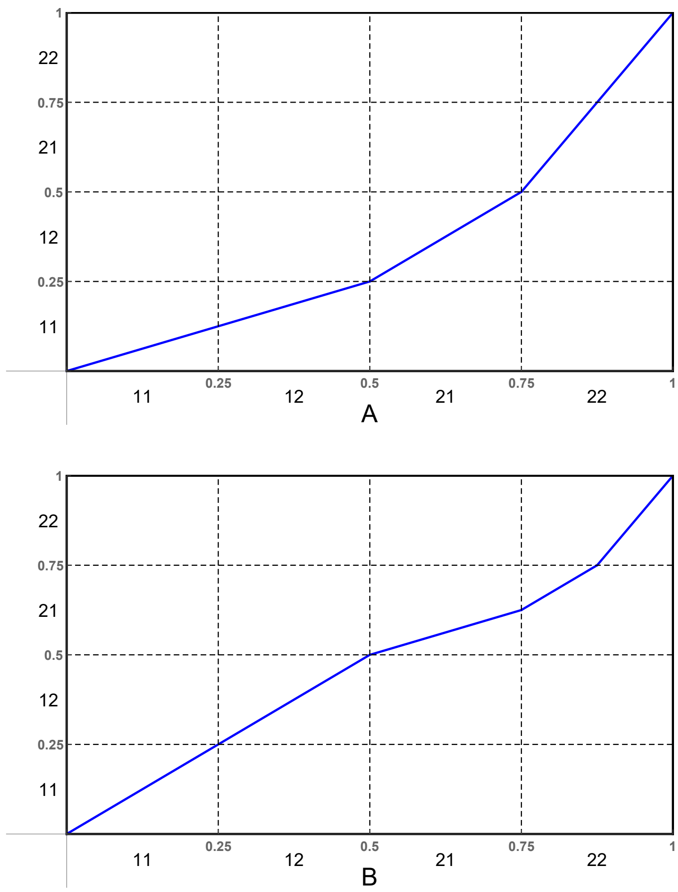

Figure 1: Graphs of the functions A and B

- map $\mathbb{Z}[1/2]\backslash\{1\}$ bijectively onto itself.

Finally, the Thompson group V is the set of right continuous piecewise linear bijections of $[0,1[$ that satisfy all of the properties of T except for the first one.

The Thompson group T is generated by A, B and C where C is defined as

$$C(x) = \begin{cases} \frac{x}{2} + \frac{3}{4}, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2x - 1, & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{4}, & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases} \qquad \pi_0(x) = \begin{cases} \frac{x}{2} + \frac{1}{2}, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2x - 1, & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ x, & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases} \tag{2}$$

whereas V is generated by A, B, C and π_0, where π_0 is defined above. Fig. 2 shows the graph of C and Fig. 3 shows the graph of π_0.

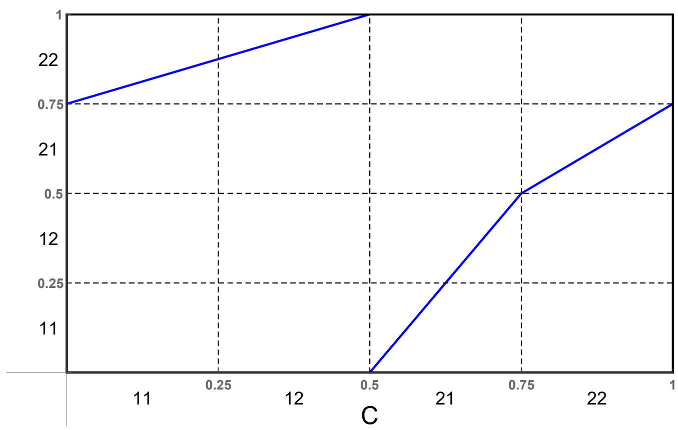

Figure 2: Graph of the function C

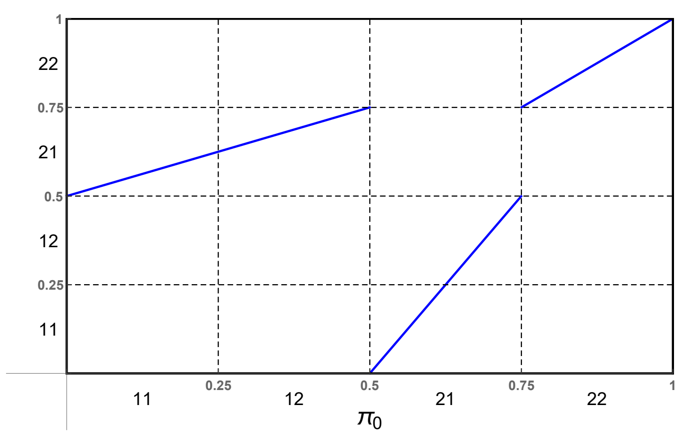

Figure 3: Graph of the function π_0

The axes of the graphs of the Figs. 1, 2 and 3 are all identified with the digits 11, 12, 21 and 22. This identification of the axes is not relevant at the moment but it will be important when we discuss how we can arrive at the image of the elements of the Thompson groups through the representations we will consider.

The Thompson groups F, T and V are not only finitely generated but also finitely presented [6]. In the sequel we need to consider the Thompson groups in terms of generators and relations since we aim to produce representations of these groups on Hilbert spaces by putting forward unitary operators and then checking that indeed these operators do verify the required relations. We use the standard presentations from [6]. The Thompson group F is the group generated by A and B that satisfies the relations

$$[AB^{-1}, X_2] = 1, \quad [AB^{-1}, X_3] = 1, \tag{3}$$

where $[g, h] = ghg^{-1}h^{-1}$ denotes the commutator, $X_2 = A^{-1}BA$ and $X_3 = A^{-2}BA^2$ (and 1 denotes the identity element of the group. The Thompson group T is generated by A, B and C and satisfies 6 relations, those in Eq. (3) together with

$$C = BC_2,\ C_2X_2 = BC_3,\ CA = C_2^2,\ C^3 = 1, \tag{4}$$

where $C_2 = A^{-1}CB$ and $C_3 = A^{-2}CB^2$. Finally, the Thompson group V is generated by A, B, C and π_0 and it satisfies 14 relations, those from T together with

$$\begin{gathered}\pi_1^2 = 1,\ [\pi_1, \pi_3] = 1,\ (\pi_2\pi_1)^3 = 1,\ [X_3, \pi_1] = 1,\\ \pi_1X_2 = B\pi_2\pi_1,\ \pi_2B = B\pi_3,\ \pi_1C_3 = C_3\pi_2,\ (\pi_1C_3)^3 = 1,\end{gathered} \tag{5}$$

where $\pi_1 = C_2^{-1}\pi_0C_2$, $\pi_2 = A^{-1}\pi_0A$ and $\pi_3 = A^{-2}\pi_0A^2$.

2.2. The Cuntz algebras

The Cuntz algebras were first introduced by Cuntz in [8]. They have been used not only in the operator algebras context but also as a host of many applications. Certain classes of Cuntz algebra representations play a fundamental role, see the seminal work [5].

Definition 2.1. *Let $n \in \mathbb{N}$. The Cuntz algebra $\mathcal{O}_n$ is the universal* C*-*algebra generated by a set of isometries $\{s_i\}_{i=1}^n$ that satisfies the relations*

$$\sum_{i=1}^{n} s_i s_i^* = \mathbf{1}, \qquad s_j^* s_j = \mathbf{1}$$

where $\mathbf{1}$ is the unit of $\mathcal{O}_n$.

If s_1, s_2 are the generators of $\mathcal{O}_2$, then $\hat{s}_1 = s_1^2, \hat{s}_2 = s_1 s_2$ and $\hat{s}_3 = s_2$ satisfy the relations of $\mathcal{O}_3$. By induction we can conclude that

$$... \subseteq \mathcal{O}_n \subset ... \subseteq \mathcal{O}_3 \subseteq \mathcal{O}_2$$

as in [8]. Our main focus here will be the Cuntz algebra $\mathcal{O}_2$. Note that the non-trivial representations of the Cuntz algebra are always injective, since $\mathcal{O}_2$ is a simple algebra, see [8]. Thus, any representation π of $\mathcal{O}_2$ on a Hilbert space H is uniquely determined by two isometries $S_1, S_2 \in B(H)$ satisfying the relations

$$S_1^* S_1 = \mathbf{1}, \qquad S_2^* S_2 = \mathbf{1}, \qquad S_1 S_1^* + S_2 S_2^* = \mathbf{1}, \tag{6}$$

where $\pi(s_1) = S_1$ and $\pi(s_2) = S_2$ are the images of the generators s_1 and s_2.

We now yield a family of representations of $\mathcal{O}_2$ which we will be considered in the sequel, see [7]. These representations are built from the orbits of points of the 1-dimensional dynamical system given by the interval map $f : [0,1] \to [0,1]$ where

$$f(x) = 2x \ (\mathrm{mod}\ 1), \tag{7}$$

see the graph of f in Fig. 4.

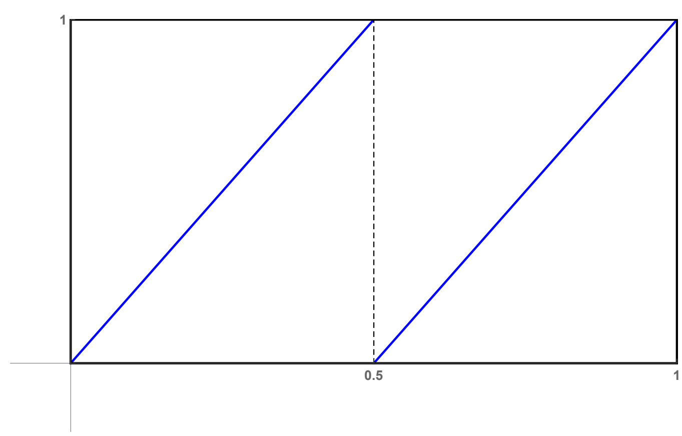

Figure 4: Graph of the function f in (7)

We can define an equivalence relation R_f on the interval $[0,1]$ as

$$x \sim y \ \text{ if and only if } \ f^n(x) = f^m(y) \text{ for some } n, m \in \mathbb{N}. \tag{8}$$

We are also interested in the orbit of a point $x \in [0,1]$ by the function f

$$\mathrm{orb}(x) = \{f^m(x) : m \in \mathbb{Z}\}. \tag{9}$$

Therefore, we have that $x \sim y$ if and only if $\mathrm{orb}(x) = \mathrm{orb}(y)$. We will also consider the Hilbert space $H_x = \ell^2(\mathrm{orb}(x))$ with the inner product defined as

$$\langle \delta_y, \delta_z \rangle = \begin{cases} 1, & \text{if } y = z \\ 0, & \text{if } y \neq z. \end{cases}$$

where $y, z \in \mathrm{orb}(x)$ and δ_y is the Dirac function on y. Notice that $\{\delta_y : y \in \mathrm{orb}(x)\}$ is an orthonormal basis for H_x. In order to construct our representation of $\mathcal{O}_2$ on H_x, let S_1 and S_2 such that

$$S_1\delta_y = \delta_{\frac{y}{2}} \qquad\qquad S_2\delta_y = \delta_{\frac{y+1}{2}} \tag{10}$$

for all $y \in \mathrm{orb}(x)$. Then we extend S_1 and S_2 by linearity and continuity to the whole Hilbert space H_x (which are clearly isometries). We may check that the adjoint operators of S_1 and S_2 are such that

$$S_1^*\delta_y = \delta_{2y} \qquad\qquad S_2^*\delta_z = \delta_{2z-1} \tag{11}$$

for all $y \in \mathrm{orb}(x) \cap [0, \frac{1}{2}[$ and $y \in \mathrm{orb}(x) \cap [\frac{1}{2}, 1[$. Moreover, it is straightforward to see that these operators satisfy the relations (6).

This observation gives us an obvious way of defining a representation of $\mathcal{O}_2$ on H_x, which we state more formally in the following (which a particular case of [7, Thm. 6]).

Lemma 2.2. *The isometries S_1 and S_2 define in Eq.* (10) *define a representation $\pi_x : \mathcal{O}_2 \to B(H_x)$ of $\mathcal{O}_2$ on H_x.*

As in [5, 7], the representation π_x is a permutative representation of $\mathcal{O}_2$ because S_1 and S_2 permute the vectors of an orthonormal of H_x. The unitary equivalence of these representations were studied in [7] within a more general context. For completeness we provide here a self-contained adaptation of this proof here to our current case of the family $(\pi_x)_{x\in[0,1]}$ considered in Lemma 2.2.

Theorem 2.3. *For $x, y \in [0,1]$, we have $\pi_x \sim \pi_y \iff x \sim y$.*

Proof. If $x \sim y$, then $H_x = H_y$. Hence, $\pi_x = \pi_y$ and the unitary equivalence is immediate. Suppose now that $x \nsim y$ and let us assume that $\pi_x \sim \pi_y$. Then, by definition of unitary equivalence, there is a unitary operator $U : H_x \to H_y$ such that, for all $a \in \mathcal{O}_2$

$$U\pi_x(a) = \pi_y(a)U.$$

Note that $U\delta_x$ can be approximated by finite linear combinations of elements $\delta_w \in H_y$, i.e. there are complex numbers c_w such that

$$U\delta_x = \sum_{w \in orb(y)} c_w \delta_w.$$

Then $||\sum_{w\in orb(y)} c_w\delta_w||_{H_y} = ||U\delta_x||_{H_y} = ||\delta_x||_{H_x} = 1$ since U is unitary. Let

$$\alpha_k = \begin{cases} 1, & \text{if } f^k(x) \in [0, \frac{1}{2}[\\ 2, & \text{if } f^k(x) \in [\frac{1}{2}, 1]. \end{cases}$$

So

$$\pi_y(s_{\alpha_k} s_{\alpha_k}^*)U\delta_x = \sum_{w \in orb(y)} c_w \pi_y(s_{\alpha_k} s_{\alpha_k}^*)\delta_w.$$

On the other hand, we also have

$$\pi_y(s_{\alpha_k} s_{\alpha_k}^*)U\delta_x = U\pi_x(s_{\alpha_k} s_{\alpha_k}^*)\delta_x = U\delta_x.$$

Since we are assuming that $x \nsim y$, then for every $\delta_w \in H_y$ there is a $k \in \mathbb{N}$ such that $\pi_y(s_{\alpha_k} s_{\alpha_k}^*)\delta_w = 0$. Then

$$\lim_{k \to +\infty} \left|\left| \sum_{w \in orb(y)} c_w \pi_y(s_{\alpha_k} s_{\alpha_k}^*)\delta_w \right|\right| = 0$$

which is a contradiction. Hence, $\pi_x \nsim \pi_y$. □

3. From Cuntz algebras to Thompson groups representations

This section shows the richness of the representations of the Thompson groups that arise from representations of Cuntz algebras.

Let $\pi : \mathcal{O}_2 \to B(H)$ be a representation of the Cuntz algebra $\mathcal{O}_2$ on a Hilbert space H and put $S_1 := \pi(s_1)$ and $S_2 := \pi(s_2)$ as the images of the generators s_1, s_2 of $\mathcal{O}_2$. Consider the 4 operators of $B(H)$ as follows:

$$A_\pi \quad := \quad S_1S_1S_1^* + S_1S_2S_1^*S_2^* + S_2S_2^*S_2^*, \tag{12}$$

$$B_\pi \quad := \quad S_1S_1^* + S_2S_1S_1S_1^*S_2^* + S_2S_1S_2S_1^*S_2^*S_2^* + S_2S_2S_2^*S_2^*S_2^*, \tag{13}$$

$$C_\pi \quad := \quad S_1S_1^*S_2^* + S_2S_1S_2^*S_2^* + S_2S_2S_1^*, \tag{14}$$

$$\pi_{0,\pi} \quad := \quad S_1S_1^*S_2^* + S_2S_1S_1^* + S_2S_2S_2^*S_2^*. \tag{15}$$

These operators were first defined in [12] (and then used in [9]) in order to define the so-called canonical representation of the Thompson group V, which is in our current notation $\pi_{\frac{1}{2}}$ (with $\pi_{\frac{1}{2}}$ as in the above Lemma 2.2) and $H_{\frac{1}{2}} = \ell^2([0,1] \cap \mathbb{Z}[\frac{1}{2}])$ where $\mathbb{Z}[\frac{1}{2}]$ denotes the dyadic numbers.

We now define the map $\rho_\pi : \{A, B, C, \pi_0\} \to B(H)$ such that $\rho_\pi(u) = u_\pi$, where $u \in \{A,\ B,\ C,\ \pi_0\}$. We would like to see that ρ_π extends to a group representation of V (still denoted by ρ_π), for which we need to check that the image of the generators of V are unitary and satisfy the relations of V. If $\pi = \pi_x$ as in Lemma 2.2, then we denote u_π simply by u_x, for $u \in \{A, B, C, \pi_0\}$ and ρ_π by ρ_x.

The groups F, T and V act by their definitions on the set $I = [0,1]$ whose actions is

$$g \cdot y = g(y) \ \text{ with } g \in V \ \text{ and } x \in [0,1]. \tag{16}$$

It is however not clear that the action leaves the orbit $\mathrm{orb}(x)$ invariant, see Eq. 9. If we prove this then clearly $g \cdot y = g(y)$ defines an action of V on $\mathrm{orb}(x)$ for every $x \in [0,1]$. We address these issues in the following result.

Theorem 3.1.

1. *Let H be a Hilbert space and $\pi : \mathcal{O}_2 \to B(H)$ be a representation of $\mathcal{O}_2$ on H and define ρ_π as above. Then ρ_π is a unitary representation of V on H.*
2. *If $y \in \mathrm{orb}(x)$ and $g \in V$,*

$$g \cdot \delta_y = \delta_{g(y)} \tag{17}$$

is an (well-defined) action of V on H_x.

3. *The representation ρ_x of V on H_x satisfies*

$$\rho_x(g)\delta_y = \delta_{g(y)} \tag{18}$$

for all $g \in V$ and $y \in \mathrm{orb}(x)$.

Proof. (1) We have to check that the images of the generators of V satisfy $\rho_\pi(g^{-1}) = \rho(g)^*$ and the relations of V (see Eqs. (3), (4) and (5)). Due to amount of computational work needed to check these statements, we did a program in $Mathematica$ that does this for us.

We start by checking that A_π, B_π, C_π and $\pi_{0\pi}$ are all unitary operators in $B(H)$. For example $A_\pi A_\pi^* = A_\pi^* A_\pi = \mathbf{1}$ is done by implementing the relations (6) into a $Mathematica$ program where $A_\pi A_\pi^*$ is reduced to the identity and the same for $A_\pi^* A_\pi$. In order to prove the relation $[A_\pi B_\pi^{-1}, X_{2,\pi}] = \mathbf{1}$, we use $A_\pi^{-1} = A_\pi^*$ (and the same for B_π^{-1} and $X_{2,\pi}^{-1}$) and compute $A_\pi B_\pi^*$ $X_{2,\pi}(A_\pi B_\pi^*)X_{2,\pi}^*$. Then the $Mathematica$ code simplifies $[A_\pi B_\pi^*, X_{2,\pi}]$ into an expression in S_1 and S_2 and their adjoints which, using the relations (6), can be further simplified to the identity as required. Each word of the form $S_{\alpha_1,\ldots,\alpha_n} S^*_{\beta_1,\ldots,\beta_m}$, where the α_i's and β_j's are in $\{1,2\}$, is represented in the code by $\{\{\alpha_1, ..., \alpha_n\},\ \{\beta_m, \beta_{m-1}, ..., \beta_1\}\}$. For example, $S_2 S_1 S_1 S_1^* S_2^*$ corresponds to $\{\{2,1,1\},\{1,2\}\}$. The other 13 relations of V are checked in a similar manner.

(2) The only non trivial thing to prove is to show that $g(y) \in \mathrm{orb}(x)$ for all $y \in \mathrm{orb}(x)$. It is enough to check this for the generators of V and, for these functions, that any of its linear sections can be written as a composition of the maps $z \mapsto 2z$, $z \mapsto \frac{z}{2}$, $z \mapsto 2z-1$ and $z \mapsto \frac{z+1}{2}$.

For A, the only section that is not obvious is for $y \in [\frac{1}{2}, \frac{3}{4}]$, but it is clear that

$$y - \frac{1}{4} = \frac{4y-1}{4} = \frac{1}{2} \cdot \frac{1}{2} \cdot (2(2y) - 1).$$

For B, for the section $y \in [\frac{1}{2}, \frac{3}{4}]$ we have

$$\frac{y}{2} + \frac{1}{4} = \frac{1}{2} \cdot \frac{2y+1}{2}$$

and for $y \in [\frac{3}{4}, \frac{7}{8}]$ we have

$$y - \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot (2(2(2y)) - 1).$$

For C, the section $y \in [0, \frac{1}{2}]$ we have

$$\frac{y}{2} + \frac{3}{4} = \frac{1}{2} \cdot \frac{2(2(\frac{y+1}{2})) + 1}{2}.$$

Finally for π_0, the section $y \in [0, \frac{1}{2}]$ we have

$$\frac{y}{2} + \frac{1}{2} = \frac{y+1}{2}.$$

(3) Expression (18) is well-defined, according to part 2 of this theorem. We now prove that (18) holds for every the generator of V and thus for every $g \in V$. First, $\rho_x(A) = S_1S_1S_1^* + S_1S_2S_1^*S_2^* + S_2S_2^*S_2^*$. So for $0 \leq y \leq \frac{1}{2}$:

$$\begin{aligned} \rho_x(A)\delta_y &= (S_1S_1S_1^*)\delta_y + (S_1S_2S_1^*S_2^*)\delta_y + (S_2S_2^*S_2^*)\delta_y \\ &= (S_1S_1)\delta_{2y} = S_1\delta_y = \delta_{\frac{y}{2}} = \delta_{A(y)}. \end{aligned}$$

For $\frac{1}{2} \leq y \leq \frac{3}{4}$:

$$\begin{aligned} \rho_x(A)\delta_y &= (S_1S_1S_1^*)\delta_y + (S_1S_2S_1^*S_2^*)\delta_y + (S_2S_2^*S_2^*)\delta_y \\ &= (S_1S_2S_1^*)\delta_{2y-1} + (S_2S_2^*)\delta_{2y-1} \\ &= (S_1S_2)\delta_{4y-2} = S_1\delta_{2y-\frac{1}{2}} = \delta_{y-\frac{1}{4}} = \delta_{A(y)} \end{aligned}$$

since $0 \leq 2y - 1 \leq \frac{1}{2}$. For $\frac{3}{4} \leq y \leq 1$:

$$\begin{aligned} \rho_x(A)\delta_y &= (S_1S_1S_1^*)\delta_y + (S_1S_2S_1^*S_2^*)\delta_y + (S_2S_2^*S_2^*)\delta_y \\ &= (S_1S_2S_1^*)\delta_{2y-1} + (S_2S_2^*)\delta_{2y-1} = S_2\delta_{4y-3} = \delta_{2y-1} = \delta_{A(y)} \end{aligned}$$

since $\frac{1}{2} \leq 2y - 1 \leq 1$.

Second, $\rho_x(B) = S_1S_1^* + S_2S_1S_1S_1^*S_2^* + S_2S_1S_2S_1^*S_2^*S_2^* + S_2S_2S_2^*S_2^*S_2^*$. So for $0 \leq y \leq \frac{1}{2}$:

$$\begin{aligned} \rho_x(B)\delta_y &= (S_1S_1^*)\delta_y + (S_2S_1S_1S_1^*S_2^*)\delta_y + (S_2S_1S_2S_1^*S_2^*S_2^*)\delta_y + (S_2S_2S_2^*S_2^*S_2^*)\delta_y \\ &= S_1\delta_{2y} = \delta_y = \delta_{B(y)}. \end{aligned}$$

For $\frac{1}{2} \leq y \leq \frac{3}{4}$:

$$\begin{aligned}
\rho_x(B)\delta_y &= (S_1S_1^*)\delta_y + (S_2S_1S_1S_1^*S_2^*)\delta_y + (S_2S_1S_2S_1^*S_2^*S_2^*)\delta_y + (S_2S_2S_2^*S_2^*S_2^*)\delta_y \\
&= (S_2S_1S_1S_1^*)\delta_{2y-1} + (S_2S_1S_2S_1^*S_2^*)\delta_{2y-1} + (S_2S_2S_2^*S_2^*)\delta_{2y-1} \\
&= (S_2S_1S_1)\delta_{4y-2} = (S_2S_1)\delta_{2y-1} \\
&= S_2\delta_{y-\frac{1}{2}} = \delta_{\frac{y}{2}+\frac{1}{4}} = \delta_{B(y)}
\end{aligned}$$

since $0 \leq 2y-1 \leq \frac{1}{2}$. For $\frac{3}{4} \leq y \leq \frac{7}{8}$:

$$\begin{aligned}
\rho_x(B)\delta_y &= (S_1S_1^*)\delta_y + (S_2S_1S_1S_1^*S_2^*)\delta_y + (S_2S_1S_2S_1^*S_2^*S_2^*)\delta_y + (S_2S_2S_2^*S_2^*S_2^*)\delta_y \\
&= (S_2S_1S_1S_1^*)\delta_{2y-1} + (S_2S_1S_2S_1^*S_2^*)\delta_{2y-1} + (S_2S_2S_2^*S_2^*)\delta_{2y-1} \\
&= (S_2S_1S_2S_1^*)\delta_{4y-3} + (S_2S_2S_2^*)\delta_{4y-3} \\
&= (S_2S_1S_2)\delta_{8y-6} = (S_2S_1)\delta_{4y-\frac{5}{2}} \\
&= S_2\delta_{2y-\frac{5}{4}} = \delta_{y-\frac{1}{8}} = \delta_{B(y)}
\end{aligned}$$

since $\frac{1}{2} \leq 2y-1 \leq \frac{3}{4}$ and $0 \leq 4y-3 \leq \frac{1}{2}$. For $\frac{7}{8} \leq y \leq 1$:

$$\begin{aligned}
\rho_x(B)\delta_y &= (S_1S_1^*)\delta_y + (S_2S_1S_1S_1^*S_2^*)\delta_y + (S_2S_1S_2S_1^*S_2^*S_2^*)\delta_y + (S_2S_2S_2^*S_2^*S_2^*)\delta_y \\
&= (S_2S_1S_1S_1^*)\delta_{2y-1} + (S_2S_1S_2S_1^*S_2^*)\delta_{2y-1} + (S_2S_2S_2^*S_2^*)\delta_{2y-1} \\
&= (S_2S_1S_2S_1^*)\delta_{4y-3} + (S_2S_2S_2^*)\delta_{4y-3} = (S_2S_2)\delta_{8y-7} \\
&= S_2\delta_{4y-3} = \delta_{2y-1} = \delta_{B(y)}
\end{aligned}$$

since $\frac{3}{4} \leq 2y-1 \leq 1$ and $\frac{1}{2} \leq 4y-3 \leq 1$.

Third, $\rho_x(C) = S_2S_2S_1^* + S_1S_1^*S_2^* + S_2S_1S_2^*S_2^*$. Thus for $0 \leq y \leq \frac{1}{2}$:

$$\begin{aligned}
\rho_{\frac{1}{p}}(C)\delta_y &= (S_2S_2S_1^*)\delta_y + (S_1S_1^*S_2^*)\delta_y + (S_2S_1S_2^*S_2^*)\delta_y \\
&= (S_2S_2)\delta_{2y} = S_2\delta_{y+\frac{1}{2}} = \delta_{\frac{y}{2}+\frac{3}{4}} = \delta_{C(y)}.
\end{aligned}$$

For $\frac{1}{2} \leq y \leq \frac{3}{4}$:

$$\begin{aligned}
\rho_x(C)\delta_y &= (S_2S_2S_1^*)\delta_y + (S_1S_1^*S_2^*)\delta_y + (S_2S_1S_2^*S_2^*)\delta_y \\
&= (S_1S_1^*)\delta_{2y-1} + (S_2S_1S_2^*)\delta_{2y-1} = S_1\delta_{4y-2} = \delta_{2y-1} = \delta_{C(y)}
\end{aligned}$$

since $0 \le 2y-1 \le \frac{1}{2}$. For $\frac{3}{4} \le y \le 1$:

$$\begin{aligned}
\rho_x(C)\delta_y &= (S_2S_2S_1^*)\delta_y + (S_1S_1^*S_2^*)\delta_y + (S_2S_1S_2^*S_2^*)\delta_y \\
&= (S_1S_1^*)\delta_{2y-1} + (S_2S_1S_2^*)\delta_{2y-1} \\
&= (S_2S_1)\delta_{4y-3} = S_2\delta_{2y-\frac{3}{2}} = \delta_{y-\frac{1}{4}} = \delta_{C(y)}
\end{aligned}$$

since $\frac{1}{2} \le 2y-1 \le 1$.

Finally, $\rho_x(\pi_0) = S_2S_1S_1^* + S_1S_1^*S_2^* + S_2S_2S_2^*S_2^*$.

So for $0 \le y \le \frac{1}{2}$:

$$\begin{aligned}
\rho_x(\pi_0)\delta_y &= (S_2S_1S_1^*)\delta_y + (S_1S_1^*S_2^*)\delta_y + (S_2S_2S_2^*S_2^*)\delta_y \\
&= (S_2S_1)\delta_{2y} = S_2\delta_y \\
&= \delta_{\frac{y}{2}+\frac{1}{2}} = \delta_{\pi_0(y)}.
\end{aligned}$$

For $\frac{1}{2} \le y \le \frac{3}{4}$:

$$\begin{aligned}
\rho_x(\pi_0)\delta_y &= (S_2S_1S_1^*)\delta_y + (S_1S_1^*S_2^*)\delta_y + (S_2S_2S_2^*S_2^*)\delta_y \\
&= (S_1S_1^*)\delta_{2y-1} + (S_2S_2S_2^*)\delta_{2y-1} \\
&= S_1\delta_{4y-2} \\
&= \delta_{2y-1} = \delta_{\pi_0(y)}
\end{aligned}$$

since $0 \le 2y-1 \le \frac{1}{2}$. For $\frac{3}{4} \le y \le 1$:

$$\begin{aligned}
\rho_x(\pi_0)\delta_y &= (S_2S_1S_1^*)\delta_y + (S_1S_1^*S_2^*)\delta_y + (S_2S_2S_2^*S_2^*)\delta_y \\
&= (S_1S_1^*)\delta_{2y-1} + (S_2S_2S_2^*)\delta 2y-1 \\
&= (S_2S_2)\delta_{4y-3} = S_2\delta_{2y-1} = \delta_y = \delta_{\pi_0(y)}
\end{aligned}$$

since $\frac{1}{2} \le 2y-1 \le 1$. □

Another proof of part (1) of the above theorem is to use the fact that $\rho_{\frac{1}{2}}$ is a unitary representation of V as in [12, 9] and then by universality of the Cuntz algebras [8] and definitions of A_π, B_π, C_π and $\pi_{0,\pi}$. These implies that indeed ρ_π is an unitary representation of V on H.

Before ending this section, we give a brief explanation of how we obtained the formulas for the images of the generators; in particular, we explain the expression for A_x. Let us consider Fig. 1 of the graph of A in the previous section, where we labelled the axes. We can interpret each summand in A_π as each one of the linear sections of A. For instance, $S_1S_1S_1^*$ corresponds to the section for $0 \leq x \leq \frac{1}{2}$, in the following way: the part S_1^* identifies the part of the x-axis where the graph is (in this case, it is 11 and 12, which is the same as only 1). The part S_1S_1 corresponds to where we are in the y-axis (in this case, the graph is in the section 11). The other linear sections are done in a similar fashion. Notice that the indices for the adjoint part are written in reverse order to the order of corresponding part in the x-axis. This a consequence of the identity $S_\beta^*S_\alpha^* = (S_\alpha S_\beta)^*$.

Generally speaking, each summand $S_\alpha S_\beta^*$, where α and β are words in $\{1, 2\}$, β identifies the linear section we are considering in the x-axis and α identifies the y-axis.

4. Unitary equivalent representations of Thompson group *V*

We are now prepared to consider the question of unitary equivalence of the representations of the Thompson groups we obtained. We will first consider the case of the representations ρ_x of V, for which we can obtain a full characterization of the unitary equivalence.

Firstly, we will need some extra results on the Thompson groups. We recall that an action of a discrete group on a set $\mathcal{X}$ is said to be amenable if there exists a finitely additive probability measure on $\mathcal{X}$ which is invariant under the action.

Theorem 4.1 (see [9]).

1. *The action $g \cdot y = g(y)$ of V on $[0, 1]$ in non-amenable.*
2. *Suppose G is a group acting on a set X and let α denote the representation of G on $\ell^2(X)$. Then, α is non-amenable if and only if there are elements $g_1, ...g_n \in G$ such that*

$$\left\| \frac{1}{n} \sum_{k=1}^{n} \pi(g_k) \right\| < 1.$$

We are now ready to study the unitary equivalence of the representations ρ_x. Firstly, we prove a auxiliary result (which is interesting on its own), adapting a proof in [9].

Theorem 4.2. *We have $C^*_{\rho_x}(V) = \pi_x(\mathcal{O}_2)$ for all $x \in [0,1]$, where $C^*_{\rho_x}(V)$ denotes the C^*-algebra generated by $\rho_x(V)$ in $\pi_x(\mathcal{O}_2)$, i.e.,*

$$C^*_{\rho_x}(V) = \overline{span\{\pi_x(g) : g \in V\}}^{||\cdot||_{B(H_x)}}.$$

Proof. We start by defining the subgroup of V, V_2, of all elements g such that $g(y) = y$ for $y \in \left[\frac{1}{2}, 1\right[$. We can define an action of V_2 on the interval $[0, \frac{1}{2}[$ as

$$g.y = g(y)$$

for $y \in \left[0, \frac{1}{2}\right[$. If we define the function $f : \left[0, \frac{1}{2}\right[\to [0,1[$ as $f(y) = 2y$, the map $g \mapsto fgf^{-1}$ from V_2 to V is a group isomorphism. Since by Theorem 4.1 the action of V on $[0,1[$ is non-amenable, the action of V_2 is also non-amenable, by isomorphism.

We begin by noticing that, since A, B, C and π_0 generate V, then

$$\rho_x(V) \subseteq \pi_x(\mathcal{O}_2),$$

thus $C^*_{\rho_x}(V) \subseteq \pi_x(\mathcal{O}_2)$.

We will prove the reverse inclusion by proving that $S_1S_1^*$ and $S_2S_2^*$ are in $C^*_{\rho_x}(V)$, for which we will use the non-amenability of action of V_2, mentioned in the first paragraph. We start by defining the sets

$$\mathcal{X}_1 = \left[0, \tfrac{1}{2}\right[\cap \operatorname{orb}(x)$$

and

$$\mathcal{X}_2 = \left[\tfrac{1}{2}, 1\right[\cap \operatorname{orb}(x).$$

Since the intervals $\left[0, \frac{1}{2}\right[$ and $\left[\frac{1}{2}, 1\right[$ are left invariant by the elements of V_2, the spaces $\ell^2(\mathcal{X}_1)$ and $\ell^2(\mathcal{X}_2)$ are invariant subspaces for $\rho_x(g)$ for any $g \in V_2$. Recall that $\rho_x(g)\delta_y = \delta_{g(y)}$, by Theorem 3.1. In particular, this gives us that the map

$$g \mapsto \rho_x(g)|_{\ell^2(\mathcal{X}_2)}$$

for $g \in V_2$ is the one associated to the action of V_2 on $[0, \frac{1}{2}[$, so, by the discussion in the first paragraph, this action is non-amenable. Thus, by Theorem 4.1, we know that the existence of a non-amenable action is equivalent to the existence of a natural number n such that there are elements $g_1, g_2, ..., g_n$ in V_2 such that

$$\left|\left|\frac{1}{n}\sum_{k=1}^{n} \rho_x(g_k)\right|\right| < 1.$$

We now proceed to prove that $S_2S_2^* \in C^*_{\rho_x}(V)$. Let $x = \frac{1}{n}\sum_{k=1}^{n} \rho_x(g_k)$ and begin by noticing that the element $S_2S_2^*$ commutes with $S_1S_1^*$ and $S_2S_2^*$. Hence, we have that

$$x^k = (xS_1S_1^*)^k + (xS_2S_2^*)^k. \tag{19}$$

for any natural number k. Notice also that, since each element of V_2 is the identity on the interval $\left[\frac{1}{2}, 1\right[$, we have $\rho_x(g)S_2S_2^* = S_2S_2^*$, for any $g \in V_2$, by definition of $S_2S_2^*$. In particular, it is the case that $xS_2S_2^* = S_2S_2^*$, which allows us to rewrite (19) as

$$x^k = (xS_1S_1^*)^k + S_2S_2^* \tag{20}$$

Since $xS_1S_1^* = x|_{\ell^2(\chi_2)}$, we conclude, recalling that $||x|| < 1$ by definition, that

$$||xS_1S_1^*|| = ||x|_{\ell^2(\mathcal{X}_2)}|| < 1.$$

By doing so, we have shown that $(xS_1S_1^*)^k \to 0$ when $n \to \infty$. This, in turn, shows that x^k converges to $S_2S_2^*$, letting us conclude that $S_2S_2^* \in C^*_{\rho_x}(V)$. The proof of the same result for $S_1S_1^*$ follows a similar line of reasoning, but using the subgroup V_1 of V of the elements that fix the interval $\left[0, \frac{1}{2}\right[$.

We now turn to proving that S_1 and S_2 are also in $C^*_{\rho_x}(V)$. We begin by recalling that

$$\begin{aligned} \rho_x(A) &= S_1S_1S_1^* + S_1S_2S_1^*S_2^* + S_2S_2^*S_2^*, \\ (\rho_x(A))^* &= S_1S_1^*S_1^* + S_2S_2S_2^* + S_2S_1S_2^*S_1^* \end{aligned}$$

and

$$\begin{aligned} \rho_x(D) &= S_2S_2S_1^*S_1^* + S_1S_1S_2^*S_1^* + S_1S_2S_1^*S_2^* + S_2S_1S_2^*S_2^*, \\ \rho_x(D^2) &= S_2S_1S_1^*S_1^* + S_1S_1S_1^*S_2^* + S_2S_2S_2^*S_1^* + S_1S_2S_2^*S_2^*. \end{aligned}$$

Then, we see that:

1. We have $\rho_x(A)(S_1S_1^*) = S_1(S_1S_1^*)$ because

$$\begin{aligned} \rho_x(A)(S_1S_1^*) &= (S_1S_1S_1^* + S_1S_2S_1^*S_2^* + S_2S_2^*S_2^*)(S_1S_1^*) \\ &= S_1S_1S_1^*S_1S_1^* + S_1S_2S_1^*S_2^*S_1S_1^* + S_2S_2^*S_2^*S_1S_1^* \\ &= S_1S_1S_1^* = S_1(S_1S_1^*). \end{aligned}$$

2. We have $\rho_x(A^{-1})(S_2S_2^*) = S_2(S_2S_2^*)$ as

$$\begin{aligned} \rho_x(A^{-1})(S_2S_2^*) &= (S_1S_1^*S_1^* + S_2S_2S_2^* + S_2S_1S_2^*S_1^*)(S_2S_2^*) \\ &= S_1S_1^*S_1^*S_2S_2^* + S_2S_2S_2^*S_2S_2^* + S_2S_1S_2^*S_1^*S_2S_2^* \\ &= S_2S_2S_2^* = S_2(S_2S_2^*). \end{aligned}$$

3. We also have $\rho_x(D^2)(S_1) = S_2$ because

$$\begin{aligned} \rho_x(D^2)(S_1) &= (S_2S_1S_1^*S_1^* + S_1S_1S_1^*S_2^* + S_2S_2S_2^*S_1^* + S_1S_2S_2^*S_2^*)(S_1) \\ &= S_2S_1S_1^*S_1^*S_1 + S_1S_1S_1^*S_2^*S_1 + S_2S_2S_2^*S_1^*S_1 + S_1S_2S_2^*S_2^*S_1 \\ &= S_2S_1S_1^* + S_2S_2S_2^* = S_2(S_1S_1^* + S_2S_2^*) = S_2. \end{aligned}$$

Using these relations, we can rewrite S_1 as

$$\begin{aligned} S_1 &= S_1(S_1S_1^* + S_2S_2^*) = S_1(S_1S_1^*) + S_1(S_2S_2^*) \\ &= \rho_x(A)(S_1S_1^*) + \rho_x(D^2)(S_2(S_2S_2^*)) \\ &= \rho_x(A)(S_1S_1^*) + \rho_x(D^2)\rho_x(A^{-1})(S_2S_2^*). \end{aligned}$$

Since all these elements are in $C^*_{\rho_x}(V)$, we conclude that $S_1 \in C^*_{\rho_x}(V)$. Finally, S_2 is also in $C^*_{\rho_x}(V)$ as $S_2 = \rho_x(D^2)S_1$. Thus, we have proven that $C^*_{\rho_x}(V) = \pi_x(\mathcal{O}_2)$. □

We are now ready to characterize the unitary equivalence classes of the family of representations $\{\rho_x\}_{x\in[0,1]}$ of the Thompson group V.

Theorem 4.3. *For $x, y \in [0,1]$ we have*

$$\rho_x \sim \rho_y \iff x \sim y.$$

Proof. If $x \sim y$, then $\mathrm{orb}(x) = \mathrm{orb}(y)$, which means that the representation ρ_x and ρ_y are equal. In particular, $\rho_x \sim \rho_y$.

Assume now that $\rho_x \sim \rho_y$. Then, there exists a unitary operator $U : B(H_y) \to B(H_x)$ such that

$$\rho_x(g) = U\rho_y(g)U^* \tag{21}$$

for all $g \in V$. Since $\rho_z = \pi_z|_V$ for all $z \in [0,1]$, we can rewrite Eq. (21) as

$$\pi_x(g) = U\pi_y(g)U^* \tag{22}$$

for $g \in V$ (where we identify V with its image under the canonical realization $\rho_{\frac{1}{2}}$). Let also $\mathcal{A}_V$ be the vector space generated by V. We will prove the desired result by first proving that Eq. (22) is true for elements of $\mathcal{A}_V$ and then, using the previous Theorem to prove that Eq. (22) indeed holds when we replace $g \in V$ by $a \in \mathcal{O}_2$.

Let $a \in \mathcal{A}_V$. Then, we can write a as

$$a = \sum_{i=1}^{n} c_i v_i$$

where $c_i \in \mathbb{C}$, $v_i \in V$ for $i \in \{1, ..., n\}$. We have

$$\begin{aligned} U\pi_x(a)U^* &= U\big(\sum_{i=1}^{n} c_i \pi_x(v_i)\big)U^* = \sum_{i=1}^{n} c_i U\pi_x(v_i)U^* \\ &= \sum_{i=1}^{n} c_i \pi_y(v_i) = \sum_{i=1}^{n} c_i \pi_y(v_i) = \pi_y(a) \end{aligned}$$

where we used that Eq. (22) is true in V and that π_x and π_y are algebra homomorphisms. Thus $\pi_x(a) = U\pi_y(a)U^*$ for any $a \in \mathcal{A}_V$.

Let now $a \in \mathcal{O}_2$. By Theorem 4.2 we know that there is a sequence (a_n) in $\mathcal{A}_V$ that converges to a. Since each a_n satisfies (22) (as just proved) then by continuity of U, π_x and π_y we get

$$\pi_x(a) = U\pi_y(a)U^* \tag{23}$$

i.e., $\pi_x \sim \pi_y$. By Theorem 2.3, $x \sim y$, which concludes the proof. □

4.1. Irreducibility of ρ_x

In this section, we prove that the group V representations ρ_x (with $x \in [0, 1]$)) are irreducible. This proof follows a very similar approach to that of the proof of unitary equivalence for the representations ρ_x, in the sense that we will use another result from [7] to transfer results from π_x to ρ_x.

We will also need the following easily-checked property of the commutator: if $A \subseteq B$ then $B' \subseteq A'$ (indeed, if something commutes with every element of B, then it also commutes with every element of A).

Theorem 4.4. *Let $x \in [0, 1]$. Then ρ_x is irreducible.*

Proof. Using [3], ρ_x is irreducible if and only if $\rho_x(V)' = \mathbb{C}\mathbf{1}$. Standard manipulations with the commutant (see [13]) leads to

$$(\rho_x(V))' = (\mathrm{span}(\rho_x(V)))' = (\overline{\mathrm{span}(\rho_x(V))})' = (\pi_x(\mathcal{O}_2))' \tag{24}$$

wherein the last equality holds by Theorem 4.2.

Thanks to [7, Thm. 6], π_x is irreducible, thus $\pi_x(\mathcal{O}_2)' = \mathbb{C}\mathbf{1}$. So Eq. (24) implies $(\rho_x(V))' = \mathbb{C}\mathbf{1}$. Therefore ρ_x is irreducible by [3]. □

Since ρ_x is irreducible by Theorem 4.4, every nonzero vector $\xi \in H_x$ is cyclic, so that $\overline{\text{span}\{\rho_x(g)\xi : g \in V\}} = H_x$. Thanks to Theorem 3.1, $\rho_x(g)$ permutes the vectors of the orthonormal basis $\{\delta_z : z \in \text{orb}(x)\}$ of H_x, where $g \in V$. In fact,

$$\langle \rho_x(g)\delta_z, \delta_{z'}\rangle = \langle \delta_{g(z)}, \delta_{z'}\rangle = \delta_{g(z),z'}, \text{ where } z, z' \in \text{orb}(x),\ g \in V.$$

Consequently the coefficient $\langle \rho_x(g)\delta_z, \delta_z\rangle$ of the vector δ_z equals 1 if and only if z is a fixed point for function $g \in V$ (and zero otherwise). Recall that the coefficients of ρ_x are the functions

$$g \mapsto \langle \rho_x(g)\xi, \eta\rangle$$

as $\xi = \sum_{z\in\text{orb}(x)} c_z\delta_z$ and $\eta = \sum_{z\in\text{orb}(x)} k_z\delta_z$ vary in H_x with c_z and k_z scalers. We can now get these coefficients as follows $\langle \rho_x(g)\xi, \eta\rangle = \sum c_z\overline{k_{g(z)}}$.

5. Unitary equivalence of representations of Thompson groups *T* and *F*

In the previous section, we saw that the situation of unitary equivalence for the representations ρ_x is the same as for the case of the representations π_x. The case of unitary equivalence of the remaining representations, σ_x and τ_x, doesn't seem to be as simple. However, we have this result for τ_x (which is also true for σ_x) which allows us to understand unitary equivalent to $\tau_{\frac{1}{2}}$.

Theorem 5.1. *If $x \in [0,1]$, then*

$$\tau_x \sim \tau_{\frac{1}{2}} \iff x \sim \frac{1}{2} \quad \textit{and} \quad \sigma_x \sim \sigma_{\frac{1}{2}} \iff x \sim \frac{1}{2}.$$

Proof. Since $F \subseteq T$, it is enough to prove the result for τ_x. If $x \sim \frac{1}{2}$, then we clearly have $\tau_x = \tau_{\frac{1}{2}}$.

Conversely, let us now assume that $\tau_x \sim \tau_{\frac{1}{2}}$ and $x \nsim \frac{1}{2}$. Then, there is a unitary operator $U : H_{\frac{1}{2}} \mapsto H_x$ such that

$$\tau_x(g)U = U\tau_{\frac{1}{2}}(g)$$

for all $g \in F$. Then

$$\tau_x(g)U\delta_0 = U\delta_{g(0)}$$

since $\tau_{\frac{1}{2}}(g)\delta_0 = \delta_{g(0)}$. In particular, let $g = A$ (the first generator of F). Since $A(0) = 0$, the previous equality simplifies to

$$\tau_x(A)U\delta_0 = U\delta_0. \tag{25}$$

Since $U\delta_0 \in H_x$, it is of the form

$$U\delta_0 = \sum_{z\in\text{orb}(x)} c_z\delta_z.$$

We can then rewrite (25) as

$$\sum_{z\in\text{orb}(x)} c_z(\tau_x(A)\delta_z) = \sum_{z\in\text{orb}(x)} c_z\delta_z \implies \sum_{z\in\text{orb}(x)} c_z\delta_{A(z)} = \sum_{z\in\text{orb}(x)} c_z\delta_z$$

which gives us the relation $c_z = c_{A(z)}$ for any $z \in orb(x)$. From the definition of A we conclude that the set $\{z, A(z), A^2(z), ...\}$ is infinite (because 0 and 1 are the only fixed points of A and 0 and 1 do not belong to $\text{orb}(x)$ as we are assuming that $x \nsim \frac{1}{2}$).

Suppose there is $w \in \text{orb}(x)$ such that $c_w \neq 0$. Since $\{w, A(w), A^2(w), ...\}$ is an infinite set, there are infinite coefficients that are equal to c_w, since $c_w = c_{A(w)}$. But $U\delta_0 \in H_x = \ell^2(\text{orb}(x))$, which implies that $\sum_{z\in\text{orb}(x)} |c_z|^2 < +\infty$. Hence, we have that $c_w = 0$, which is a contradiction.

Thus, we have that $U\delta_0 = 0$. But, since U is a unitary operator, this would imply that $||\delta_0|| = 0$, which is absurd. Therefore, $\tau_x \nsim \tau_{\frac{1}{2}}$ if $x \nsim \frac{1}{2}$, as desired. □

Remark 5.2. *In the same way as it happens for unitary equivalence, the proof of Theorem 4.3 is not directly adaptable to the cases of* σ_x *and* τ_x *since it relies on Theorem 4.2. We note that* $\mathbb{C}\delta_0$ *is an* F*-invariant subspace of* $\tau_{\frac{1}{2}}$.

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