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GRASSMAN MANIFOLDS AS SUBSETS OF EUCLIDEAN SPACES

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1. Introduction

Let E be a Euclidean space. Following Palais, we identify each vector subspace F of E with the orthogonal projection $\pi_F : E \to F$. In this way, the Grassman manifold $G(E)$ of all vector subspaces of E appears as a submanifold of the Euclidean space $L(E;E)$ of all linear maps from E into E (with the Hilbert-Schmidt inner product). The aim of this paper is to present some explicit formulas concerning the differential geometry of $G(E)$ as a submanifold of $L(E;E)$. Most of these formulas extend naturally to the case where E is an infinite dimensional Hilbert space, although in this case there is no natural inner product in $L(E;E)$.

2. Notation and Preliminaries

Let E and F be finite or infinite dimensional Hilbert spaces. We will denote by $L(E;F)$ the vector space of all continuous linear maps from E into F. If $\xi \in L(E;F)$, we will denote by $\xi^* \in L(F;E)$ its *adjoint* linear map, the one defined by the identity

$$\langle \xi(x), y\rangle = \langle x, \xi^*(y)\rangle.$$

The following identities will be used quite often:

$$\xi^{**} = \xi; \quad (\eta\circ\xi)^* = \xi^*\circ\eta^*; \quad id_E^* = id_E. \tag{2.1}$$

A linear map $\xi \in L(E;E)$ is *self-adjoint* if $\xi^* = \xi$. The map $L(E;F) \to L(F;E)$, $\xi \to \xi^*$, is a *real* linear map (even if E and F are complex spaces, it is not a complex linear map) and the set $L_{sa}(E;E)$ of self-adjoint linear maps is a real vector subspace of $L(E;F)$.

In case E or F is infinite dimensional, we will look on $L(E;F)$ merely as a Banach space (with the sup norm). In case E and F are finite dimensional, we take in the finite dimensional vector space $L(E;F)$ the *Hilbert-Schmidt inner product,* defined by

$$\langle \xi, \eta\rangle = \sum_{1\le k\le n} \langle \xi(x_k), \eta(x_k)\rangle, \tag{2.2}$$

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where $x_1, \ldots, x_k$ is an arbitrary orthonormal basis of E. We will use the following identities concerning these inner products,

$$\langle \xi, \eta \rangle = \langle \eta^*, \xi^* \rangle; \quad \langle \lambda, \mu^* \circ \eta \rangle = \langle \mu \circ \lambda, \eta \rangle = \langle \mu, \eta \circ \lambda^* \rangle. \tag{2.3}$$

The word "manifold" will always mean an embedded submanifold of some finite dimensional or Banach vector space B and the tangent vector spaces will be considered as vector subspaces of the ambient vector space B. In fact, one can even define, for each point a of an arbitrary subset M of B, a notion of tangent vector subspace $T_a(M)$, which behaves well with respect to differentiability (see, for example, [2]). In the same spirit, by vector bundle we will mean a vector sub-bundle of a constant one. A vector bundle $\underline{E}$ with basis M will be a family $(E_x)_{x\in M}$, where each E_x is a vector subspace of a fixed finite dimensional or Banach vector space E, verifying the usual properties, and we will use the same symbol $\underline{E}$ to denote the corresponding subset of $M \times E$. It will be useful to allow a vector bundle to have as basis an arbitrary subset M of a finite dimensional or Banach vector space B.

If $\underline{E} = (E_x)_{x\in M}$ is a vector bundle with $E_x \subset E$, we identify a *connection* in $\underline{E}$ by its second fundamental form at each point $x \in M$, which is a bilinear map $\theta_x : E_x \times T_x(M) \to E$ such that

$$(u, \theta_x(w, u)) \in T_{(x,w)}(\underline{E}). \tag{2.4}$$

For each smooth section $W = (W_x)_{x\in M}$ of $\underline{E}$, the covariant derivative $\nabla W_x(u)$ is given by the formula

$$\nabla W_x(u) = DW_x(u) - \theta_x(W_x, u). \tag{2.5}$$

If E is a Hilbert space, the *metric connection* of $\underline{E}$ is the one defined by the condition that $\theta_x(w, u)$ is orthogonal to the fibre E_x; if $\pi_x : E \to E_x$ is the orthogonal projection, then $x \to \pi_x$ is a smooth map from M into $L(E; E)$ and we have the following formula for this connection,

$$\theta_x(w, u) = D\pi_x(u)(w). \tag{2.6}$$

We will use also the following characterization of the curvature tensor of a connection θ in the vector bundle $\underline{E} = (E_x)_{x\in M}$, where $M \subset B$ is a manifold and $E_x \subset E$: assuming that $x \to \hat{\theta}_x$ is a smooth map from M into the space $L(E, B; E)$ of bilinear maps, such that each θ_x is a restriction of $\hat{\theta}_x$, the curvature tensor is the trilinear map

$$R_x : T_x(M) \times T_x(M) \times E_x \to E_x$$

defined by

$$R_x(u, v, w) = D\hat{\theta}_x(u)(w, v) - D\hat{\theta}_x(v)(w, u) + \hat{\theta}_x(\theta_x(w, u), v) - \hat{\theta}_x(\theta_x(w, v), u). \tag{2.7}$$

3. The Grassman Manifolds

Let E be a finite or infinite dimensional real Hilbert space. For each closed vector subspace $F \subset E$, we will denote by π_F the orthogonal projection from E onto F. We have hence a natural bijective map between the set of closed vector subspaces of E and the set of orthogonal projections. We will denote by $G(E)$ the subset of $L(E;E)$ whose elements are the orthogonal projections onto closed subspaces, and we will call $G(E)$ the *Grassman manifold* of E. The fact that $G(E)$ is indeed a manifold is proved in Akin [1], who attributes this result to Palais (unpublished preprint), but we will sketch here an independent proof.

The following characterization of the elements of $G(E)$ is well known:

3.1. A linear map $\xi \in L(E;E)$ belongs to $G(E)$ if and only if it is self-adjoint and verifies $\xi \circ \xi = \xi$.

We can consider a morphism from the constant vector bundle $E_{G(E)}$, with basis $G(E)$ and fibre E, into itself, associating to each $\xi \in G(E)$ the linear map $\xi : E \to E$. The fact that the image of an idempotent morphism is a vector bundle allows us to state:

3.2. There exists a *tautological vector bundle* with basis $G(E)$, whose fibre in each π_F is F.

Using formula (2.6) for the metric connection, we deduce:

3.3. The metric connection of the tautological vector bundle is defined by

$$\theta_\xi(w,n) \;=\; \eta(w),$$

for each $\xi \in G(E)$, $w \in \xi(E)$ and $\eta \in T_\xi(G(E))$.

As a corollary of the local constancy of the dimension of the fibres of a vector bundle, we see that, for each n, the subset $G_n(E)$ of $G(E)$, whose elements are the π_F such that F is n-dimensional, is open in $G(E)$.

Let $F \subset E$ be a fixed closed vector subspace. It is a well known simple linear algebra result that, for each closed vector subspace $G \subset E$, the following two properties are equivalent:

$$(a) \quad E = F^\perp \oplus G \quad \text{(direct sum)};$$
$$(b) \quad \pi_F|_G \text{ is an isomorphism from } G \text{ onto } F;$$

and that, if they are verified, the projection $E \to G$ associated to the direct sum is $(\pi_F|_G)^{-1} \circ \pi_F$. To each $\alpha \in L(F;F^\perp)$ we associate its graphic $G = \{x + \alpha(x)\}_{x\in F}$, which is a closed vector subspace of E verifying the conditions above. Inversely, for each closed vector subspace $G \subset E$ verifying the conditions above, there exists one and only one $\alpha \in L(F;F^\perp)$ whose graphic is G, namely $\alpha = \pi_{F^\perp} \circ (\pi_F|_G)^{-1}$.

We will use the preceding well-known considerations in the proof of the following result:

3.4. Let E be a real Hilbert space and let $F \subset E$ be a closed vector subspace. Let $\mathcal{U}_F \subset G(E)$ be the set of the orthogonal projections $\xi \in G(E)$ such that $E = F^\perp \oplus \xi(E)$. Then $\mathcal{U}_F$ is an open subset in $G(E)$, containing π_F, and there exists a diffeomorphism $\psi_F : \mathcal{U}_F \to L(F;F^\perp)$, defined by $\psi_F(\xi) = \pi_{F^\perp} \circ (\pi_F|_{\xi(E)})^{-1}$, that verifies $\psi_F(\pi_F) = 0$.

Proof. The considerations before the statement show that ψ_F is a bijective map from $\mathcal{U}_F$ onto $L(F;F^\perp)$, whose inverse $\psi_F^{-1} : L(F;F^\perp) \to \mathcal{U}_F$ associates to each α the orthogonal projection onto the closed vector subspace $\{x + \alpha(x)\}_{x\in F}$. All we have to show is that

$\mathcal{U}_F$ is open in $G(E)$ and that both ψ_F and ψ_F^{-1} are smooth maps. For that, we consider the morphism from the tautological vector bundle $(\xi(E))_{\xi\in G(E)}$ into the constant vector bundle $F_{G(E)}$ whose value at $\xi \in G(E)$ is $\pi_F|_{\xi(E)} : \xi(E) \to F$; the fact that $\xi \in \mathcal{U}_F$ if and only if the "fibre" of the morphism at ξ is an isomorphism implies that $\mathcal{U}_F$ is open in $G(E)$; taking the restrictions of the vector bundles to $\mathcal{U}_F$, the fact that the inverse of a (smooth) isomorphism is smooth implies that the map $\mathcal{U}_F \to L(F;E)$, $\xi \to (\pi_F|_{\xi(E)})^{-1}$ is smooth, hence $\psi_F : \mathcal{U}_F \to L(F;F^\perp)$ is also smooth. Now, we have an injective morphism from the constant vector bundle $F_{L(F,F^\perp)}$ into the constant vector bundle $E_{L(F,F^\perp)}$, whose fibre at $\alpha \in L(F,F^\perp)$ is the linear map $F \to E$, $x \to x + \alpha(x)$, hence the image of this morphism is a vector bundle with basis $L(F;F^\perp)$ and this implies that the map $\psi_F^{-1} : L(F,F^\perp) \to L(E;E)$ is smooth. □

As a corollary, we have:

3.5. If E is a real Hilbert space, then $G(E)$ is a manifold in $L(E;E)$. If E is N-dimensional and $F \subset E$ is n-dimensional, then the dimension of $G(E)$ at π_F is $n(N-n)$.

3.6. Let E be a real Hilbert space, $F \subset E$ be a closed vector subspace and $\psi_F : \mathcal{U}_F \to L(F,F^\perp)$ be the diffeomorphism defined in 3.4. For each $\xi \in \mathcal{U}_F$ and $\eta \in T_\xi(G(E))$, we have

$$D\psi_F(\xi)(\eta) \;=\; \eta \circ (\pi_F|_{\xi(E)})^{-1} - (\pi_F|_{\xi(E)})^{-1} \circ \pi_F \circ \eta \circ (\pi_F|_{\xi(E)})^{-1}.$$

In particular, $D\psi_F(\xi)(\eta) = \eta|_F$.

Proof. Let $\phi_F : \mathcal{U}_F \to L(F;E)$ be the smooth map defined by $\phi_F(\xi) = (\pi_F|_{\xi(E)})^{-1}$ (see the proof of 3.4). Let $w \in F$ arbitrary. Differentiating the identity $\pi_F(\phi_F(\xi)(w)) = w$, we obtain

$$\pi_F(D\phi_F(\xi)(\eta)(w)) = 0,$$

hence $D\phi_F(\xi)(\eta)(w) \in F^\perp$. On the other hand, we have a smooth section of the tautological vector bundle $(\xi(E))_{\xi\in G(E)}$ associating to each ξ, $\phi_F(\xi)(w)$; its covariant derivative with respect to the metric connection, which, by (2.5) and 3.3, is equal to

$$D\phi_F(\xi)(\eta)(w) - \eta(\phi_F(\xi)(w)),$$

must hence belong to $\xi(E)$. We can now conclude that $D\phi_F(\xi)(\eta)(w)$ is the projection of $\eta(\phi_F(\xi)(w))$ onto $F^\perp$ associated to the direct sum $E = F^\perp \oplus \xi(E)$. The fact that $\psi_F(\xi)(w) = \pi_{F^\perp}(\phi_F(\xi)(w))$ shows that $D\psi_F(\xi)(\eta)(w) = \pi_{F^\perp}(D\phi_F(\xi)(w))$, hence $D\psi_F(\xi)(\eta)(w)$ is also the projection of $\eta(\phi_F(\xi)(w))$ onto $F^\perp$ associated to the direct sum $E = F^\perp \oplus \xi(E)$ and, by the considerations made before 3.4, this projection is equal to

$$\eta\left((\pi_F|_{\xi(E)})^{-1}(w)\right) - (\pi_F|_{\xi(E)})^{-1}\left(\pi_F(\eta((\pi_F|_{\xi(E)})^{-1}(w)))\right).$$

To show that $D\psi_F(\pi_F)(\eta) = \eta|_F$ it will be enough to know that each $\eta \in T_{\pi_F}(G(E))$ maps F into $F^\perp$. To see this, we differentiate the identity $\xi\circ\xi = \xi$ and obtain $\eta\circ\pi_F + \pi_F\circ\eta = \eta$, hence $\eta\circ\pi_F = \eta - \pi_F\circ\eta = \pi_{F^\perp}\circ\eta$ and the proof is complete. □

We present now several equivalent characterizations of the tangent vector spaces to G(E).

3.7. Let E be a real Hilbert space and let $F \subset E$ be a closed vector subspace. The tangent vector space $T_{\pi_F}(G(E))$ is then contained in the vector space $L_{sa}(E;E)$ of self adjoint maps and, for each $\eta \in L_{sa}(E;E)$, the following conditions are equivalent:

(a) $\eta \in T_{\pi_F}(G(E))$;
(b) $\eta(F) \subset F^\perp$ and $\eta(F^\perp) \subset F$;
(c) $\eta \circ \pi_F + \pi_F \circ \eta = \eta$;
(d) $\eta \circ \pi_F = (Id - \pi_F) \circ \eta$;
(e) $\eta \circ (Id - \pi_F) = \pi_F \circ \eta$;
(f) $\eta \circ (2\pi_F - Id) = -(2\pi_F - Id) \circ \eta$.

Proof. The fact that each $T_{\pi_F}(G(E))$ is contained in $L_{sa}(E;E)$ is a consequence of the fact that $G(E) \subset L_{sa}(E;E)$. The equivalence between the four last conditions is trivial. Assuming (a), we obtain (c) simply by differentiating the identity $\xi \circ \xi = \xi$ at π_F in the direction of η. It is readily seen that condition (d) implies that $\eta(F) \subset F^\perp$ and that condition (e) implies that $\eta(F^\perp) \subset F$ $(Id - \pi_F = \pi_{F^\perp})$. Let us prove now that condition (b) implies condition (a). The fact that ψ_F is a diffeomorphism from the open set $\mathcal{U}_F$ in $G(E)$ onto $L(F;F^\perp)$ implies that $D\psi_F(\pi_F) : T_{\pi_F}(G(E)) \to L(E;F^\perp)$ is an isomorphism. We can hence take $\eta' \in T_{\pi_F}(G(E))$ such that

$$\eta|_F = D\psi_F(\pi_F)(\eta') = \eta'|_F.$$

Then η' is self-adjoint and verifies condition (b), hence $\eta'|_{F^\perp} : F^\perp \to F$ is the adjoint map to $\eta'|_F : F \to F^\perp$ and $\eta|_{F^\perp} : F^\perp \to F$ is the adjoint map to $\eta|_F : F \to F^\perp$. We deduce now that $\eta'|_{F^\perp} = \eta|_{F^\perp}$, hence $\eta = \eta'$ and the proof is complete. $\square$

Remark. To feel what is happening, assume that E is finite dimensional and take an orthonormal basis $x_1, \dots, x_N$ of E, whose first n vectors constitute a basis for F. Then the matrices of π_F, $id - \pi_F$ and $2\pi_F - Id$ are respectively

$$\begin{bmatrix} Id & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & Id \end{bmatrix} \qquad \begin{bmatrix} Id & 0 \\ 0 & -Id \end{bmatrix}$$

and condition (b) says that the elements of $T_{\pi_F}(G(E))$ are the linear maps whose matrix has the form

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}.$$

4. The Differential Geometry of Grassman manifolds

4.1. Let E be a real Hilbert space and let $F \subset E$ be a closed vector subspace. For each $\eta \in L_{sa}(E;E)$ the following conditions are then equivalent:

(a) $\eta(F) \subset F$ and $\eta(F^\perp) \subset F^\perp$;
(b) $\eta \circ \pi_F = \pi_F \circ \eta$;
(c) $\eta \circ (Id - \pi_F) = (Id - \pi_F) \circ \eta$.

We will denote by $T_{\pi_F}(G(E))^\perp$ the set of self-adjoint linear maps $\eta \in L_{sa}(E; E)$ verifying the preceding conditions.

Proof. The fact that (b) and (c) are equivalent is trivial. It is readily seen that (b) implies $\eta(F) \subset F$ and that (c) implies $\eta(F^\perp) \subset F^\perp$. Assuming (a), one sees that $\eta \circ \pi_F(x) = \eta(x) = \pi_F \circ \eta(x)$ for $x \in F$ and $\eta \circ \pi_F(x) = 0 = \pi_F \circ \eta(x)$ for $x \in F^\perp$, hence $\eta \circ \pi_F(x) = \pi_F \circ \eta(x)$ for arbitrary x and (b) is proved. □

4.2. Let E be a real Hilbert space and let $F \subset E$ be a closed vector subspace. Then $L_{sa}(E; E)$ is the direct sum of the closed vector subspaces $T_{\pi_F}(G(E))$ and $T_{\pi_F}(G(E))^\perp$ and the projections $\bar{\pi}_{\pi_F} : L_{sa}(E; E) \to T_{\pi_F}(G(E))$ and $\bar{\pi}^\perp_{\pi_F} : L_{sa}(E; E) \to T_{\pi_F}(G(E))^\perp$ associated to this direct sum are defined by

$$\bar{\pi}_{\pi_F}(\eta) = (Id - \pi_F) \circ \eta \circ \pi_F + \pi_F \circ \eta \circ (Id - \pi_F),$$
$$\bar{\pi}^\perp_{\pi_F}(\eta) = (Id - \pi_F) \circ \eta \circ (Id - \pi_F) + \pi_F \circ \eta \circ \pi_F.$$

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Proof. Conditions (a) of 4.1 and (b) of 3.7 show that the intersection $T_{\pi_F}(G(E)) \cap T_{\pi_F}(G(E))^\perp$ is $\{0\}$. It is readily seen that, for each $\eta \in L_{sa}(E; E)$, $\bar{\pi}_{\pi_F}(\eta)$ applies F into $F^\perp$ and $F^\perp$ into F and $\bar{\pi}^\perp_{\pi_F}(\eta)$ applies F into F and $F^\perp$ into $F^\perp$, hence $\bar{\pi}_{\pi_F}(\eta) \in T_{\pi_F}(G(E))$ and $\bar{\pi}^\perp_{\pi_F}(\eta) \in T_{\pi_F}(G(E))^\perp$. All we have to note now is that, for each η, $\bar{\pi}_{\pi_F}(\eta) + \bar{\pi}^\perp_{\pi_F}(\eta) = \eta$. □

4.3. If E is a finite dimensional real Hilbert space and if we consider in $L_{sa}(E; E)$ the Hilbert-Schmidt inner product, then, for each vector subspace $F \subset E$, the subspaces $T_{\pi_F}(G(E))$ and $T_{\pi_F}(G(E))^\perp$ of $L_{sa}(E; E)$ are mutually orthogonal, hence each one is the orthgonal complement of the other.

Proof. Assume $\eta \in T_{\pi_F}(G(E))$ and $\eta' \in T_{\pi_F}(G(E))^\perp$. Choose an orthonormal basis $x_1, \ldots, x_N$ of E such that the first n vectors constitute a basis of F and the last $N - n$ vectors constitute a basis of $F^\perp$. Conditions (b) of 3.7 and (a) of 4.1 assure that, for each $1 \le k \le N$, $\langle \eta(x_k), \eta'(x_k) \rangle = 0$, hence $\langle \eta, \eta' \rangle = 0$ (cf. (2.2)). □

The preceding result explains why we employ the notation $T_{\pi_F}(G(E))^\perp$ and $\bar{\pi}_{\pi_F}$.

If E is a finite or infinite dimensional real Hilbert space we will define the *canonical connection* in the manifold $G(E)$ as the one that verifies the condition that $\theta_{\pi_F}(\eta, \alpha)$ belongs to the kernel $T_{\pi_F}(G(E))^\perp$ of the linear map $\bar{\pi}_{\pi_F} : L_{sa}(E; E) \to T_{\pi_F}(G(E))$, for each η and α in $T_{\pi_F}(G(E))$. In an analogous way to that used in the case of a metric connection, it is easily seen that this connection is symmetric and also is defined by the formula

$$\theta_{\pi_F}(\eta, \alpha) = D\bar{\pi}_{\pi_F}(\alpha)(\eta). \qquad (4.1)$$

This is the connection that we will always consider in the Grassman manifold $G(E)$. Of, course, in case E is finite dimensional, this connection is the metric connection with respect to the Hilbert-Schmidt inner product.

We can obtain a more explicit formula for the connection on $G(E)$ by calculating the derivative in (4.1), using the formula in 4.2, $\bar{\pi}_\xi(\eta) = (Id - \xi) \circ \eta \circ \xi + \xi \circ \eta \circ (Id - \xi)$. This gives

$$\begin{aligned} \theta_\xi(\eta, \alpha) &= -\alpha \circ \eta \circ \xi + (Id - \xi) \circ \eta \circ \alpha + \alpha \circ \eta \circ (Id - \xi) - \xi \circ \eta \circ \alpha, \\ &= (Id - 2\xi) \circ \eta \circ \alpha + \alpha \circ \eta \circ (Id - 2\xi). \end{aligned} \tag{4.2}$$

Let us now obtain, using (2.7), two formulas for the curvature, the first for the metric connection of the tautological vector bundle, and the second for the canonical connection of the Grassman manifold. In the first case, we take $\hat{\theta}_\xi(w, \eta) = \eta(w)$ for each $w \in E$ and $\eta \in L(E; E)$ (cf. 3.3) obtaining

$$R_\xi(\alpha, \beta, w) = \beta(\alpha(w)) - \alpha(\beta(w)) \tag{4.3}$$

for each α and β in $T_\xi(G(E))$ and $w \in \xi(E)$. In the second case, we take $\hat{\theta}_\xi(\eta, \alpha) = (Id - 2\xi) \circ \eta \circ \alpha + \alpha \circ \eta \circ (Id - 2\xi)$ and obtain, noting that $(Id - 2\xi) \circ (Id - 2\xi) = Id$ and that, by 3.7(f), $Id - 2\xi$ commutes with the composite of any two elements of $T_\xi(G(E))$,

$$R_\xi(\alpha, \beta, \eta) = \eta \circ \alpha \circ \beta - \eta \circ \beta \circ \alpha + \beta \circ \alpha \circ \eta - \alpha \circ \beta \circ \eta. \tag{4.4}$$

Assuming that E is finite dimensional, we obtain, for the sectional curvatures:

$$\mathrm{Riem}_\xi(\alpha, \beta) = \langle R_\xi(\alpha, \beta, \alpha), \beta \rangle = 2\langle \alpha \circ \beta, \alpha \circ \beta \rangle - 2\langle \alpha \circ \beta, \beta \circ \alpha \rangle. \tag{4.5}$$

To prove it, all we have to do is to apply the formulas in (2.3), remembering that α and β are self-adjoint. The fact that $(\alpha \circ \beta)^* = \beta \circ \alpha$ implies that $\alpha \circ \beta$ and $\beta \circ \alpha$ have the same norm and we can hence apply Cauchy-Schwartz to conclude that $\mathrm{Riem}_\xi(\alpha, \beta) \geq 0$ and $\mathrm{Riem}_\xi(\alpha, \beta) = 0$ if and only if $\alpha \circ \beta = \beta \circ \alpha$.

One can also establish easily the following formula for the Ricci curvature:

$$\mathrm{Ricci}_\xi(\alpha, \beta) = \frac{N-2}{2} \langle \alpha, \beta \rangle \tag{4.6}$$

where N is the dimension of E.

Grassman manifolds (or, more precisely, their connected components) are sometimes represented as homogeneous spaces of the orthogonal group. The following considerations will compare this approach with the one we are using.

Let E be a real Hilbert space and let $O(E) \subset L(E; E)$ be the orthogonal group, i.e. the set of the toplinear isomorphisms $\xi : E \to E$ such that $\xi^* = \xi^{-1}$. It is well known that $O(E)$ is a manifold (a Lie group) and that, for each $\xi \in O(E)$ and $\alpha \in L(E; E)$, we have

$$\alpha \in T_\xi(O(E)) \text{ if and only if } \alpha^* \circ \xi + \xi^* \circ \alpha = 0. \tag{4.7}$$

In the case where E is finite dimensional, the Riemann structure in $O(E)$ induced by the Hilbert-Schmidt inner product is readily seen to be bi-invariant. The orthogonal projections $\pi_\xi : L(E; E) \to T_\xi(O(E))$ are defined by

$$\pi_\xi(\lambda) = \frac{1}{2}(\lambda - \xi \circ \lambda^* \circ \xi). \tag{4.8}$$

Even in the case E is infinite dimensional, we define projection maps $\pi_\xi : L(E;E) \to T_\xi(O(E))$ by formula (4.8) and we have an associated symmetric connection in $O(E)$ defined by the bilinear maps $\theta_\xi : T_\xi(O(E)) \times T_\xi(O(E)) \to L(E;E)$,

$$\theta_\xi(\alpha,\beta) = D\pi_\xi(\beta)(\alpha) = -\frac{1}{2}\left(\beta\circ\alpha^*\circ\xi + \xi\circ\alpha^*\circ\beta\right). \tag{4.9}$$

Of course, in the finite dimensional case, this will be the metric connection.

Now assume that E is a finite or infinite dimensional real Hilbert space and $H \subset E$ is a fixed closed vector subspace. We can define a smooth map $\Phi : O(E) \to G(E)$ associating to each $\xi \in O(E)$ the orthogonal projection onto $\xi(H)$; denoting by $\pi : E \to H$ the orthogonal projection, it is easy to see that we have

$$\Phi(\xi) = \xi\circ\pi\circ\xi^*. \tag{4.10}$$

Although Φ is not a totally geodesic map, we can nevertheless state:

4.4. $\Phi : O(E) \to O(E)$ has totally geodesic fibres and, in case E is finite dimensional, is a Riemannian submersion.

Proof. The derivative linear map $D\Phi_\xi : T_\xi(O(E)) \to T_{\Phi(\xi)}(O(E))$ is defined by

$$D\Phi_\xi(\alpha) = \alpha\circ\pi\circ\xi^* + \xi\circ\pi\circ\alpha^*.$$

Given $\alpha \in T_\xi(O(E))$ and $\beta \in T_{\Phi(\xi)}(O(G(E))$ arbitrary, we obtain, using (4.7), 3.7(c) and (2.3),

$$\begin{aligned}
\langle D\Phi_\xi(\alpha),\beta\rangle &= \langle\alpha\circ\pi\circ\xi^*,\beta\rangle + \langle\xi\circ\pi\circ\alpha^*,\beta\rangle \\
&= \langle\alpha\circ\pi\circ\xi^*,\beta\rangle + \langle-\xi\circ\pi\circ\xi^*\circ\alpha\circ\xi^*,\beta\rangle \\
&= \langle\alpha,\beta\circ\xi\circ\pi\rangle + \langle\alpha,-\xi\circ\pi\circ\xi^*\circ\beta\circ\xi\rangle \\
&= \langle\alpha,\beta\circ\xi\circ\pi - \beta\circ\xi + \beta\circ\xi\circ\pi\rangle \quad = \quad \langle\alpha, 2\beta\circ\xi\circ\pi - \beta\circ\xi\rangle,
\end{aligned}$$

where, using (4.7), we can see that $2\beta\circ\xi\circ\pi - \beta\circ\xi \in T_\xi(O(E))$. Hence, the adjoint linear map $D\Phi_\xi^* : T_{\Phi(\xi)}(O(E)) \to T_\xi(O(E))$ is defined by

$$D\Phi^*_\xi(\beta) = 2\beta\circ\xi\circ\pi - \beta\circ\xi.$$

It is not difficult to verify now that

$$D\Phi_\xi(D\Phi^*_\xi(\beta)) = \beta,$$

which means precisely that Φ is a Riemannian submersion.

Let $\xi \in O(E)$ and let $O_o(E)$ be the fibre of Φ over $\Phi(\xi)$. To prove that $O_o(E)$ is a totally geodesic submanifold of $O(E)$, all we have to see is that, for each α and β in $T_\xi(O_o(E))$, we have $(\beta, \theta_\xi(\alpha,\beta)) \in T_{(\xi,\alpha)}(T(O_o(E)))$, where θ_ξ is the connection on $O(E)$. Using the formula for $D\Phi_\xi$, we see that the fact that α and β are in $T_\xi(O_o(E))$ is equivalent to

$$\alpha\circ\pi\circ\xi^* + \xi\circ\pi\circ\alpha^* = 0, \quad \beta\circ\pi\circ\xi^* + \xi\circ\pi\circ\beta^* = 0$$

and, using the same formula, one concludes easily that, if $(\beta,\lambda) \in T_{(\xi,\alpha)}(T(O(E)))$, then $(\beta,\lambda) \in T_{(\xi,\alpha)}(T(O_o(E)))$ if and only if

$$\alpha\circ\pi\circ\beta^* + \beta\circ\pi\circ\alpha^* + \lambda\circ\pi\circ\xi^* + \xi\circ\pi\circ\lambda^* = 0.$$

Now, using formula (4.9) for $\theta_\xi(\alpha,\beta)$ and the characterization of $T_\xi(O(E))$given in 4.7, we obtain

$$\begin{aligned}
&\alpha\circ\pi\circ\beta^*+\beta\circ\pi\circ\alpha^*+\theta_\xi(\alpha,\beta)\circ\pi\circ\xi^*+\xi\circ\pi\circ\theta_\xi(\alpha,\beta)^*\\
&\quad= \alpha\circ\pi\circ\beta^*+\beta\circ\pi\circ\alpha^*\\
&\qquad -\frac{1}{2}\left(\beta\circ\alpha^*\circ\xi+\xi\circ\alpha^*\circ\alpha^*\circ\beta\right)\circ\pi\circ\xi^*-\frac{1}{2}\xi\circ\pi\left(\xi^*\circ\alpha\circ\beta^*+\beta^*\circ\alpha\circ\xi^*\right)\\
&\quad= \alpha\circ\pi\circ\beta^*+\beta\circ\pi\circ\alpha^*+\frac{1}{2}\beta\circ\xi^*\circ\alpha\circ\pi\circ\xi^*+\frac{1}{2}\alpha\circ\xi^*\circ\beta\circ\pi\circ\xi^*\\
&\qquad +\frac{1}{2}\xi\circ\pi\circ\alpha^*\circ\xi\circ\beta^*+\frac{1}{2}\xi\circ\pi\circ\beta^*\circ\xi\circ\alpha^*\\
&\quad= \alpha\circ\pi\circ\beta^*+\beta\circ\pi\circ\alpha^*-\frac{1}{2}\beta\circ\pi\circ\alpha^*-\frac{1}{2}\alpha\circ\pi\circ\beta^*-\frac{1}{2}\alpha\circ\pi\circ\beta^*-\frac{1}{2}\beta\circ\pi\circ\alpha^*\\
&\quad= 0,
\end{aligned}$$

and the proof is complete. □

We are going now to present a formula for the geodesics in $G(E)$ with arbitrary initial conditions. Let E be a real Hilbert space.

4.5. For each $\xi\in G(E)$ and $\eta\in T_\xi(G(E))$, there exists a smooth map $f:\mathbb{R}\to G(E)$ defined by

$$f(t)=\frac{1}{2}\Big(Id+(2\xi-Id)\circ\cos(2t\eta)+\sin(2t\eta)\Big)$$

and f is a geodesic of $G(E)$ that verifies $f(0)=\xi$ and $f'(0)=\eta$.

Proof. We note first that, from 3.7(f), we conclude that $(2\xi-Id)$ commutes with $\cos(2t\eta)$ and anti-commutes with $\sin(2t\eta)$. It is now trivial that $f(t)$ is self-adjoint and, noting that $(2\xi-Id)\circ(2\xi-Id)=Id$ and

$$\cos(2t\eta)\circ\cos(2t\eta)+\sin(2t\eta)\circ\sin(2t\eta)=Id,$$

we obtain

$$\begin{aligned}
f(t)\circ f(t) &= \frac{1}{4}\Big(Id+(2\xi-Id)\circ\cos(2t\eta)+\sin(2t\eta)\quad+(2\xi-Id)\circ\cos(2t\eta)\\
&\qquad+(2\xi-Id)\circ\cos(2t\eta)\circ(2\xi-Id)\circ\cos(2t\eta)+(2\xi-Id)\circ\cos(2t\eta)\circ\sin(2t\eta)\\
&\qquad+\sin(2t\eta)+\sin(2t\eta)\circ(2\xi-Id)\circ\cos(2t\eta)+\sin(2t\eta)\circ\sin(2t\eta)\Big)\\
&= \frac{1}{4}\Big(Id+(2\xi-Id)\circ\cos(2t\eta)+\sin(2t\eta)+(2\xi-Id)\circ\cos(2t\eta)\\
&\qquad+\cos(2t\eta)\circ\cos(2t\eta)+\sin(2t\eta)+\sin(2t\eta)\circ\sin(2t\eta)\Big)\quad=\quad f(t),
\end{aligned}$$

whence we conclude that $f(t) \in G(E)$. Next we see that

$$\begin{aligned} f'(t) &= \frac{1}{2}\Big(-2(2\xi - Id) \circ \sin(2t\eta) \circ \eta + 2\cos(2t\eta) \circ \eta\Big) \\ &= \Big(\cos(2t\eta) - (2\xi - Id) \circ \sin(2t\eta)\Big) \circ \eta, \end{aligned}$$

in particular $f'(0) = \eta$. Next we obtain

$$f''(t) = \Big(-2\sin(2t\eta) - 2(2\xi - Id) \circ \cos(2t\eta)\Big) \circ \eta^2.$$

On the other side, remembering 3.7(f), we have

$$\begin{aligned} f'(t) \circ f'(t) &= \cos(2t\eta)^2 \circ \eta^2 - \cos(2t\eta) \circ \eta \circ (2\xi - Id) \circ \sin(2t\eta) \circ \eta \\ &\quad -(2\xi - Id) \circ \sin(2t\eta) \circ \eta \circ \cos(2t\eta) \circ \eta \\ &\quad +(2\xi - Id) \circ \sin(2t\eta) \circ \eta \circ (2\xi - Id) \circ \sin(2t\eta) \circ \eta \\ &= \cos(2t\eta)^2 \circ \eta^2 + (2\xi - Id) \circ \cos(2t\eta) \circ \sin(2t\eta) \circ \eta^2 \\ &\quad -(2\xi - Id) \circ \sin(2t\eta) \circ \cos(2t\eta) \circ \eta^2 \\ &\quad +(2\xi - Id) \circ (2\xi - Id) \circ \sin(2t\eta)^2 \circ \eta^2 \\ &= \Big(\cos(2t\eta)^2 + \sin(2t\eta)^2\Big) \circ \eta^2 \quad = \quad \eta^2, \end{aligned}$$

and, using (4.2), we have now

$$\begin{aligned} &\theta_{f(t)}(f'(t), f'(t)) \\ &\quad = (Id - 2f(t)) \circ f'(t) \circ f'(t) + f'(t) \circ f'(t) \circ (Id - 2f(t)) \\ &\quad = \Big(- (2\xi - Id) \circ \cos(2t\eta) - \sin(2t\eta)\Big) \circ \eta^2 \\ &\qquad + \eta^2 \circ \Big(- (2\xi - Id) \circ \cos(2t\eta) - \sin(2t\eta)\Big) \quad = \quad f''(t), \end{aligned}$$

whence we conclude that f is indeed a geodesic. □

4.6. Let E be a real Hilbert space. $G(E)$ is then a symmetric space and, for each $\pi \in G(E)$, the symmetry $Sym : G(E) \to G(E)$ with respect to π is defined by

$$Sym(\xi) = (Id - 2\pi) \circ \xi \circ (Id - 2\pi).$$

Proof. It is trivial that $Sym(\xi)$ is a self-adjoint map and the fact that $(Id-2\pi)\circ(Id-2\pi) = Id$ shows that $Sym(\xi) \circ Sym(\xi) = Sym(\xi)$, hence $Sym(\xi) \in G(E)$. It is trivial to see that $Sym(\pi) = \pi$ and that $Sym(Sym(\xi)) = \xi$. We have

$$DSym_\xi(\alpha) = (Id - 2\pi) \circ \alpha \circ (Id - 2\pi),$$

hence, remembering (4.2),

$$\begin{aligned}
&\nabla DSym_\xi(\alpha,\beta) = (Id-2\pi)\circ\theta_\xi(\beta,\alpha)\circ(Id-2\pi) - \theta_{Sym(\xi)}(DSym_\xi(\beta), DSym_\xi(\alpha))\\
&= (Id-2\pi)\circ(Id-2\xi)\circ\beta\circ\alpha\circ(Id-2\pi) + (Id-2\pi)\circ\alpha\circ\beta\circ(Id-2\xi)\circ(Id-2\pi)\\
&\quad -\Big(Id-2(Id-2\pi)\circ\xi\circ(Id-2\pi)\Big)\circ(Id-2\pi)\circ\beta\circ(Id-2\pi)\circ(Id-2\pi)\circ\alpha\circ(Id-2\pi)\\
&\quad -(Id-2\pi)\circ\alpha\circ(Id-2\pi)\circ(Id-2\pi)\circ\beta\circ(Id-2\pi)\circ\Big(Id-2(Id-2\pi)\circ\xi\circ(Id-2\pi)\Big)\\
&= (Id-2\pi)\circ\beta\circ\alpha\circ(Id-2\pi) - 2(Id-2\pi)\circ\xi\circ\beta\circ\alpha\circ(Id-2\pi)\\
&\quad +(Id-2\pi)\circ\alpha\circ\beta\circ(Id-2\pi) - 2(Id-2\pi)\circ\alpha\circ\beta\circ\xi\circ(Id-2\pi)\\
&\quad -(Id-2\pi)\circ\beta\circ\alpha\circ(Id-2\pi) + 2(Id-2\pi)\circ\xi\circ\beta\circ\alpha\circ(Id-2\pi)\\
&\quad -(Id-2\pi)\circ\alpha\circ\beta\circ(Id-2\pi) + 2(Id-2\pi)\circ\alpha\circ\beta\circ\xi\circ(Id-2\pi) \quad = \quad 0,
\end{aligned}$$

that is to say, Sym is a totally geodesic diffeomorphism. Now, if $f:\mathbb{R}\to G(E)$ is a geodesic with $f(0)=\pi$ and $f'(0)=\eta$, we have

$$f(t) = \frac{1}{2}\left(Id + (2\pi - Id)\circ\cos(2t\eta) + \sin(2t\eta)\right),$$

hence

$$\begin{aligned}
Sym(f(t)) &= \frac{1}{2}(Id-2\pi)\circ\Big(Id + (2\pi - Id)\circ\cos(2t\eta) + \sin(2t\eta)\Big)\circ(Id-2\pi)\\
&= \frac{1}{2}\Big(Id + (Id-2\pi)\circ\cos(2t\eta) - \sin(2t\eta)\Big) \quad = \quad f(-t),
\end{aligned}$$

and the proof is complete. □

5. The complex Grassman manifolds

Assume that E is a complex Hilbert space, whose inner product will always be denoted by $\langle,\rangle_{\mathbb{C}}$. Then E is also a real Hilbert space, with the inner product

$$\langle x,y\rangle = \mathrm{Re}\langle x,y\rangle_{\mathbb{C}} \tag{5.1}$$

and the following two facts are trivial:

5.1. If $F\subset E$ is a complex vector subspace, then the orthogonal projection $\pi : E\to F$ is the same when we consider in E either the complex or the real inner product.

5.2. If $\xi : E\to E$ is a complex linear map, then the adjoint map $\xi^* : E\to E$ is the same when we consider E to be either a complex or a real Hilbert space.

We will denote by $L(E;E)$ the vector space of all continuous *real* linear maps and by $L_{\mathbb{C}}(E;E)$ its vector subspace whose elements are the complex linear maps. In the case where E is finite dimensional the Hilbert-Schmidt inner product that we will consider in $L(E;E)$ will be the one associated to the real structure of E and we will consider in the closed subspace $L_{\mathbb{C}}(E;E)$ the induced inner product.

If E is a complex Hilbert space, we will denote by $G_{\mathbb{C}}(E)$ the set of the orthogonal projections onto closed complex vector subspaces, and we call $G_{\mathbb{C}}(E)$ the *complex Grassman* *manifold* of E. $G(E)$ will denote the real Grassman manifold of E, i.e. the Grassman manifold of E, when considered as a real Hilbert space. It is trivial to conclude that

$$G_{\mathbb{C}}(E) = G(E) \cap L_{\mathbb{C}}(E; E). \tag{5.2}$$

All that has been said in Section 3 applies *mutatis mutandis* to the complex Grassman manifolds, but one must be aware that $G_{\mathbb{C}}(E)$ is only a *real* manifold within the complex vector space $L_{\mathbb{C}}(E; E)$. The essential reason for this is the fact that the map $\xi \to \xi^*$ is not $\mathbb{C}$-linear, but it was natural to anticipate this because, in case E is finite dimensional, $G_{\mathbb{C}}(E)$ (like $G(E)$) is compact (because it is closed and bounded) and it is well known that there exists no compact nontrivial complex submanifold of a complex vector space.

For each closed complex vector subspace $F \subset E$, we still have a diffeomorphism $\psi_F : \mathcal{U}_F \to L_{\mathbb{C}}(F, F^{\perp})$, where $\mathcal{U}_F$ is open in $G_{\mathbb{C}}(E)$ and contains π_F (cf. 3.4), hence:

5.3. If E has complex dimension N and $F \subset E$ has complex dimension n, then the real manifold $G_{\mathbb{C}}(E)$ has dimension $2n(N-n)$ in π_F.

The tangent vector space $T_{\pi_F}(G_{\mathbb{C}}(E))$ is contained in the real vector space $L_{\mathbb{C}sa}(E; E)$, whose elements are the self-adjoint complex linear maps and, for each $\eta \in L_{\mathbb{C}sa}(E; E)$, the fact that $\eta \in T_{\pi_F}(G_{\mathbb{C}}(E))$ is equivalent to each of the conditions (b) to (f) of 3.7; in other words:

$$T_{\pi_F}(G_{\mathbb{C}}(E)) = T_{\pi_F}(G(E)) \cap L_{\mathbb{C}}(E; E). \tag{5.3}$$

Although $G_{\mathbb{C}}(E)$ is only a real submanifold of $L_{\mathbb{C}}(E; E)$, it admits a complex structure:

5.4. Let E be a complex Hilbert space. Then the real manifold $G_{\mathbb{C}}(E)$ admits a complex structure defined by the linear maps

$$J_\xi : T_\xi(G_{\mathbb{C}}(E)) \to T_\xi(G_{\mathbb{C}}(E)), \quad J_\xi(\eta) = i\eta \circ (2\xi - Id).$$

For this structure the real diffeomorphisms $\psi_F : \mathcal{U}_F \to L_{\mathbb{C}}(F, F^{\perp})$ are in fact holomorphic.

Proof. To see that J_ξ applies $T_\xi(G_{\mathbb{C}}(E))$ into itself we use 3.7(f), remembering that $(2\xi - Id) \circ (2\xi - Id) = Id$ and noting that

$$(i\eta \circ (2\xi - Id))^* = -i(\eta \circ (2\xi - Id))^* = -i(2\xi - Id) \circ \eta = i\eta \circ (2\xi - Id).$$

It is also trivial that $J_\xi(J_\xi(\eta)) = -\eta$. The fact that this almost complex structure is indeed a complex one comes from the fact that the real diffeomorphisms $\psi_F : \mathcal{U}_F \to L_{\mathbb{C}}(F, F^{\perp})$ are holomorphic; this is a simple consequence of the formula in 3.6,

$$D\psi_F(\xi)(\eta) = \eta \circ (\pi_F|_{\xi(E)})^{-1} - (\pi_F|_{\xi(E)})^{-1} \circ \pi_F \circ \eta \circ (\pi_F|_{\xi(E)})^{-1},$$

the formula $D\psi_F(\xi)(J_\xi(\eta)) = iD\psi_F(\xi)(\eta)$ being a simple consequence of the fact that the restriction of $(2\xi - Id)$ to $\xi(E)$ is the identity. $\square$

Note that, in case E is finite dimensional, if we choose a complex orthonormal basis $x_1, \ldots, x_N$ of E such that $x_1, \ldots, x_n$ is a basis of $\xi(E)$, then, if $\eta \in T_\xi(G_{\mathbb{C}}(E))$ has matrix $\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$, $J_\xi(\eta)$ has matrix $\begin{bmatrix} 0 & -iA^* \\ iA & 0 \end{bmatrix}$.

The considerations in 4.1-4.3 and (4.1)-(4.5) apply *mutatis mutandis* to the complex Grassman manifolds and we have in particular a *canonical symmetric connection* in $G_{\mathbb{C}}(E)$ defined also by

$$\theta_\xi(\eta,\alpha) = (Id - 2\xi)\circ\eta\circ\alpha + \alpha\circ\eta\circ(Id-2\xi). \tag{5.4}$$

This implies in particular that:

5.5. $G_{\mathbb{C}}(E)$ is a totally geodesic submanifold of $G(E)$. We see now that:

5.6. If E is a complex Hilbert space, then the morphism $J = (J_\xi)$, from the vector bundle $T(G_{\mathbb{C}}(E))$ into itself, is parallel.

Proof. From $J_\xi(\beta) = i\beta\circ(2\xi - Id)$, we obtain

$$\begin{aligned}
\nabla J_\xi(\alpha)(\beta) &= 2i\beta\circ\alpha + i\theta_\xi(\beta,\alpha)\circ(2\xi - Id) - \theta_\xi(i\beta\circ(2\xi-Id),\alpha)\\
&= 2i\beta\circ\alpha + i(Id-2\xi)\circ\beta\circ\alpha\circ(2\xi-Id) + i\alpha\circ\beta\circ(Id-2\xi)\circ(2\xi-Id)\\
&\quad -i(Id-2\xi)\circ\beta\circ(2\xi-Id)\circ\alpha - i\alpha\circ\beta\circ(2\xi-Id)\circ(Id-2\xi)\\
&= 2i\beta\circ\alpha - i\beta\circ\alpha - i\alpha\circ\beta - i\beta\circ\alpha + i\alpha\circ\beta \quad = \quad 0.
\end{aligned}$$

□

In the case where the complex Hilbert space E is finite dimensional, we note that the Hilbert-Schmidt inner product in $L(E;E)$ is the real part of the complex inner product (the one defined by (2.2) with $\langle,\rangle_{\mathbb{C}}$ instead of $\langle,\rangle$) and we see that

$$\begin{aligned}
\langle J_\xi(\alpha), J_\xi(\beta)\rangle &= \langle i\alpha\circ(2\xi-Id), i\beta\circ(2\xi-Id)\rangle \quad = \quad \langle\alpha\circ(2\xi-Id),\beta\circ(2\xi-Id)\rangle\\
&= \langle\alpha\circ(2\xi-Id)\circ(2\xi-Id),\beta\rangle \quad = \quad \langle\alpha,\beta\rangle,
\end{aligned}$$

hence:

5.7. If E is a finite dimensional complex Hilbert space, then $G_{\mathbb{C}}(E)$ is a Kähler manifold.

We end with the remark that (4.7)-(4.10) and 4.4 work equally well in the complex case, the usual notation for $O_{\mathbb{C}}(E)$ being $U(E)$ (the unitary group).

References

[1] Akin, E. *The Metric Theory of Banach Manifolds*, Lecture Notes in Mathematics, **662**, Springer (1978).

[2] Machado, A. *A locally closed set with a smooth group structure is a Lie group.* Proc. Amer. Math. Soc. **84** (2) (1982).

FIGURE 1. Scanned page 85 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

A MACHADO & I SALAVESSA

Grassman manifolds as subsets of Euclidean spaces

1. INTRODUCTION

Let E be a Euclidean space. Following Palais, we identify each vector subspace F of E with the orthogonal projection $\pi_F : E \to F$. In this way, the Grassman manifold G(E) of all vector subspaces of E appears as a submanifold of the Euclidean space L(E;E) of all linear maps from E into E (with the Hilbert-Schmidt inner product). The aim of this paper is to present some explicit formulas concerning the differential geometry of G(E) as a submanifold of L(E;E). Most of these formulas extend naturally to the case where E is an infinite dimensional Hilbert space, although in this case there is no natural inner product in L(E;E).

2. NOTATION AND PRELIMINARIES

Let E and F be finite or infinite dimensional Hilbert spaces. We will denote by L(E;F) the vector space of all continuous linear maps from E into F. If $\xi \in L(E;F)$, we will denote by $\xi^* \in L(F;E)$ its *adjoint* linear map, the one defined by the identity

$$\langle \xi(x), y \rangle = \langle x, \xi^*(y) \rangle.$$

The following identities will be used quite often:

$$\xi^{**} = \xi; \quad (\eta \circ \xi)^* = \xi^* \circ \eta^*; \quad id_E^* = id_E. \tag{2.1}$$

A linear map $\xi \in L(E;E)$ is *self-adjoint* if $\xi^* = \xi$. The map $L(E;F) \to L(F;E)$, $\xi \to \xi^*$, is a *real* linear map (even if E and F are complex spaces, it is not a complex linear map) and the set $L_{sa}(E;E)$ of self-adjoint linear maps is a real vector subspace of L(E;F).

In case E or F is infinite dimensional, we will look on L(E;F) merely as a Banach space (with the sup norm). In case E and F are finite dimensional, we take in the finite dimensional vector space L(E;F) the *Hilbert-Schmidt inner product*, defined by

FIGURE 2. Scanned page 86 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

$$\langle\xi,\eta\rangle = \sum_{1\le k\le n} \langle\xi(x_k), \eta(x_k)\rangle, \tag{2.2}$$

where $x_1,\ldots,x_k$ is an arbitrary orthonormal basis of E. We will use the following identities concerning these inner products:

$$\langle\xi,\eta\rangle = \langle\eta^*,\xi^*\rangle; \ \langle\lambda,\mu^*\circ\eta\rangle = \langle\mu\circ\lambda,\eta\rangle = \langle\mu,\eta\circ\lambda^*\rangle. \tag{2.3}$$

The word "manifold" will always mean an embedded submanifold of some finite dimensional or Banach vector space B and the tangent vector spaces will be considered as vector subspaces of the ambient vector space B. In fact, one can even define, for each point a of an arbitrary subset M of B, a notion of tangent vector subspace $T_a(M)$, which behaves well with respect to differentiability (see, for example, [2]). In the same spirit, by vector bundle we will mean a vector sub-bundle of constant one. A vector bundle $\underline{E}$ with basis M will be a family $(E_x)_{x\in M}$, where each E_x is a vector subspace of a fixed finite dimensional or Banach vector space E, verifying the usual properties, and we will use the same symbol $\underline{E}$ to denote the corresponding subset of $M \times E$. It will be useful to allow a vector bundle to have as basis an arbitrary subset M of a finite dimensional or Banach vector space B.

If $\underline{E} = (E_x)_{x\in M}$ is a vector bundle with $E_x \subset E$, we identify a *connection* in $\underline{E}$ by its second fundamental form at each point $x \in M$, which is a bilinear map $\theta_x : E_x \times T_x(M) \to E$ such that

$$(u,\theta_x(w,u)) \in T_{(x,w)}(\underline{E}). \tag{2.4}$$

For each smooth section $W = (W_x)_{x\in M}$ of $\underline{E}$, the covariant derivative $\nabla W_x(u)$ is given by the formula

$$\nabla W_x(u) = DW_x(u) - \theta_x(W_x,u). \tag{2.5}$$

If E is a Hilbert space, the *metric connection* of $\underline{E}$ is the one defined by the condition that $\theta_x(w,u)$ is orthogonal to the fibre E_x; if $\pi_x : E \to E_x$ is the orthogonal projection, then $x \to \pi_x$ is a smooth map from M into L(E;E) and we have the following formula for this connection:

$$\theta_x(w,u) = D\pi_x(u)(w). \tag{2.6}$$

We will use also the following characterization of the curvature tensor of a connection θ in the vector bundle $\underline{E} = (E_x)_{x\in M}$, where $M \subset B$ is a manifold

FIGURE 3. Scanned page 87 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

and $E_x \subset E$: assuming that $x \to \hat{\theta}_x$ is a smooth map from M into the space L(E,B;E) of bilinear maps, such that each θ_x is a restriction of $\hat{\theta}_x$, the curvature tensor is the trilinear map

$$R_x : T_x(M) \times T_x(M) \times E_x \to E_x$$

defined by

$$R_x(u,v,w) = D\hat{\theta}_x(u)(w,v) - D\hat{\theta}_x(v)(w,u) + \hat{\theta}_x(\theta_x(w,u),v) - \hat{\theta}_x(\theta_x(w,v),u). \tag{2.7}$$

3. THE GRASSMAN MANIFOLDS

Let E be a finite or infinite dimensional real Hilbert space. For each closed vector subspace $F \subset E$, we will denote π_F the orthogonal projection from E onto F. We have hence a natural bijective map between the set of closed vector subspaces of E and the set of orthogonal projections. We will denote by G(E) the subset of L(E;E) whose elements are the orthogonal projections onto closed subspaces, and we will call G(E) the *Grassman manifold* of E. The fact that G(E) is indeed a manifold is proved in Akin [1], who attributes this result to Palais (unpublished preprint), but we will sketch here an independent proof.

The following characterization of the elements of G(E) is well known:

3.1. A linear map $\xi \in L(E;E)$ belongs to G(E) if and only if it is self-adjoint and verifies $\xi \circ \xi = \xi$.

We can consider a morphism from the constant vector bundle $E_{G(E)}$, with basis G(E) and fibre E, into itself, associating to each $\xi \in G(E)$ the linear map $\xi : E \to E$. The fact that the image of an idempotent morphism is a vector bundle allows us to state:

3.2. There exists a *tautological vector bundle* with basis G(E), whose fibre in each π_F is F.

Using formula (2.6) for the metric connection, we deduce:

3.3 The metric connection of the tautological vector bundle is defined by

$$\theta_\xi(w,\eta) = \eta(w),$$

for each $\xi \in G(E)$, $w \in \xi(E)$ and $\eta \in T_\xi(G(E))$.

As a corollary of the local constancy of the dimension of the fibres of a vector bundle, we see that, for each n, the subset $G_n(E)$ of $G(E)$, whose elements are the π_F such that F is n-dimensional, is open in $G(E)$.

Let $F \subset E$ be a fixed closed vector subspace. It is a well known simple linear algebra result that, for each closed vector subspace $G \subset E$, the following two properties are equivalent:

(a) $E = F^\perp \oplus G$ (direct sum);

(b) $\pi_{F/G}$ is an isomorphism from G onto F;

and that, if they are verified, the projection $E \to G$ associated to the direct sum is $(\pi_{F/G})^{-1} \cdot \pi_F$. To each $\alpha \in L(F;F^\perp)$ we associate its graphic $G = \{x + \alpha(x)\}_{x \in F}$, which is a closed vector subspace of E verifying the conditions above. Inversely, for each closed vector subspace $G \subset E$ verifying the conditions above, there exists one and only one $\alpha \in L(F;F^\perp)$ whose graphic is G, namely $\alpha = \pi_{F^\perp} \circ (\pi_{F/G})^{-1}$.

We will use the preceding well-known considerations in the proof of the following result:

3.4. Let E be a real Hilbert space and let $F \subset E$ be a closed vector subspace. Let $\mathcal{U}_F \subset G(E)$ be the set of the orthogonal projections $\xi \in G(E)$ such that $E = F^\perp \oplus \xi(E)$. Then $\mathcal{U}_F$ is an open subset in $G(E)$, containing π_F, and there exists a diffeomorphism $\psi_F : \mathcal{U}_F \to L(F;F^\perp)$, defined by $\psi_F(\xi) = \pi_{F^\perp} \circ (\pi_{F/\xi(E)})^{-1}$, that verifies $\psi_F(\pi_F) = 0$.

Proof The considerations before the statement show that ψ_F is a bijective map from $\mathcal{U}_F$ onto $L(F;F^\perp)$, whose inverse $\psi_F^{-1} : L(F;F^\perp) \to \mathcal{U}_F$ associates to each α the orthogonal projection onto the closed vector subspace $\{x + \alpha(x)\}_{x \in F}$. All we have to show is that $\mathcal{U}_F$ is open in $G(E)$ and that both ψ_F and ψ_F^{-1} are smooth maps. For that, we consider the morphism from the tautological vector bundle $(\xi(E))_{\xi \in G(E)}$ into the constant vector bundle $F_{G(E)}$ whose value at $\xi \in G(E)$ is $\pi_{F/\xi(E)} : \xi(E) \to F$; the fact that $\xi \in \mathcal{U}_F$ if and only if the "fibre" of the morphism at ξ is an isomorphism implies that $\mathcal{U}_F$ is open in $G(E)$; taking the restrictions of the vector bundles to $\mathcal{U}_F$, the fact that the inverse of a (smooth) isomorphism is smooth implies that the map $\mathcal{U}_F \to L(F;E)$, $\xi \to (\pi_{F/\xi(E)})^{-1}$ is smooth, hence $\psi_F : \mathcal{U}_F \to L(F;F^\perp)$ is also

FIGURE 5. Scanned page 89 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

smooth. Now, we have an injective morphism from the constant vector bundle $F_{L(F;F^\perp)}$ into the constant vector bundle $E_{L(F;F^\perp)}$, whose fibre at $\alpha \in L(F;F^\perp)$ is the linear map $F \to E$, $x \to x + \alpha(x)$, hence the image of this morphism is a vector bundle with basis $L(F;F^\perp)$ and this implies that the map $\psi_F^{-1}: L(F;F^\perp) \to L(E;E)$ is smooth.

As a corollary, we have:

3.5. If E is a real Hilbert space, then $G(E)$ is a manifold in $L(E;E)$. If E is N-dimensional and $F \subset E$ is n-dimensional, then the dimension of $G(E)$ at π_F is $n(N-n)$.

3.6. Let E be a real Hilbert space, $F \subset E$ be a closed vector subspace and $\psi_F: \mathcal{U}_F \to L(F;F^\perp)$ be the diffeomorphism defined in 3.4. For each $\xi \in \mathcal{U}_F$ and $\eta \in T_\xi(G(E))$, we have

$$D\psi_F(\xi)(\eta) = \eta\circ(\pi_{F/\xi(E)})^{-1} - (\pi_{F/\xi(E)})^{-1}\circ\pi_F\circ\eta\circ(\pi_{F/\xi(E)})^{-1}.$$

In particular, $D\psi_F(\pi_F)(\eta) = \eta_{/F}$.

<u>Proof</u> Let $\Phi_F: \mathcal{U}_F \to L(F;E)$ be the smooth map defined by $\Phi_F(\xi) = (\pi_{F/\xi(E)})^{-1}$ (see the proof of 3.4). Let $w \in F$ arbitrary. Differentiating the identity $\pi_F(\Phi_F(\xi)(w)) = w$, we obtain

$$\pi_F(D\Phi_F(\xi)(\eta)(w)) = 0,$$

hence $D\Phi_F(\xi)(\eta)(w) \in F^\perp$. On the other hand, we have a smooth section of the tautological vector bundle $(\xi(E))_{\xi\in G(E)}$ associating to each ξ, $\Phi_F(\xi)(w)$; its covariant derivative with respect to the metric connection, which, by (2.5) and 3.3, is equal to

$$D\Phi_F(\xi)(\eta)(w) - \eta(\Phi_F(\xi)(w)),$$

must hence belong to $\xi(E)$. We can now conclude that $D\Phi_F(\xi)(\eta)(w)$ is the projection of $\eta(\Phi_F(\xi)(w))$ onto $F^\perp$ associated to the direct sum $E = F^\perp \oplus \xi(E)$. The fact that $\psi_F(\xi)(w) = \pi_{F^\perp}(\Phi_F(\xi)(w))$ shows that $D\psi_F(\xi)(\eta)(w) = \pi_{F^\perp}(D\Phi_F(\xi)(w))$, hence $D\psi_F(\xi)(\eta)(w)$ is also the projection of $\eta(\Phi_F(\xi)(w))$ onto $F^\perp$ associated to the direct sum $E = F^\perp \oplus \xi(E)$ and, by the considerations made before 3.4, this projection is equal to

FIGURE 6. Scanned page 90 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

$$\eta((\pi_{F/\xi(E)})^{-1}(w)) - (\pi_{F/\xi(E)})^{-1}(\pi_F(\pi((\eta_{F/\xi(E)})^{-1}(w)))).$$

To show that $D\psi_F(\pi_F)(\eta) = \eta_{/F}$ it will be enough to know that each $\eta \in T_{\pi_F}(G(E))$ maps F into $F^{\perp}$. To see this, we differentiate the identity $\xi \circ \xi = \xi$ and obtain $\eta \circ \pi_F + \pi_F \circ \eta = \eta$, hence $\eta \circ \pi_F = \eta - \pi_F \circ \eta = \pi_{F^{\perp}} \circ \eta$ and the proof is complete.

We present now several equivalent characterizations of the tangent vector spaces to G(E).

3.7. Let E be a real Hilbert space and let $F \subset E$ be a closed vector subspace. The tangent vector space $T_{\pi_F}(G(E))$ is then contained in the vector space $L_{sa}(E;E)$ of self adjoint maps and, for each $\eta \in L_{sa}(E;E)$, the following conditions are equivalent:

(a) $\eta \in T_{\pi_F}(G(E))$;

(b) $\eta(F) \subset F^{\perp}$ and $\eta(F^{\perp}) \subset F$;

(c) $\eta \circ \pi_F + \pi_F \circ \eta = \eta$;

(d) $\eta \circ \pi_F = (Id-\pi_F) \circ \eta$;

(e) $\eta \circ (Id-\pi_F) = \pi_F \circ \eta$;

(f) $\eta \circ (2\pi_F-Id) = -(2\pi_F-Id) \circ \eta$.

Proof The fact that each $T_{\pi_F}(G(E))$ is contained in $L_{sa}(E;E)$ is a consequence of the fact that $G(E) \subset L_{sa}(E;E)$. The equivalence between the four last conditions is trivial. Assuming (a), we obtain (c) simply by differentiating the identity $\xi \circ \xi = \xi$ at π_F in the direction of η. It is readily seen that condition (d) implies that $\eta(F) \subset F^{\perp}$ and that condition (e) implies that $\eta(F^{\perp}) \subset F$ $(Id-\pi_F = \pi_{F^{\perp}})$. Let us prove now that condition (b) implies condition (a). The fact that ψ_F is a diffeomorphism from the open set U_F in G(E) onto $L(F;F^{\perp})$ implies that $D\psi_F(\pi_F):T_{\pi_F}(G(E)) \to L(E;F^{\perp})$ is an isomorphism. We can hence take $\eta' \in T_{\pi_F}(G(E))$ such that

$$\eta_{/F} = D\psi_F(\pi_F)(\eta') = \eta'_{/F}.$$

Then η' is self-adjoint and verifies condition (b), hence $\eta'_{/F^{\perp}}:F^{\perp} \to F$ is

FIGURE 7. Scanned page 91 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

the adjoint map to $\eta'_{/F}:F \to F^{\perp}$ and $\eta_{/F^{\perp}}:F^{\perp} \to F$ is the adjoint map to $\eta_{/F}:F \to F^{\perp}$. We deduce now that $\eta'_{/F^{\perp}} = \eta_{/F^{\perp}}$, hence $\eta = \eta'$ and the proof is complete.

Remark. To feel what is happening, assume that E is finite dimensional and take an orthonormal basis $x_1,\dots,x_N$ of E, whose first n vectors constitute a basis for F. Then the matrices of π_F, $Id-\pi_F$ and $2\pi_F-Id$ are respectively

$$\begin{bmatrix} Id & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & Id \end{bmatrix} \quad \begin{bmatrix} Id & 0 \\ 0 & -Id \end{bmatrix}$$

and condition (b) says that the elements of $T_{\pi_F}(G(E))$ are the linear maps whose matrix has the form

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}.$$

4. THE DIFFERENTIAL GEOMETRY OF GRASSMAN MANIFOLDS

4.1. Let E be a real Hilbert space and let $F \subset E$ be a closed vector subspace. For each $\eta \in L_{sa}(E;E)$ the following conditions are then equivalent:

(a) $\eta(F) \subset F$ and $\eta(F^{\perp}) \subset F^{\perp}$;

(b) $\eta \circ \pi_F = \pi_F \circ \eta$;

(c) $\eta \circ (Id-\pi_F) = (Id-\pi_F) \circ \eta$.

We will denote by $T_{\pi_F}(G(E))^{\perp}$ the set of self-adjoint linear maps $\eta \in L_{sa}(E;E)$ verifying the preceding conditions.

Proof The fact that (b) and (c) are equivalent is trivial. It is readily seen that (b) implies $\eta(F) \subset F$ and that (c) implies $\eta(F^{\perp}) \subset F^{\perp}$. Assuming (a), one sees that $\eta \circ \pi_F(x) = \eta(x) = \pi_F \circ \eta(x)$ for $x \in F$ and $\eta \circ \pi_F(x) = 0 = \pi_F \circ \eta(x)$ for $x \in F^{\perp}$, hence $\eta \circ \pi_F(x) = \pi_F \circ \eta(x)$ for arbitrary x and (b) is proved.

4.2 Let E be a real Hilbert space and let $F \subset E$ be a closed vector subspace. Then $L_{sa}(E;E)$ is the direct sum of the closed vector subspaces $T_{\pi_F}(G(E))$ and $T_{\pi_F}(G(E))^{\perp}$ and the projections $\pi^{-}_{\pi_F}:L_{sa}(E;E) \to T_{\pi_F}(G(E))$ and $\pi^{-\perp}_{\pi_F}\, L_{sa}(E;E) \to T_{\pi_F}(G(E))^{\perp}$ associated to this direct sum are defined by

FIGURE 8. Scanned page 92 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

$$\bar{\pi}_{\pi_F}(\eta) = (\mathrm{Id}-\pi_F)\circ\eta\circ\pi_F + \pi_F\circ\eta\circ(\mathrm{Id}-\pi_F)$$

$$\bar{\pi}^{\perp}_{\pi_F}(\eta) = (\mathrm{Id}-\pi_F)\circ\eta\circ(\mathrm{Id}-\pi_F) + \pi_F\circ\eta\circ\pi_F$$

Proof Conditions (a) of 4.1 and (b) of 3.7 show that the intersection $T_{\pi_F}(G(E)) \cap T_{\pi_F}(G(E))^{\perp}$ is $\{0\}$. It is readily seen that, for each $\eta \in L_{sa}(E;E)$, $\bar{\pi}_{\pi_F}(\eta)$ applies F into $F^{\perp}$ and $F^{\perp}$ into F and $\bar{\pi}^{\perp}_{\pi_F}(\eta)$ applies F into F and $F^{\perp}$ into $F^{\perp}$, hence $\bar{\pi}_{\pi_F}(\eta) \in T_{\pi_F}(G(E))$ and $\bar{\pi}^{\perp}_{\pi_F}(\eta) \in T_{\pi_F}(G(E))^{\perp}$. All we have to note now is that, for each η, $\bar{\pi}_{\pi_F}(\eta) + \bar{\pi}^{\perp}_{\pi_F}(\eta) = \eta$.

4.3 If E is a finite dimensional real Hilbert space and if we consider in $L_{sa}(E;E)$ the Hilbert-Schmidt inner product, then, for each vector subspace $F \subset E$, the subspaces $T_{\pi_F}(G(E))$ and $T_{\pi_F}(G(E))^{\perp}$ of $L_{sa}(E;E)$ are mutually orthogonal, hence each one is the orthogonal complement of the other.

Proof Assume $\eta \in T_{\pi_F}(G(E))$ and $\eta' \in T_{\pi_F}(G(E))^{\perp}$. Choose an orthonormal basis $x_1,\dots,x_N$ of E such that the first n vectors constitute a basis of F and the last $N-n$ vectors constitute a basis of $F^{\perp}$. Conditions (b) of 3.7 and (a) of 4.1 assure that, for each $1 \leq k \leq N$, $\langle \eta(x_k), \eta'(x_k)\rangle = 0$, hence $\langle \eta,\eta'\rangle = 0$ (cf. (2.2)).

The preceding result explains why we employ the notation $T_{\pi_F}(G(E))^{\perp}$ and $\bar{\pi}_{\pi_F}$.

If E is a finite or infinite dimensional real Hilbert space we will define the *canonical connection* in the manifold $G(E)$ as the one that verifies the condition that $\theta_{\pi_F}(\eta,\alpha)$ belongs to the kernel $T_{\pi_F}(G(E))^{\perp}$ of the linear map $\bar{\pi}_{\pi_F}: L_{sa}(E;E) \to T_{\pi_F}(G(E))$, for each η and α in $T_{\pi_F}(G(E))$. In an analogous way to that used in the case of a metric connection, it is easily seen that this connection is symmetric and also is defined by the formula

$$\theta_{\pi_F}(\eta,\alpha) = D\bar{\pi}_{\pi_F}(\alpha)(\eta). \tag{4.1}$$

This is the connection that we will always consider in the Grassman manifold $G(E)$. Of, course, in case E is finite dimensional, this connection is the metric connection with respect to the Hilbert-Schmidt inner product.

We can obtain a more explicit formula for the connection on $G(E)$ by calculating the derivative in (4.1), using the formula in 4.2,

FIGURE 9. Scanned page 93 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

$\bar{\pi}_\xi(\eta) = (\mathrm{Id}-\xi) \circ \eta \circ \xi + \xi \circ \eta \circ (\mathrm{Id}-\xi)$. This gives

$$\theta_\xi(\eta,\alpha) = -\alpha\circ\eta\circ\xi + (\mathrm{Id}-\xi)\circ\eta\circ\alpha + \alpha\circ\eta\circ(\mathrm{Id}-\xi) - \xi\circ\eta\circ\alpha, \tag{4.2}$$

$$\theta_\xi(\eta,\alpha) = (\mathrm{Id}-2\xi)\circ\eta\circ\alpha + \alpha\circ\eta\circ(\mathrm{Id}-2\xi).$$

Let us now obtain, using (2.7), two formulas for the curvature, the first for the metric connection of the tautological vector bundle, and the second for the canonical connection of the Grassman manifold. In the first case, we take $\hat{\theta}_\xi(w,\eta) = \eta(w)$ for each $w \in E$ and $\eta \in L(E;E)$ (cf. 3.3) obtaining

$$R_\xi(\alpha,\beta,w) = \beta(\alpha(w)) - \alpha(\beta(w)), \tag{4.3}$$

for each α and β in $T_\xi(G(E))$ and $w \in \xi(E)$. In the second case, we take $\hat{\theta}_\xi(\eta,\alpha) = (\mathrm{Id}-2\xi)\circ\eta\circ\alpha + \alpha\circ\eta\circ(\mathrm{Id}-2\xi)$ and obtain, noting that $(\mathrm{Id}-2\xi)\circ(\mathrm{Id}-2\xi) = \mathrm{Id}$ and that, by 3.7(f), $\mathrm{Id}-2\xi$ commutes with the composite of any two elements of $T_\xi(G(E))$,

$$R_\xi(\alpha,\beta,\eta) = \eta\circ\alpha\circ\beta - \eta\circ\beta\circ\alpha + \beta\circ\alpha\circ\eta - \alpha\circ\beta\circ\eta. \tag{4.4}$$

Assuming that E is finite dimensional, we obtain, for the sectional curvatures:

$$\mathrm{Riem}_\xi(\alpha,\beta) = \langle R_\xi(\alpha,\beta,\alpha),\beta\rangle = 2\langle\alpha\circ\beta,\alpha\circ\beta\rangle - 2\langle\alpha\circ\beta,\beta\circ\alpha\rangle. \tag{4.5}$$

To prove it, all we have to do is to apply the formulas in (2.3), remembering that α and β are self-adjoint. The fact that $(\alpha\circ\beta)^* = \beta\circ\alpha$ implies that $\alpha\circ\beta$ and $\beta\circ\alpha$ have the same norm and we can hence apply Cauchy-Schwartz to conclude that $\mathrm{Riem}_\xi(\alpha,\beta) \geq 0$ and $\mathrm{Riem}_\xi(\alpha,\beta) = 0$ if and only if $\alpha\circ\beta = \beta\circ\alpha$.

One can also establish easily the following formula for the Ricci curvature:

$$\mathrm{Ricci}_\xi(\alpha,\beta) = \frac{N-2}{2}\langle\alpha,\beta\rangle, \tag{4.6}$$

where N is the dimension of E.

Grassman manifolds (or, more precisely, their connected components) are sometimes represented as homogeneous spaces of the orthogonal group. The following considerations will compare this approach with the one we are using.

Let E be a real Hilbert space and let $O(E) \subset L(E;E)$ be the orthogonal

FIGURE 10. Scanned page 94 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

group, i.e. the set of the toplinear isomorphisms $\xi:E \to E$ such that $\xi^* = \xi^{-1}$. It is well known that $O(E)$ is a manifold (a Lie group) and that, for each $\xi \in (E)$ and $\alpha \in L(E;E)$, we have

$$\alpha \in T_\xi(O(E)) \text{ if and only if } \alpha^* \circ \xi + \xi^* \circ \alpha = 0. \tag{4.7}$$

In the case where E is finite dimensional, the Riemann structure in $O(E)$ induced by the Hilbert-Schmidt inner product is readily seen to be bi-invariant. The orthogonal projections $\pi_\xi:L(E;E) \to T_\xi(O(E))$ are defined by

$$\pi_\xi(\lambda) = \frac{1}{2}(\lambda - \xi \circ \lambda^* \circ \xi). \tag{4.8}$$

Even in the case E is infinite dimensional, we define projection maps $\pi_\xi:L(E;E) \to T_\xi(O(E))$ by formula (4.8) and we have an associated symmetric connection in $O(E)$ defined by the bilinear maps $\theta_\xi:T_\xi(O(E)) \times T_\xi(O(E)) \to L(E;E)$,

$$\theta_\xi(\alpha,\beta) = D\pi_\xi(\beta)(\alpha) = -\frac{1}{2}(\beta\circ\alpha^*\circ\xi+\xi\circ\alpha^*\circ\beta). \tag{4.9}$$

Of course, in the finite dimensional case, this will be the metric connection.

Now assume that E is a finite or infinite dimensional real Hilbert space and $H \subset E$ is a fixed closed vector subspace. We can define a smooth map $\Phi:O(E) \to G(E)$ associating to each $\xi \in O(E)$ the orthogonal projection onto $\xi(H)$; denoting by $\pi:E \to H$ the orthogonal projection, it is easy to see that we have

$$\Phi(\xi) = \xi \circ \pi \circ \xi^*. \tag{4.10}$$

Although Φ is not a totally geodesic map, we can nevertheless state:

4.4. $\Phi:O(E) \to G(E)$ has totally geodesic fibres and, in case E is finite dimensional, is a Riemannian submersion.

<u>Proof</u> The derivative linear map $D\Phi_\xi:T_\xi(O(E)) \to T_{\Phi(\xi)}(G(E))$ is defined by

$$D\Phi_\xi(\alpha) = \alpha \circ \pi \circ \xi^* + \xi \circ \pi \circ \alpha^*.$$

Given $\alpha \in T_\xi(O(E))$ and $\beta \in T_{\Phi(\xi)}(G(E))$ arbitrary, we obtain, using (4.7), 3.7(c) and (2.3),

FIGURE 11. Scanned page 95 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

$$\langle D\Phi_\xi(\alpha),\beta\rangle = \langle\alpha\circ\pi\circ\xi^*,\beta\rangle + \langle\xi\circ\pi\circ\alpha^*,\beta\rangle$$

$$= \langle\alpha\circ\pi\circ\xi^*,\beta\rangle + \langle-\xi\circ\pi\circ\xi^*\circ\alpha\circ\xi^*,\beta\rangle$$

$$= \langle\alpha,\beta\circ\xi\circ\pi\rangle + \langle\alpha,-\xi\circ\pi\circ\xi^*\circ\beta\circ\xi\rangle$$

$$= \langle\alpha,\beta\circ\xi\circ\pi-\beta\circ\xi + \beta\circ\xi\circ\pi\rangle = \langle\alpha,2\beta\circ\xi\circ\pi-\beta\circ\xi\rangle,$$

where, using (4.7), we can see that $2\beta\circ\xi\circ\pi-\beta\circ\xi \in T_\xi(O(E))$. Hence, the adjoint linear map $D\Phi_\xi{}^*:T_{\Phi(\xi)}(G(E)) \to T_\xi(O(E))$ is defined by

$$D\Phi_\xi{}^*(\beta) = 2\beta\circ\xi\circ\pi - \beta\circ\xi.$$

It is not difficult to verify now that

$$D\Phi_\xi(D\Phi_\xi{}^*(\beta)) = \beta,$$

which means precisely that Φ is a Riemannian submersion.

Let $\xi \in O(E)$ and let $O_o(E)$ be the fibre of Φ over $\Phi(\xi)$. To prove that $O_o(E)$ is a totally geodesic submanifold of $O(E)$, all we have to see is that, for each α and β in $T_\xi(O_o(E))$, we have $(\beta,\theta_\xi(\alpha,\beta)) \in T_{(\xi,\alpha)}(T(O_o(E)))$, where θ_ξ is the connection of $O(E)$. Using the formula for $D\Phi_\xi$, we see that the fact that α and β are in $T_\xi(O_o(E))$ is equivalent to

$$\alpha\circ\pi\circ\xi^* + \xi\circ\pi\circ\alpha^* = 0,\ \beta\circ\pi\circ\xi^* + \xi\circ\pi\circ\beta^* = 0$$

and, using the same formula, one concludes easily that, if $(\beta,\lambda) \in T_{(\xi,\alpha)}(T(O(E)))$, then $(\beta,\lambda) \in T_{(\xi,\alpha)}(T(O_o(E)))$ if and only if

$$\alpha\circ\pi\circ\beta^* + \beta\circ\pi\circ\alpha^* + \lambda\circ\pi\circ\xi^* + \xi\circ\pi\circ\lambda^* = 0.$$

Now, using formula (4.9) for $\theta_\xi(\alpha,\beta)$ and the characterization of $T_\xi(O(E))$ given in 4.7, we obtain

$$\alpha\circ\pi\circ\beta^* + \beta\circ\pi\circ\alpha^* + \theta_\xi(\alpha,\beta)\circ\pi\circ\xi^* + \xi\circ\pi\circ\theta_\xi(\alpha,\beta)^*$$

$$= \alpha\circ\pi\circ\beta^* + \beta\circ\pi\circ\alpha^*$$

$$- \frac{1}{2}(\beta\circ\alpha^*\circ\xi+\xi\circ\alpha^*\circ\alpha^*\circ\beta)\circ\pi\circ\xi^* - \frac{1}{2}\xi\circ\pi\circ(\xi^*\circ\alpha\circ\beta^* + \beta^*\circ\alpha\circ\xi^*)$$

$$=\alpha\circ\pi\circ\beta^* + \beta\circ\pi\circ\alpha^* + \frac{1}{2}\beta\circ\xi^*\circ\alpha\circ\pi\circ\xi^* + \frac{1}{2}\alpha\circ\xi^*\circ\beta\circ\pi\circ\xi^* + \frac{1}{2}\xi\circ\pi\circ\alpha^*\circ\xi\circ\beta^*$$
$$+ \frac{1}{2}\xi\circ\pi\circ\beta^*\circ\xi\circ\alpha^* = \alpha\circ\pi\circ\beta^* + \beta\circ\pi\circ\alpha^* - \frac{1}{2}\beta\circ\pi\circ\alpha^* - \frac{1}{2}\alpha\circ\pi\circ\beta^* - \frac{1}{2}\alpha\circ\pi\circ\beta^* - \frac{1}{2}\beta\circ\pi\circ\alpha^* = 0$$

FIGURE 12. Scanned page 96 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

and the proof is complete.

We are going now to present a formula for the geodesics in $G(E)$ with arbitrary initial conditions. Let E be a real Hilbert space.

4.5. For each $\xi \in G(E)$ and $\eta \in T_\xi(G(E))$, there exists a smooth map $f:\mathbb{R} \to G(E)$ defined by

$$f(t) = \frac{1}{2}\,(\mathrm{Id} + (2\xi-\mathrm{Id})\circ\cos(2t\eta) + \sin(2t\eta))$$

and f is a geodesic of $G(E)$ that verifies $f(0) = \xi$ and $f'(0) = \eta$.

<u>Proof</u> We note first that, from 3.7(f), we conclude that $2\xi-\mathrm{Id}$ commutes with $\cos(2t\eta)$ and anti-commutes with $\sin(2t\eta)$. It is now trivial that $f(t)$ is self-adjoint and, noting that $(2\xi-\mathrm{Id})\circ(2\xi-\mathrm{Id}) = \mathrm{Id}$ and

$$\cos(2t\eta)\circ\cos(2t\eta) + \sin(2t\eta)\circ\sin(2t\eta) = \mathrm{Id},$$

we obtain

$$\begin{aligned} f(t)\circ f(t) = \frac{1}{4}\,(&\mathrm{Id} + (2\xi-\mathrm{Id})\circ\cos(2t\eta) + \sin(2t\eta) + (2\xi-\mathrm{Id})\circ\cos(2t\eta) \\ &+ (2\xi-\mathrm{Id})\circ\cos(2t\eta)\circ(2\xi-\mathrm{Id})\circ\cos(2t\eta) + (2\xi-\mathrm{Id})\circ\cos(2t\eta)\circ\sin(2t\eta) \\ &+ \sin(2t\eta) + \sin(2t\eta)\circ(2\xi-\mathrm{Id})\circ\cos(2t\eta) + \sin(2t\eta)\circ\sin(2t\eta)) \\ = \frac{1}{4}\,(&\mathrm{Id} + (2\xi-\mathrm{Id})\circ\cos(2t\eta) + \sin(2t\eta) + (2\xi-\mathrm{Id})\circ\cos(2t\eta) \\ &+ \cos(2t\eta)\circ\cos(2t\eta) + \sin(2t\eta) + \sin(2t\eta)\circ\sin(2t\eta)) = f(t), \end{aligned}$$

whence we conclude that $f(t) \in G(E)$. Next, we see that

$$\begin{aligned} f'(t) &= \frac{1}{2}(-2(2\xi-\mathrm{Id})\circ\sin(2t\eta)\circ\eta + 2\cos(2t\eta)\circ\eta) \\ &= (\cos(2t\eta) - (2\xi-\mathrm{Id})\cdot\sin(2t\eta))\circ\eta, \end{aligned}$$

in particular $f'(0) = \eta$. Next, we obtain

$$f''(t) = (-2\sin(2t\eta) - 2(2\xi-\mathrm{Id})\cos(2t\eta))\circ\eta^2.$$

On the other side, remembering 3.7(f), we have

FIGURE 13. Scanned page 97 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

$$f'(t)\circ f'(t) = \cos(2t\eta)^2\circ\eta^2 - \cos(2t\eta)\circ\eta\circ(2\xi-\mathrm{Id})\circ\sin(2t\eta)\circ\eta$$

$$- (2\xi-\mathrm{Id})\circ\sin(2t\eta)\circ\eta\circ\cos(2t\eta)\circ\eta$$

$$+ (2\xi-\mathrm{Id})\circ\sin(2t\eta)\circ\eta\circ(2\xi-\mathrm{Id})\circ\sin(2t\eta)\circ\eta$$

$$= \cos(2t\eta)^2\circ\eta^2 + (2\xi-\mathrm{Id})\circ\cos(2t\eta)\circ\sin(2t\eta)\circ\eta^2$$

$$- (2\xi-\mathrm{Id})\circ\sin(2t\eta)\circ\cos(2t\eta)\circ\eta^2$$

$$+ (2\xi-\mathrm{Id})\circ(2\xi-\mathrm{Id})\circ\sin(2t\eta)^2\circ\eta^2$$

$$= (\cos(2t\eta)^2 + \sin(2t\eta)^2)\circ\eta^2 = \eta^2.$$

and, using (4.2), we have now

$$\theta_{f(t)}(f'(t),f'(t))$$

$$= (\mathrm{Id}-2f(t))\circ f'(t)\circ f'(t) + f'(t)\circ f'(t)\circ(\mathrm{Id}-2f(t))$$

$$= (-(2\xi-\mathrm{Id})\circ\cos(2t\eta) - \sin(2t\eta))\circ\eta^2$$

$$+ \eta^2\circ(-(2\xi-\mathrm{Id})\circ\cos(2t\eta) - \sin(2t\eta)) = f''(t),$$

whence we conclude that f is indeed a geodesic.

4.6. Let E be a real Hilbert space. $G(E)$ is then a symmetric space and, for each $\pi\in G(E)$, the symmetry $\mathrm{Sym}:G(E)\to G(E)$ with respect to π is defined by

$$\mathrm{Sym}(\xi) = (\mathrm{Id}-2\pi)\circ\xi\circ(\mathrm{Id}-2\pi).$$

<u>Proof</u> It is trivial that $\mathrm{Sym}(\xi)$ is a self-adjoint map and the fact that $(\mathrm{Id}-2\pi)\circ(\mathrm{Id}-2\pi) = \mathrm{Id}$ shows that $\mathrm{Sym}(\xi)\circ\mathrm{Sym}(\xi) = \mathrm{Sym}(\xi)$, hence $\mathrm{Sym}(\xi)\in G(E)$. It is trivial to see that $\mathrm{Sym}(\pi) = \pi$ and that $\mathrm{Sym}(\mathrm{Sym}(\xi)) = \xi$. We have

$$D\,\mathrm{Sym}_\xi(\alpha) = (\mathrm{Id}-2\pi)\circ\alpha\circ(\mathrm{Id}-2\pi),$$

FIGURE 14. Scanned page 98 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

hence, remembering (4.2),

$$\nabla D\ \mathrm{Sym}_\xi(\alpha,\beta) = (\mathrm{Id}-2\pi)\circ\theta_\xi(\beta,\alpha)\circ(\mathrm{Id}-2\pi)-\theta_{\mathrm{Sym}(\xi)}(D\mathrm{Sym}_\xi(\beta),D\mathrm{Sym}_\xi(\alpha))$$

$$= (\mathrm{Id}-2\pi)\circ(\mathrm{Id}-2\xi)\circ\beta\circ\alpha\circ(\mathrm{Id}-2\pi)+(\mathrm{Id}-2\pi)\circ\alpha\circ\beta\circ(\mathrm{Id}-2\xi)\circ(\mathrm{Id}-2\pi)$$

$$- (\mathrm{Id}-2(\mathrm{Id}-2\pi)\circ\xi\circ(\mathrm{Id}-2\pi))\circ(\mathrm{Id}-2\pi)\circ\beta\circ\ (\mathrm{Id}-2\pi)\circ(\mathrm{Id}-2\pi)\ \circ\alpha\circ(\mathrm{Id}-2\pi)$$

$$- (\mathrm{Id}-2\pi)\circ\alpha\circ(\mathrm{Id}-2\pi)\circ(\mathrm{Id}-2\pi)\circ\beta\circ(\mathrm{Id}-2\pi)\circ(\mathrm{Id}-2(\mathrm{Id}-2\pi)\circ\xi\circ(\mathrm{Id}-2\pi))$$

$$= (\mathrm{Id}-2\pi)\circ\beta\circ\alpha\circ(\mathrm{Id}-2\pi) - 2(\mathrm{Id}-2\pi)\circ\xi\circ\beta\circ\alpha\circ(\mathrm{Id}-2\pi)$$

$$+ (\mathrm{Id}-2\pi)\circ\alpha\circ\beta\circ(\mathrm{Id}-2\pi) - 2(\mathrm{Id}-2\pi)\circ\alpha\circ\beta\circ\xi\circ(\mathrm{Id}-2\pi)$$

$$- (\mathrm{Id}-2\pi)\circ\beta\circ\alpha\circ(\mathrm{Id}-2\pi) + 2(\mathrm{Id}-2\pi)\circ\xi\circ\beta\circ\alpha\circ(\mathrm{Id}-2\pi)$$

$$- (\mathrm{Id}-2\pi)\circ\alpha\circ\beta\circ(\mathrm{Id}-2\pi) + 2(\mathrm{Id}-2\pi)\circ\alpha\circ\beta\circ\xi\circ(\mathrm{Id}-2\pi) = 0,$$

that is to say, Sym is a totally geodesic diffeomorphism. Now, if $f:\mathbb{R}\to G(E)$ is a geodesic with $f(0)=\pi$ and $f'(0)=\eta$, we have

$$f(t) = \frac{1}{2}\ (\mathrm{Id} + (2\pi-\mathrm{Id})\circ\cos(2t\eta) + \sin(2t\eta)),$$

hence

$$\mathrm{Sym}(f(t)) = \frac{1}{2}\ (\mathrm{Id}-2\pi)\circ(\mathrm{Id}+(2\pi-\mathrm{Id})\circ\cos(2t\eta) + \sin(2t\eta))\circ(\mathrm{Id}-2\pi)$$

$$= \frac{1}{2}\ (\mathrm{Id}+(2\pi-\mathrm{Id})\circ\cos(2t\eta) - \sin(2t\eta)) = f(-t)$$

and the proof is complete.

5. THE COMPLEX GRASSMAN MANIFOLDS

Assume that E is a complex Hilbert space, whose inner product will always be denoted by $\langle\ ,\ \rangle_{\mathbb{C}}$. Then E is also a real Hilbert space, with the inner product

$$\langle x,y\rangle = \mathrm{Re}\ \langle x,y\rangle_{\mathbb{C}} \tag{5.1}$$

and the following two facts are trivial:

FIGURE 15. Scanned page 99 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

5.1 If $F \subset E$ is a complex vector subspace, then the orthogonal projection $\pi : E \to F$ is the same when we consider in E either the complex or the real inner product.

5.2. If $\xi : E \to E$ is a complex linear map, then the adjoint map $\xi^* : E \to E$ is the same when we consider E to be either a complex or a real Hilbert space.

We will denote by $L(E;E)$ the vector space of all continuous *real* linear maps and by $L_{\mathbb{C}}(E;E)$ its vector subspace whose elements are the complex linear maps. In the case where E is finite dimensional the Hilbert-Schmidt inner product that we will consider in $L(E;E)$ will be the one associated to the real structure of E and we will consider in the closed subspace $L_{\mathbb{C}}(E;E)$ the induced inner product.

If E is a complex Hilbert space, we will denote by $G_{\mathbb{C}}(E)$ the set of the orthonormal projections onto closed complex vector subspaces, and we call $G_{\mathbb{C}}(E)$ the *complex Grassman manifold* of E. $G(E)$ will denote the real Grassman manifold of E, i.e. the Grassman manifold of E, when considered as a real Hilbert space. It is trivial to conclude that

$$G_{\mathbb{C}}(E) = G(E) \cap L_{\mathbb{C}}(E;E). \tag{5.2}$$

All that has been said in Section 3 applies *mutatis mutandis* to the complex Grassman manifolds, but one must be aware that $G_{\mathbb{C}}(E)$ is only a *real* manifold *within* the complex vector space $L_{\mathbb{C}}(E;E)$. The essential reason for this is the fact that the map $\xi \to \xi^*$ is not $\mathbb{C}$-linear, but it was natural to anticipate this because, in case E is finite dimensional, $G_{\mathbb{C}}(E)$ (like $G(E)$) is compact (because it is closed and bounded) and it is well known that there exists no compact nontrivial complex submanifold of a complex vector space.

For each closed complex vector subsapce $F \subset E$, we still have a diffeomorphism $\psi_F : \mathcal{U}_F \to L_{\mathbb{C}}(F;F^{\perp})$, where $\mathcal{U}_F$ is open in $G_{\mathbb{C}}(E)$ and contains π_F (cf. 3.4), hence:

5.3. If E has complex dimension N and $F \subset E$ has complex dimension n, then the real manifold $G_{\mathbb{C}}(E)$ has dimension $2n(N-n)$ in π_F.

The tangent vector space $T_{\pi_F}(G_{\mathbb{C}}(E))$ is contained in the real vector space $L_{\mathbb{C}sa}(E;E)$, whose elements are the self-adjoint complex linear maps and, for each $\eta \in L_{\mathbb{C}sa}(E;E)$, the fact that $\eta \in T_{\pi_F}(G_{\mathbb{C}}(E))$ is equivalent to each of the conditions (b) to (f) of 3.7; in other words:

FIGURE 16. Scanned page 100 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

$$T_{\pi_F}(G_{\mathbb{C}}(E)) = T_{\pi_F}(G(E)) \cap L_{\mathbb{C}}(E;E). \tag{5.3}$$

Although $G_{\mathbb{C}}(E)$ is only a real submanifold of $L_{\mathbb{C}}(E;E)$, it admits a complex structure:

5.4. Let E be a complex Hilbert space. Then the real manifold $G_{\mathbb{C}}(E)$ admits a complex structure defined by the linear maps

$$J_\xi : T_\xi(G_{\mathbb{C}}(E)) \to T_\xi(G_{\mathbb{C}}(E)), \quad J_\xi(\eta) = i\eta \circ (2\xi - \mathrm{Id}).$$

For this structure the real diffeomorphisms $\psi_F : \mathcal{U}_F \to L_{\mathbb{C}}(F;F^\perp)$ are in fact holomorphic.

<u>Proof</u> To see that J_ξ applies $T_\xi(G_{\mathbb{C}}(E))$ into itself we use 3.7(f), remembering that $(2\xi-\mathrm{Id})\circ(2\xi-\mathrm{Id}) = \mathrm{Id}$ and noting that

$$(i\eta\circ(2\xi-\mathrm{Id}))^* = -i(\eta\circ(2\xi-\mathrm{Id}))^* = -i(2\xi-\mathrm{Id})\circ\eta = i\eta\circ(2\xi-\mathrm{Id}).$$

It is also trivial that $J_\xi(J_\xi(\eta)) = -\eta$. The fact that this almost complex structure is indeed a complex one comes from the fact that the real diffeomorphisms $\psi_F : \mathcal{U}_F \to L(F;F^\perp)$ are holomorphic; this is a simple consequence of the formula in 3.6,

$$D\psi_F(\xi)(\eta) = \eta\circ(\pi_{F/\xi(E)})^{-1} - (\pi_{F/\xi(E)})^{-1}\circ\pi_F\circ\eta\circ(\pi_{F/\xi(E)})^{-1},$$

the formula $D\psi_F(\xi)(J_\xi(\eta)) = i\, D\psi_F(\xi)(\eta)$ being a simple consequence of the fact that the restriction of $(2\xi-\mathrm{Id})$ to $\xi(E)$ is the identity.

Note that, in case E is finite dimensional, if we choose a complex orthonormal basis $x_1,\dots,x_N$ of E such that $x_1,\dots,x_n$ is a basis of $\xi(E)$, then, if $\eta \in T_\xi(G_{\mathbb{C}}(E))$ has matrix $\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$, $J_\xi(\eta)$ has matrix $\begin{bmatrix} 0 & -iA^* \\ iA & 0 \end{bmatrix}$.

The considerations in 4.1-4.3 and (4.1)-(4.5) apply *mutatis mutandi* to the complex Grassman manifolds and we have in particular a *canonical symmetric connection* in $G_{\mathbb{C}}(E)$ defined also by

$$\theta_\xi(\eta,\alpha) = (\mathrm{Id}-2\xi)\circ\eta\circ\alpha + \alpha\circ\eta\circ(\mathrm{Id}-2\xi). \tag{5.4}$$

This implies in particular that:

5.5. $G_{\mathbb{C}}(E)$ is a totally geodesic submanifold of $G(E)$. We see now that:

5.6. If E is a complex Hilbert space, then the morphism $J = (J_\xi)$, from the vector bundle $T(G_{\mathbb{C}}(E))$ into itself, is parallel.

Proof From $J_\xi(\beta) = i\,\beta \circ (2\xi - Id)$, we obtain

$$\nabla J_\xi(\alpha)(\beta) = 2i\beta\circ\alpha + i\theta_\xi(\beta,\alpha)\circ(2\xi-Id) - \theta_\xi(i\,\beta\circ(2\xi-Id),\alpha)$$

$$= 2i\beta\circ\alpha + i(Id-2\xi)\circ\beta\circ\alpha(2\xi-Id) + i\alpha\circ\beta\circ(Id-2\xi)\circ(2\xi-Id)$$

$$- i(Id-2\xi)\circ\beta\circ(2\xi-Id)\circ\alpha - i\,\alpha\circ\beta\circ(2\xi-Id)\circ(Id-2\xi)$$

$$= 2i\,\beta\circ\alpha - i\beta\circ\alpha - i\,\alpha\circ\beta - i\,\beta\circ\alpha + i\alpha\circ\beta = 0.$$

In the case where the complex Hilbert space E is finite dimensional, we note that the Hilbert-Schmidt inner product in $L(E;E)$ is the real part of a complex inner product (the one defined by (2.2) with $\langle\ ,\ \rangle_{\mathbb{C}}$ instead of $\langle\ ,\ \rangle$) and we see that

$$\langle J_\xi(\alpha), J_\xi(\beta)\rangle = \langle i\alpha\circ(2\xi-Id),\ i\beta\circ(2\xi-Id)\rangle$$

$$= \langle\alpha\circ(2\xi-Id),\ \beta\circ(2\xi-Id)\rangle = \langle\alpha\circ(2\xi-Id)\circ(2\xi-Id),\beta\rangle$$

$$= \langle\alpha,\beta\rangle,$$

hence:

5.7. If E is a finite dimensional complex Hilbert space, then $G_{\mathbb{C}}(E)$ is a Kähler manifold.

We end with the remark that (4.7)-(4.10) and 4.4 work equally well in the complex case, the usual notation for $O_{\mathbb{C}}(E)$ being $U(E)$ (the unitary group).

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FIGURE 18. Scanned page 102 of the original paper published in *Res. Notes Math.* **131** (1985), 85–102

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