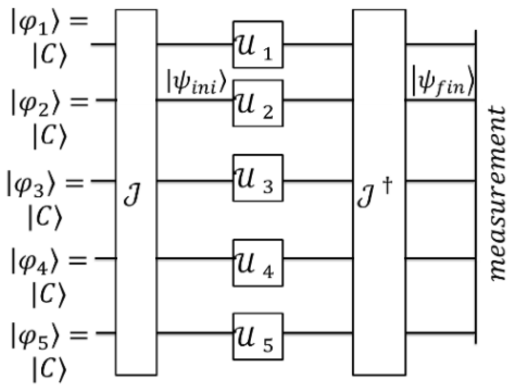

Quantum Pirates

A Quantum Game-Theory Approach to The Pirate Game

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I can live with doubt and uncertainty and not knowing. I think it is much more interesting to live not knowing than to have answers that might be wrong. If we will only allow that, as we progress, we remain unsure, we will leave opportunities for alternatives. We will not become enthusiastic for the fact, the knowledge, the absolute truth of the day, but remain always uncertain... In order to make progress, one must leave the door to the unknown ajar.

Richard P. Feynman

Acknowledgments

I heard the term “Quantum Computing” for the first time at a presentation in the Department of Physics of the University of Coimbra, and the idea of using particles to simulate physical systems and outperform some classical algorithms fascinated me ever since.

This work is a story of a path started naively our of curiosity for the world around me, a path filled hardships but also ripe with good moments. This journey was marked by self-doubt, but it was also rich in moments where wonderful travel companions walked alongside giving me strength to carry on. I want to thank all these travel companions.

Namely I would like to thank my advisor, Professor Andreas Wichert, for giving me the freedom to fail and learn from my mistakes, while teaching me how to make Science.

I would also like to thank my co-workers at Connect Coimbra for teaching me how to balance work and life, and the desire to make a difference in my community.

Last but definitely not the least I give thanks to my loved ones who never ceased supporting me.

Abstract

In this document, we develop a model and a simulation of a quantization scheme for the mathematical puzzle created by Omohundro and Stewart - “A puzzle for pirates ”, also known as Pirate Game. This game is a multi-player version of the game “Ultimatum ”, where the players (Pirates), must distribute fixed number of gold coins acording to some rules.

The Quantum Theory of Games is a field that seeks to introduce the mathematical formalism of Quantum Mechanics in order to explore models of conflict that arise when rational beings make decisions. These models of conflict are pervasive in the structural make-up of our society. The combination of game theory and Quantum Probability, despite not having a practical application, can help in the development of new quantum algorithms. Furthermore the fact that Game Theory is transversal to many areas of knowledge can provide insights to future application of these models.

In this dissertation we focused on the role of quantum entanglement and the use of quantum strategies in the game system. We found that when there is no entanglement the game behaved as the original problem even when the players adopted quantum strategies. When using a unrestricted strategic space and the game system is maximally entangled we found that the game is strictly determined (like the original problem). We also found that when only a the captain has access to quantum strategies in the Pirate Game, she can obtain all the gold coins. These results corroborate similar findings in the field.

Keywords

Quantum Game Theory; Pirate Game; Quantum Mechanics; Quantum Computing; Game Theory; Probability Theory

Resumo

Neste trabalho desenvolvemos e simulámos um modelo quâtico para o puzzle matemático criado por Omohundro e Stewart, “Um puzzle para piratas”(original em inglês “A Puzzle for Pirates”). Este jogo consiste numa versão multi-jogador do jogo “Ultimato”, no qual os jogadores (Piratas), distribuem um número limitado de moeadas de ouro.

A Teoria de Jogos Quântica é uma área que procura introduzir o formalismo matemático na base da Mecânica Quântica para explorar modelos de conflito que surgem quando seres racionais tomam decisões. Estes modelos de conflito estão na base da estrutura da nossa sociedade. A combinação de Teoria de Jogos e a Teoria de Probabilidade Quântica apesar de ainda não ter uma aplicação prática pode ajudar no desenvolvimento de novos algoritmos quânticos.

Nesta dissertação focámo-nos sobretudo no papel do fenómeno quântico entrelaçamento e existência de estratégias quânticas no sistema do jogo. Verificámos que quando não existe entrelaçamento o jogo se comporta como um jogo clássico, mesmo quando os jogadores utilizam estratégias quânticas. Quando utilizamos um espaço estratégico não restrito e o sistema está maximamente entrelaçado descobrimos que o jogo é estrictamente determinado (como no problema original). Também se verificou que quando apenas o capitão tem acesso a estratégias quânticas no jogo, este consegue obter todas as moedas. Estes resultados corroboram resultados similares na literatura.

Palavras Chave

Teoria de Jogos Quântica; Jogo dos Piratas; Mecânica Quântica; Teoria de Jogos; Computação Quântica; Teoria de Probabilidade

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Standard Notation

$\mathbf{i}$	is the imaginary constant $\sqrt{-1}$.
e	is natural constant $2.71828...$
$\|x\|$	represents the absolute value of x
$\mathbb{R}$	represents the real numbers.
$\mathbb{C}$	represents the complex numbers.
$P(A)$	represents the probability of an event A occurring.

Sets and Spaces

$\mathcal{H}^n$	represents a n-dimension Hilbert Space.
$\mathcal{B}$	$= \{\|b_i\rangle, i = 0, n-1\}$ is a set of orthonormal basis.

Vectors

$\|\psi\rangle$	is a quantum state labeled ψ represented by an abstract vector, can also be represented by a $n \times 1$ matrix relative to a basis $\mathcal{B}$.
$\otimes$	represents the tensor product between two vectors.
$\|0\rangle, \|1\rangle$	are the standard basis for a two-state quantum system (qubit).

Matrices

$exp(A)$	, where A is a square matrix, is a matrix exponential function, equivalent to e^A.
$\|A\|$	determinant of the matrix A.
I	represents a 2×2 identity matrix.
$\sigma_x, \sigma_y, \sigma_z$	are 2×2 matrices that form a set known as Pauli Operators.
C, D	represent Cooperate and Defect strategies in a game, in a quantum game theory environment they are 2×2 matrices.

ρ	represents a density matrix.
$\mathcal{U}$	is a unitary matrix.
$\mathcal{U}^{-1}$	is the inverse of the unitary matrix $\mathcal{U}$.
$\mathcal{U}^{*}$	is the conjugate of $\mathcal{U}$.
$\mathcal{U}^{T}$	is the transpose of $\mathcal{U}$.
$\mathcal{U}^{\dagger}$	is the conjugate transpose of $\mathcal{U}$.
$\otimes$	represents the Kronecker product between two matrices.

Game Theory

Γ	represents a game.
N	in a game denotes the number of players.
$u_i(A)$	is the expected utility for player i when the outcome A occurs.
E_i	is utility functional that specifies the expected utility in a quantum game.

1

Introduction

Contents

In this section we lay the motivation for this work, its relevance, our objectives, and the general outline of this document.

1.1 Motivation

It is not always easy to understand and introduce new paradigms. Some things we find natural nowadays, were once the source of controversy. Taking for example the number zero 0. As numbers were introduced to help count physical objects, the idea of representing “nothingness” was once considered strange. In the beginning there were numerous ways devised to deal with this mathematical inconvenience, a special case. However the need to use “zero” as a number in its own right lead to the popularization of its concept as a number and opened door to new breakthroughs in mathematics [1].

Quantum mechanics shared this same problem and its mathematical formalization grew out of the need to explain phenomena at the atomic scale [2]. Nowadays we accepted quantum mechanics as a tool, to analyse and predict behaviour at a microscopic level. However at our scale some quantum mechanics phenomena seem almost “ridicule” and paradoxical.

The Quantum Computing started attracting interest in the decade of 1980, with the works of Yuri Manin and Richard Feynman. The objective was to create a computational machine that could use the entanglement and superposition of the wave function to perform calculations that are currently impossible with classic computers. A popular example of that kind of computation is the prime factorization, which is in the base for modern cryptography. However we must bear in mind that our current computers are in fact quantic at the microscopic level. Silicon, a semiconductor, in the base of modern chips, and its properties arise from quantum mechanics (it is neither a pure conductor, not a isolator material). Silicon is used to construct transistors that are based on the concept of accurate measuring; in this process the heat production is unavoidable. It is the heat produced by the microchips that make them hard to integrate and scale, and despite the Moore’s Law that predicts that the number of transistors double every year (in the industry), there are physical limits that cannot be surpassed. When the transistors become too small they will start behaving in a quantum way, introducing errors in our deterministic computations [3].

The idea of trying to develop in quantum computing arises from the desire to explore this paradigm. The standard curriculum of an undergraduate computer scientist is focused on the current computational paradigm, which has its roots in von Neumman Architecture [4]. While the construction and the inner-workings of computational systems that rely on new paradigms may fall outside of the scope of computer science, understanding it from the point of view of Information Technology and pushing boundaries on model representation are areas where a computer scientist might contribute.

1.2 Problem Description

A model provides a way to abstract the reality, which is often too complex to analyse in its breadth. Game Theory is an area that tried to find ways to represent and analyse situations of conflict generated when intelligent rational beings make decisions. This discipline has major applications in Economics, Biology, Political Science, and Artificial Intelligence. Combining both the mathematical foundations of Quantum Mechanics, and the Game Theory is an idea that is starting to attract attention. These two areas share the same founding father, von Neumman, and creating a Quantum Model for a game seems to be a relevant way to explore the theory behind Quantum Mechanics while having a controlled and creative way to apply it. It is also interesting to analyse how these quantum games differ from their well defined classical counterparts. Furthermore simulating quantum algorithms on a classical computer is usually impractical because this systems grow exponentially with the number of qubits. In game theory we have relevant problems that can be modelled using few qubits, thus making it possible to simulate in a classical computer.

The Pirate Game is a mathematical, Game Theory, problem; with this work want to explore it in the light of quantum game theory. This means developing a quantization scheme, which is a way to transform the original game into a quantum simulation. Our simulations were implemented on Matlab.

Despite not having a clear "real-world" application yet, modelling games with quantum mechanics rules may aid the development of new algorithms that would be ideally deployed using quantum computers.

Furthermore applying quantum probabilities to a well established area as Game Theory might introduce new insights and even relevant practical applications [5].

1.3 Objectives

The main objectives with this work are to learn how to quantize a classical Game Theory problem. We also aim to investigate and compile a set of relevant works on Quantum Models. Moreover we seek to compare the Quantum and the Classical versions of a game. Namely to observe how quantum phenomena, such entanglement, and the usage of quantum strategic space may alter the Nash equilibria in a quantum game.

1.4 Contributions

This dissertation contributes to the area of Quantum Game Theory and Quantum Computing. The main contributions expected from this work are:

- To provide a state of the art and related work research with examples and executable simulations that could be used to facilitate the entry in this field;
- The design and simulation of an unique Quantum Model, and a comparative study with the original version. The Pirate Game is an example of a mathematical game that has not been modelled in a quantum domain yet;
- To study the the possible outcomes, given the initial set-up in a quantum game. Specifically to study the role of entanglement, and the quantum strategies.

1.5 Thesis Outline

This document is organized in six main chapters: Introduction, Background, Related Work, Pirate Game, Analysis and Results, and Conclusion.

In the introductory chapter we lay out our problem, its relevance, the mains objectives of this work.

The second chapter, “Background”, will pose an overview of theoretical concepts, and definitions needed to understand this work. We will start by making a comparative study between the Classical Probability Theory (Bayesian Probability), and the Quantum Probability (also known as von Neumann Probability). Then we will lay the fundamental concepts needed to work and understand quantum computing, namely the Superposition and Entanglement phenomena, the role of Operators, how to scale quantum systems. Finally we will present a Game Theory background, and some Quantum Game Theory concepts.

After laying the theoretical construct which is the foundation of this work we will analyse some “Related Work”. In this chapter we select, analyse, and experiment with some Quantum Computing and Quantum Game Theory Models. We tried to privilege a practical approach which might help an unfamiliar reader consolidate some theoretical concepts from the previous chapter.

In the fourth chapter we present the original Pirate Game and our quantum version, emphasizing our chosen definition of quantum game.

The fifth chapter will be reserved for analysing and discussing results obtained from our implementation of the Quantum Pirate Game.

Finally we will wrap up this work by reflecting on our principle results, the relevance of this model. We will also try to set new starting points for future work.

2

Background

Contents

In this section we present the theorectical concepts needed in order to understand this work.

2.1 Bayesian Probability

The Probability theory has its roots in the 16^{th} century with attempts to analyse games of chance by Cardano. It is not hard to understand why games of chance are in the foundations of the probability theory, as throughout History the concept of probability has fascinated the Human being. Luck, fate were words that reflect things we feel we have no control upon and are associated with those games, and almost paradoxically we evolved in a way that we do not feel comfortable around them.

2.1.1 Kolmogorov axioms

In spite of the fact that the problem of games of chance kept attracting numerous mathematicians (with some of the most influential ones being Fermat, Pascal and Laplace), it was not until the 20^{th} century that the Russian mathematician Kolmogorov laid the foundations of the modern probability theory (first published in 1933) introducing three axioms [6]:

1. The probability of an event is a non-negative real number:

$$P(A) \in \mathbb{R} \ \wedge \ P(A) \geq 0 \tag{2.1}$$

   This number represents the likelihood of that event happening, the greater the probability the more certain is its associated outcome.

2. The sum of probabilities of all possible outcomes in a space is always 1 ($P(\Omega) = 1$). These first two axioms leave us with the corollary that probabilities are bounded:

$$0 \leq P(A) \leq 1 \tag{2.2}$$

3. The probability of a sequence of pairwise disjoint events is the sum of the probabilitis of each of those events. A corollary of this axiom is:

$$P(A_1 \vee A_2 \vee ... \vee A_n) = P(A_1) + P(A_2) + ... + P(A_n) \tag{2.3}$$

Another important corollary derived from the axioms is the addition law of probability:

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B) \tag{2.4}$$

2.1.2 Conditional Probability

When some evidence is presented we have what we call conditional probability (or posterior probability). Conditional probability is represented as $P(A|B)$, that could be read as: the probability of A after evidence B is presented.

The product rule is used to calculate posterior probability:

$$P(A|B) = \frac{P(A \wedge B)}{P(B)} \tag{2.5}$$

2.1.3 Joint Distribution

When we need to deal with more than a one Boolean variable, the joint distribution is used to define events in terms of those variables. This distribution grows exponentially (for n variables, there are 2^n combinations), with the probability space of the various variables considered. If we consider the joint probability distribution between all the variables in a given domain we call this full joint probability.

Assuming that have three random variables X, Y and W representing medical inferences. X represents if a person has fever, x being the answer 1 (true) or 0 (false). Y represents if a person has been in a tropical region. W stands for if a person has headaches. Their joint distribution can be considered a vector with length 8 accounting for the multiple combinations of X, Y, and W. For example $P(101)$ represents the probability of having fever and seizures without having been in a tropical region. The notation $P(xyw)$ is equivalent to $P(x, y, w)$, and to $P(X = x \vee Y = y \vee W = w)$

The joint distribution can be calculated using conditional probabilities by the chain rule [7]. Given a set of variables $x_1, x_2, ..., x_n$:

$$P(x_1, ..., x_n) = P(x_1) \prod_{i=2}^{n} P(x_i|x_{i-1}, ..., x_1) \tag{2.6}$$

For the last example a chain rule to calculate the joint probability would be:

$$P(x, y, w) = P(x|y, w).P(y|w).P(w) \tag{2.7}$$

To calculate P(x) from a joint distribution we need the marginal distribution $P_X : x \rightarrow \sum_{w \in W} \sum_{y \in Y} P(x, y, w)$. This would require 2^{n-1} sums.

2.1.4 Conditional Independence

When variables are independent they are uncorrelated and their marginal distributions are equal to their prior probability distribution [8]. For example:

$$P_X(x) = \sum_{w \in W} \sum_{y \in Y} P(x, y, w) = P(x) \quad (2.8)$$

So, two variables are independent iif [9] [7]:

$$P(X|W) = P(X) \quad (2.9)$$

That means if we are presented with three variables X, Y and W that are conditional independent relatively to each other, their joint distribution would only account their probability prior probability functions, thus simplifying the chain rule (Equation (2.7)):

$$P(x, y, w) = P(x).P(y).P(w) \quad (2.10)$$

Although it isn't always possible to decouple variables and assume that they are conditionally independent, doing so is a way to counter the "curse of dimensionality".

2.1.5 Markov Chains

Markov Chains define a system in terms of states and the probabilistic transitions from one state to another.

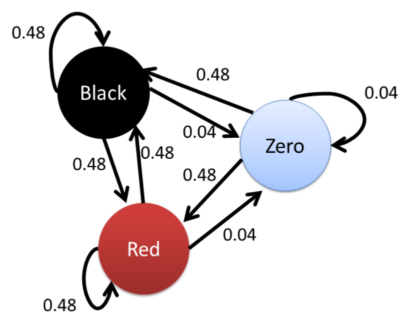

Figure 2.1: Markov Chain of a perspective on a roulette.

In 1913, at Monte Carlo Casino (Monaco), black came up twenty-six times in succession in roulette. Knowing that the probability of the ball landing on a red or on a black house is approximately 0.48 (the zero is a neutral house), many a gambler lost enormous sums while betting red as the they believed that the roulette was ripe with red, given the history. Players didn't want to believe that this and insisted that the roulette was biased; this became known as the Gamblers fallacy.

In the Markov Chain represented on Figure 2.1, we can see that in the state black the probability of transitioning to red is the same as to stay on the state black.

Given a system represented by the states $\{x_0, x_1, ..., x_i, ..., x_n\}$, and considering p_{ij} the probability of being in the state j and transitioning to the state i, the mixed state vector 2.11, which represents the probabilities of the system in the that i to transition to the other states. .

$$\overrightarrow{x_i} = \{p_{0i}x_0, p_{1i}x_1, ..., p_{ii}x_i, ..., p_{ni}x_n\} \tag{2.11}$$

The law of total probability is verified as $\sum_{j=0}^{n} p_{ji} = 1$, by specifying the every transition we get a stochastic matrix P, named the Markov matrix.

To illustrate how to construct a Markov Chain we will pick up on the example of Figure 2.1.

In this simplification of the Roulette we have 3 states:

- Black (B);
- Red (R);
- Zero (0).

We indifferently assign an index to each state, in order to construct the mixed state vector as in 2.11. Having the mixed state vectors defined the next step is to use them to create the stochastic matrix that has specified every transition 2.12.

$$R = \begin{bmatrix} p_{BB} & p_{RB} & p_{0B} \\ p_{BR} & p_{RR} & p_{0R} \\ p_{B0} & p_{R0} & p_{00} \end{bmatrix} = \begin{bmatrix} 0.48 & 0.48 & 0.48 \\ 0.48 & 0.48 & 0.48 \\ 0.04 & 0.04 & 0.04 \end{bmatrix} \tag{2.12}$$

2.2 Von Neumann Probability

In the beginning on the 20^{th} century the nature of light was on the spotlight of scientific investigation. The question whether light would be a particle (corpuscular theory), or a wave (undulatory theory), had been posed throughout History. Newton, notoriously, considered light to be a particle and presented arguments such as the fact that light travels in a straight like, not bending when presented with obstacles, unlike waves, and gave an interpretation of the diffraction mechanism by resorting to a special medium (aether), where the light corpuscles could create a localized wave [10].

The idea of light as a particle stood up until the 18^{th}century as many scientists (Robert Hooke, Christian Huygens and Leonard Euler to name a few) tried to explain contradictions found in corpuscular theory. This brought back the idea that light behaves like a wave.

One of the most famous experiments that corroborates the undulatory theory is the Young's experiments (19^{th} century), or the double-slit interferometer.

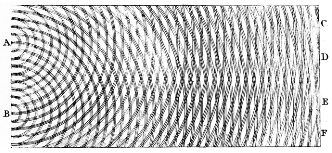

Figure 2.2: Thomas Young's sketch of two-slit diffraction of light.

The apparatus for the double-slit experiment can be seen in Figure 2.2. A light source is placed in such a way that "two portions" of light arrive at same time at the slits. Behind the barrier is a "wall" placed to intercept the light. The light captured at a wall will sport an interference pattern similar to the pattern when two waves interfere. The double-slit experiment was considered for a while the full stop on the discussion on the nature of light. However with experiments on the spectres of the light emitted by diverse substances and its relation with temperature, a new problem was posed.

The black body radiation problem was the theoretical problem where a body that absorbs light in all the electromagnetic spectrum, this makes the body acting as an natural vibrator, where each mode would have the same energy, according to the classic theory.

When a black body is at a determined temperature the frequency of the radiation it emits depends on the temperature. The classic theory predicted that most of the energy of the body would be in the high frequency part of the spectrum (violet part) where most modes would be found, this led to a prediction called the ultraviolet catastrophe. According to the classic theory the black body would emit radiation with an infinite power for temperatures above approximately $5000K$. Max Plank(1901), provided an explanation where the light was exchanged in discrete amounts called quanta, so that each frequency

[0]Source: Young, Thomas: Probability. http://en.wikipedia.org/wiki/File:Young_Diffraction.png(1803)

would only have specific levels of energy. Plank also determined through experimentation the value of the energy of the quanta that became known as photons later, that value became the physical constant called Plank constant:

$$h = 6.62606957(29) \times 10^{-34} J.s \tag{2.13}$$

In 1905, Einstein used the concept of quanta (photons) to explain the photoelectric effect. De Broglie(1924), suggested that all matter has a wave behaviour. This prediction was confirmed by studying the interference patterns caused by electron diffraction.

In Quantum Mechanics, a particle is deemed as a system that is described by a wave-function (quantum system). The wave function is usually represented by the Greek letter ψ. The main purpose of Quantum Mechanics is to give predictions for the outcomes derived from measurements on quantum systems [11].

2.2.1 Mathematical Foundations of Quantum Probability

As previously explained, Quantum Theory is a branch of physics that has evolved from the need to explain certain phenomena that could not be explained with the current classical theory. In the beginning of the 20$^{\text{th}}$ century Dirac and von Neumann helped to create the mathematical formalisms for this theory [12] [13].

Von Neumann's contributions revolved around the mathematical rigour that enabled to explain and manipulate the quantum phenomena. His framework is strongly based on Hilbert's theory of operators. Dirac's concerns were more of a practical nature, as his notation provided a compact way to represent quantum states. Their combined contributions were invaluable to establish this area.

A Hilbert Space is an extension of a vector space which requires the definition of inner product. A quantum system is represented by a n-dimensional complex Hilbert Space - $\mathcal{H}^n$-; a complex vector space in which the inner product is defined.

A 2-dimensional Hilbert Space - $\mathcal{H}^2$- (corresponding to a qubit for example), can be represented by a Bloch Sphere. In the Figure 2.3 we have a representation of quantum state $|\psi\rangle$ in a two-dimensional Hilbert Space.

From Dirac it is important to point the Dirac's notation (also known as Bra-ket notation or $\langle Bra|c|ket\rangle$) (introduced in 1939 [12]), that is widely used in literature based on quantum theory. This notation uses angle brackets and vertical bars to represent quantum states (or abstract vectors) as it can be seen

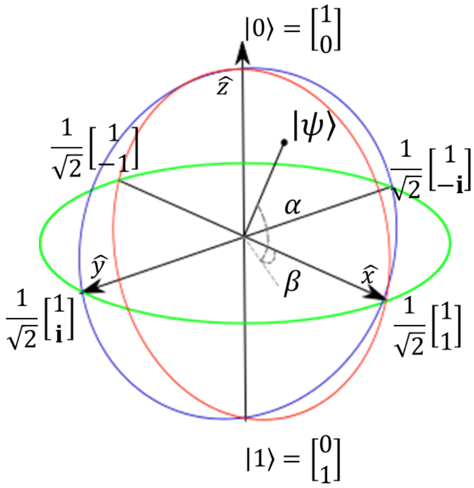

Figure 2.3: Representation of a quantum state $|\psi\rangle$ in a two-dimensional Hilbert Space($\mathcal{H}^2$)

bellow:

$$|z\rangle = \begin{bmatrix} z_1 \\ z_2 \\ \ldots \\ z_n \end{bmatrix} \tag{2.14}$$

$$\langle z| = (|z\rangle)^* = \begin{bmatrix} z_1^* & z_2^* & \ldots & z_n^* \end{bmatrix} \tag{2.15}$$

This notation provides for an elegant representation of the inner product (2.16), and as the linearity that arises from the inner product, in equation (2.17). While the Bra-ket notation can be useful in terms of condensing information, using vectors and matrices to represent the states turns out to be a more approachable way to understand and manipulate data.

$$\langle z|z\rangle = \sum_{i=1}^{n} \bar{z}_i z_i \tag{2.16}$$

where $\bar{z}_i$ is the complex conjugate of z_i.

$$\langle z|(\alpha|x\rangle + \beta|y\rangle) = \alpha\langle z|x\rangle + \beta\langle z|y\rangle \tag{2.17}$$

2.2.2 Measurement

The Born rule was formulated by Born in 1926. This law allows to predict the probability that a measurement on a quantum system will yield a certain result. This law provides a link between the mathematical foundation of Quantum Mechanics and the experimental evidence [14] [15].

The Born rule states that if there is a system which is in a state $|\psi\rangle$ (in a given n-dimensional Hilbert Space $\mathcal{H}$), the probability of measuring a specific eigenvalue λ_i associated with the i-th eigenvector of a set of orthonormal basis $\mathcal{B}$, ψ_i, will be given by [14]:

$$P_\psi(\lambda_i) = \langle\psi|Proj_i|\psi\rangle \tag{2.18}$$

where $Proj_i$ is a projection matrix corresponding to ψ_i :

$$Proj_i = |\psi_i\rangle\langle\psi_i| \tag{2.19}$$

A projection matrix (or a projection operator) corresponding to a single eigenvector of $\mathcal{B}$, can be understood as a ”yes” or ”no” question because it has two eigenvalues (0 or 1).

As the set of eigenvectors $\mathcal{B} = \{\psi_0, \psi_1, ..., \psi_i, ..., \psi_{n-1}\}$ forms a orthogonal basis of the n-dimensional Hilbert Space considered, the state $|\psi\rangle$ can be written as a linear combination of those eigenvectors:

$$|\psi\rangle = \alpha_0\psi_0 + \alpha_1\psi_1 + ... + \alpha_i\psi_i + ... + \alpha_{n-1}\psi_{n-1} \tag{2.20}$$

The coefficients α_i are complex numbers called probability amplitudes, and their squared sum is equal to 1:

$$\sum_{i=0}^{n} |\alpha_i|^2 = 1 \tag{2.21}$$

which satisfies the classic law of total probability (referred previously in the Kolmogorov axioms), which states that the sum of probabilities of all possible outcomes in a space is always 1. This brings us to:

$$P_\psi(\lambda_i) = \langle\psi|\psi_i\rangle\langle\psi_i|\psi\rangle = |\langle\psi|\psi_i\rangle|^2 = |\alpha_i^*\alpha_i| \tag{2.22}$$

So the determination of the probability of an event ($P(A)$), is made by projecting the quantum state labled ψ on the eigenvectors of $\mathcal{B}$ corresponding to the operator A (represented by a matrix), and measuring the squared length of the projection. [16]

$$P(A) = (A|\psi\rangle)^2 \tag{2.23}$$

The squared length of the projection gives a prediction or an expected value for the system, if we would peak and measure the state of the system at that moment. When we measure the system its wave function ψ collapses onto a well defined state. If we think of an analogy, in fair coin we know that the probability of getting Heads or Tails is respectively $P(H) = 0.5$ and $P(T) = 0.5$, however when we toss it we either get Heads or Tails. The probability conveys the meaning that if we repeat the experiment *ad infinitum* we would get Heads half the time.

If A can be described by an Hermitian operator ($A = A^\dagger$), it can be decomposed as a linear combination of projectors $Proj_i$, thus being a set of "yes" "no" questions.

According to Leiffer [17] "quantum theory can be thought of as a non-commutative, operator-valued, generalization of classical probability theory".

As in the classical probability theory where from a random variable it is possible to establish a probability distribution, also known as density function, in the Hilbert space there is a equivalent density operator. The density operator (ρ) is a Hermitian operator that has the particularity of having its trace equal to 1 [14].

$$\rho = \sum_{i=0}^{n-1} p_i |\psi_i\rangle\langle\psi_i| \tag{2.24}$$

where p_i represents a probability associated with the eigenstate ψ_i, so the trace of the matrix ρ (the sum of the elements on the main diagonal the matrix) is

$$tr(\rho) = 1 \tag{2.25}$$

2.2.3 Example: Double-slit experiment with electrons

Like the Young's Experiment where light created an interference pattern similar to a wave, firing electrons one at the time produces a similar pattern. The unobserved fired electron behaved like a wave and after passing the slits the wavelets interfered with one another to create a interference pattern. However if a measuring device was active while the electron was fired the interference pattern wasn't registered.

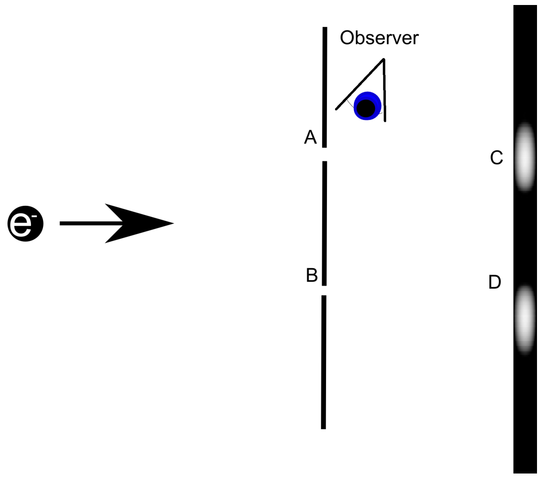

Figure 2.4: Double-slit experiment where there is a measuring device that allows to know through which slit the electron passed.

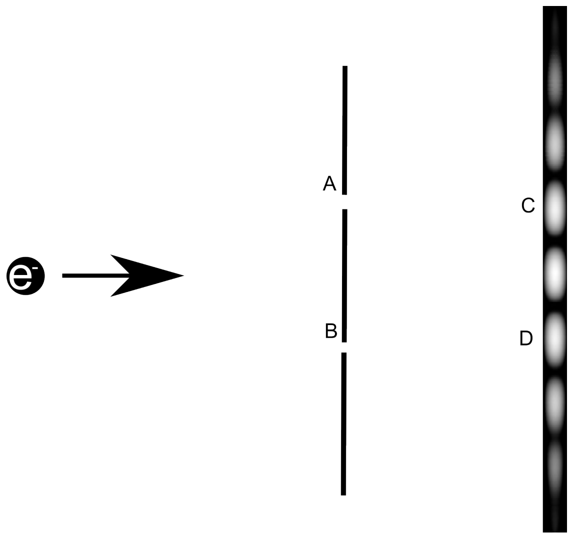

Figure 2.5: Double-slit experiment, where electrons exhibit the interference pattern characteristic in waves

The fact that the electron was measured while passing through a slit produced a particle behaviour, explained by the classical theory (Figure 2.4).

In this experiment a single electron is shoot at a time. So in the start of the experiment (S), we know the initial position of the electron. The labels F and S help us identify the state of the electron before passing through the slit and afterwards, however it is more accurate to describe the state of the electron as a wave function ψ, and $|A\rangle$ and $|B\rangle$ a pair of orthonormal eigenvectors, which can be used to ask whether or not the electron has passed through the slit A or B, this leads us to the direct application of the Born rule (2.18).

A final measurement is made when the electron hits the wall behind the slits, where we know the final position (F) of the electron. [1]

If this experiment is observed, there is an intermediate measure that tells us whether the electron went through the slit A or B. The corresponding probability amplitudes related to this measurement are ω_A and ω_B, and:

$$\omega_A = \langle F|A\rangle\langle A|S\rangle \tag{2.26}$$

$$\omega_B = \langle F|B\rangle\langle B|S\rangle \tag{2.27}$$

If we consider the intermediate measurement the probability $P(F|S)$ will be:

$$P(F|S) = |\langle F|A\rangle\langle A|S\rangle|^2 + |\langle F|B\rangle\langle B|S\rangle|^2 \tag{2.28}$$

But if we only measure the position of the electron at the end of the experiment that probability will be:

$$P(F|S) = |\langle F|A\rangle\langle A|S\rangle + \langle F|B\rangle\langle B|S\rangle|^2 \tag{2.29}$$

The latter equation will be dependent on a interference coefficient that will be responsible the interference pattern observed in the unobserved experiment.

This experiment reveals the non-commutative nature and the importance of the measurement in quantum probability [17] [18]. In classic probability the moment when we measure where is the electron was would not affect the experiment, when using von Neumman probabilities we could obtain the same result as long as $|\langle F|A\rangle\langle A|S\rangle|^2 + |\langle F|B\rangle\langle B|S\rangle|^2 \approx |\langle F|A\rangle\langle A|S\rangle + \langle F|B\rangle\langle B|S\rangle|^2$, which can happen if $2\langle F|A\rangle\langle A|S\rangle\langle F|B\rangle\langle B|S\rangle = 0$.

2.2.4 Example: Polarization of Light

The photons in a beam of light don't vibrate all the same direction in most of the natural sources of light. To filter the light polaroids are used. A polaroid only allows the passage of light in a well-defined direction and thus reducing the intensity of the light. In the Figure 2.6 we can observe that the introduction of the oblique polaroid in the third situation led to a passage of light. Although there is a classical explanation to this phenomenon if we consider waves when we are considering a beam of light, if our light source emits one photon at the time a quantum mechanical explanation is needed [19].

To model the polarization of the photon in a quantum setting, we will use a vector $|v\rangle$ in a two-dimensional Hilbert Space:

$$|v\rangle = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.30}$$

[1] Mohrhoff, U.: Two Slits. http://thisquantumworld.com/wp/the-mystique-of-quantum-mechanics/two-slit-experiment/#fn1back

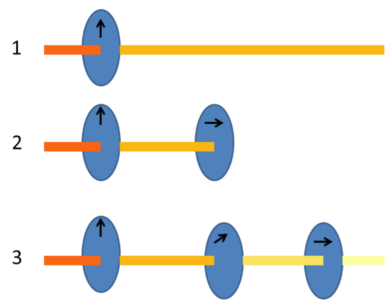

Figure 2.6: 1. With one vertical polaroid the unpolarized light is attenuated by a half. 2. Vertical polarization followed by a horizontal polarization will block all the passing light. 3. Inserting a oblique polaroid between the vertical and horizontal polaroids will allow light to pass.

where $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ would represent the vertical direction (could also be represented by the state vector $|\uparrow\rangle$), and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the horizontal one (another possible representation to this basis could be $|\rightarrow\rangle$). We can consider the Figure 2.3 as a graphical representation for this system.

In the first situation if $a = \frac{1}{\sqrt{2}}$, that would mean that the probability of passing the vertical polaroid would be $a = (\frac{1}{\sqrt{2}})^2 = 0.5$, that would light to the expected reduction of a half of the intensity of light. After passing through the vertical polaroid the photon will have a polarization of $|v\rangle = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Considering now the second situation after we have our photon polarized vertically (like in the end on the first situation), the probability of being vertically polarized is 1, thus making the probability of passing through the horizontal polaroid 0. In the third situation, after being vertically polarized the photon will pass through an oblique polaroid that makes its direction

$$|v\rangle = cos(\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbf{i}.sin(\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.31}$$

θ being the angle of the polaroid. The photon filtered by the vertical polaroid will pass this second polaroid with a probability of $(cos(\theta))^2$, becoming polarized according to the filter, as we can observe depending on the value of θ we will now have a horizontal component in the vector that describes the state of the photon. This will make the photon pass the horizontal polaroid with a probability of $sin(\theta)^2$.

2.3 Quantum Computing

Quantum Computing is an area that tries to take advantage of Quantum Mechanics phenomena to perform calculations. This idea of using the properties of a wave function to compute was first introduced by the works of Feynman [20]. Shor (at Bell Labs), developed an algorithm that, running on a quantum computer, could factor large numbers that can provide theoretical speed-ups over classical algorithms [19].

The quantum equivalent of a bit (the basic unit of information in computers), is a qubit (or quantum bit). A qubit is a two-state quantum system that can be interpreted as normalized vectors in a 2-dimensional Hilbert space. This Hilbert Space($\mathcal{H}^2$), an element in that space can be uniquely specified by resorting to a base ($\mathcal{B}$), with two orthonormal basis (also known as pure states). In quantum computing the base used is $\mathcal{B} = \{|0\rangle, |1\rangle\}$ or $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$.

2.3.1 Superposition

The qubit can be described in terms of linear transformations of pure states as in

$$|\psi\rangle = \omega_0|0\rangle + \omega_1|1\rangle = \begin{bmatrix} \omega_0 \\ \omega_1 \end{bmatrix} \tag{2.32}$$

where ω_0 and ω_1 are complex numbers called probability amplitudes, because when their module is squared they represent a probability or a probability density function associated with the system being in a certain state. When a system is in a mixture of pure states, meaning ω_0 and ω_1 are both different than 0, it is in a state known as a superposition. Measuring forces the system to collapse and assume one of the pure states with a certain probability.

In the example stated in the representation 2.32 the probability of the system falling into the state $|0\rangle$ would be $|\omega_0|^2$. The probability of the system falling into $|1\rangle$ would be $|\omega_1|^2$. The second axiom of Kolmogorov (law of total probability) is verified (2.33).

$$|\omega_0|^2 + |\omega_1|^2 = 1 \tag{2.33}$$

ω_0 and ω_1 (2.32) are complex numbers, the so-called probability amplitudes. When squared, the probability amplitude represents a probability.

2.3.2 Compound Systems

The representation of a system comprising multiple qubits grows exponentially. If to represent a single qubit system there is a 2-dimensional Hilbert space($\mathcal{H}^2$), to represent a system with m qubits a 2^m-dimension space would be required. To represent a higher dimension multiple-qubit system composed by single-qubits, one can perform a tensor product of single-qubit systems.

The tensor product is an operation denoted by the symbol $\otimes$. Given two vector spaces V and W with basis (2.34) and (2.35) respectively, their tensor product would be the mn-dimensional vector space with a basis with elements from the set (2.36) [19].

$$A = \{|\alpha_1\rangle, |\alpha_2\rangle, ..., |\alpha_m\rangle\} \tag{2.34}$$

$$B = \{|\beta_1\rangle, |\beta_2\rangle, ..., |\beta_n\rangle\} \tag{2.35}$$

$$C = \{|\alpha_i\rangle \otimes |\beta_j\rangle\} \tag{2.36}$$

For example, if we consider two Hilbert spaces $\mathcal{H}^2$ with basis $A = \{|0\rangle, |1\rangle\}$ and $B = \{|-\rangle, |+\rangle\}$, their tensor product would be a $\mathcal{H}^4$ with basis (2.37).

$$AB = \{|0-\rangle, |0+\rangle, |1-\rangle, |1+\rangle\} \tag{2.37}$$

Now taking the former Hilbert space, supposing we have the qubits

$$|v\rangle = a_0|0\rangle + a_1|1\rangle \tag{2.38}$$

and

$$|w\rangle = b_0|-\rangle + b_1|+\rangle \tag{2.39}$$

then their tensor product would be

$$|v\rangle \otimes |w\rangle = a_0 b_0 |0-\rangle + a_0 b_1 |0+\rangle + a_1 b_0 |1-\rangle + a_1 b_1 |1+\rangle \quad (2.40)$$

The Bra-ket notation provides a way to prevent the escalation of the basis notation. When specified the vector space the basis can be specified in base 10 for simplicity sake. According to this the basis of the last example would be $AB = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ and a system with 3 qubit could be represented in a Hilbert space, $\mathcal{H}^8$, with basis $\mathcal{B} = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle, |7\rangle\}$, however it is important to have the Hilbert dimension (the cardinal of $\mathcal{B}$), in mind.

When working with a compound system wich holds n qubits (two-state quantum system), represented by the quantum state $|\psi\rangle$ we often want to be able to separate its components $\psi_0, \psi_1, ..\psi_{n-1}$. This way we use the following notations as equivalent $|\psi\rangle = |\psi_0, \psi_1, ...\psi_{n-1}\rangle = |\psi_0\rangle|\psi_1\rangle...|\psi_{n-1}\rangle = |\psi_0\rangle \otimes |\psi_1\rangle \otimes ... \otimes |\psi_{n-1}\rangle$ [19]

2.3.3 Operators

In order to perform transformations in the qubit systems, we can define operators in the Hilbert space that preserve the inner product. Unitary operators are used in the majority of quantum algorithms because they are reversible, thus insuring that no rule of quantum mechanics (like the time evolution of the system), is violated while applying the transformations. This helps preserve symmetries and basis changes. Unitary matrices ($UU^* = U^*U = I$), represent a unitary operator on a finite-dimensional Hilbert Space.

As Unitary operators are a subset of linear operators, they preserve the inner product, this leads us to an interesting difference between classical computing and quantum computing: it is impossible to copy or clone unknown quantum states [19]. This is known as: The No-Cloning Principle.

We can prove the "The No-Cloning Principle" by *reductio ad impossibilem*. Suppose we have an operator U which is unitary and clones quantum states, and two unknown quantum states $|a\rangle$ and $|b\rangle$. The transformation U means that if we apply it to $|a\rangle$ we have $U(|a\rangle|0\rangle) = |a\rangle|a\rangle$. The result when applying to $|b\rangle$ is $U(|b\rangle|0\rangle) = |b\rangle|b\rangle$.

If we consider a state $|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$, by the principle of linearity $U(|c\rangle|0\rangle) = \frac{1}{\sqrt{2}}(U(|a\rangle|0\rangle) + U(|b\rangle|0\rangle))$, giving the final result $U(|c\rangle|0\rangle) = \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |b\rangle|b\rangle)$. However if U is a cloning operator then $U(|c\rangle|0\rangle) = |c\rangle|c\rangle$.

$|c\rangle|c\rangle = \frac{1}{2}(|a\rangle|a\rangle + |a\rangle|b\rangle + |b\rangle|a\rangle + |b\rangle|b\rangle)$ is different from $\frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |b\rangle|b\rangle)$, thus we can affirm there is no unitary operator U that can clone unknown quantum states [19].

Other important class of operators are the self-adjoint operators, that have the property stated in Equation 2.41. The Hermitian operator is one that satisfies the property of being equal to its conjugate transposed, $A = A^{*T} = A^{\dagger}$. In a finite-dimensional Hilbert space defined by a set of orthonormal basis every self-adjoint operator is Hermitian [14].

$$\langle A^{\dagger} z | x \rangle = \langle z | A x \rangle \tag{2.41}$$

As discussed before Hertitian Operators can be decomposed as a linear combination of projectors ($Proj_i$), and their eigenvalues are real, which makes them useful to represent physical properties that can be measured. For example in a particle system composed by an electron they could represent the spin of an electron "clockwise" or "counter-clockwise". An Hermitian operator (H), is also relevant because it can be used to generate a unitary operator U, as in

$$U = exp(\mathbf{i}H) \tag{2.42}$$

When presented with q qubits, a q-dimensional square matrix is called a Quantum Gate.

Some important Quantum Gates that operate on a single qubit are:

- The Identity Matrix (2.43).

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.43}$$

- The Pauli Operators. This operators are used to perform the NOT operation.
  - The Bit-Flip Operator

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.44}$$

  - The Phase-Flip Operator

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.45}$$

  - The Bit and Phase-Flip Operator

$$\sigma_y = \begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix} \tag{2.46}$$

- The Hadamard Gate, belongs to a general class of Fourier Transforms. This 2×2 particular case is also a Discrete Fourier Transform matrix.

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.47}$$

The Pauli operators with the identity matrix I, form an orthogonal basis for the complex Hilbert space of all 2×2 matrices, also known as Special Unitary Group 2 - $\mathsf{SU}(2)$ [21].

2.3.4 Entanglement

In multiple qubit systems, qubits can interfere with each other, thus making impossible to determine the state of part of the system without “disturbing” the whole. In other words, there are states in a multi-qubit system that cannot be described as a probabilistic mixture of the tensor product of single-qubit systems; when this happens this state is not separable (with respect to the tensor product decomposition), this phenomenon is called quantum entanglement [19].

If a mixed quantum state ψ is separable, with ψ being a quantum system constructed by n sub-states defined each defined in its vector space (an Hilbert space) with dimension m, and basis $V_0, V_1, ..., V_n$, then it can be written as:

$$\rho = \sum_{j=0}^{m-1} p_j |\varphi_j\rangle_0 \langle\varphi_j|_0 \otimes ... \otimes |\varphi_j\rangle_n \langle\varphi_j|_n, \sum_i p_i = 1, |\varphi_j\rangle_i \in V_i \tag{2.48}$$

According to the latter formula an example of a density matrix for a separable 2-qubit system could be

$$\rho = 0.5 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{\{0\}} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{\{1\}} + 0.5 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_{\{0\}} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_{\{1\}} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \tag{2.49}$$

On the other hand a maximally entangled 2-qubit system would be caracterized by a density matrix that would correspond to a uniform probability distribution like [19]

$$\rho = \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix} \tag{2.50}$$

Quantum entanglement is one of the main differences from the classical theory [19]. In an entangled system each member has a quantum state that is described with relation to the other members, thus cannot being described independently. This property is not local as transformations that act separately in different parts of an entangled system cannot break the entanglement. However, if we measure a part of the entangled system, the system collapses, and if we measure the otherpart in any point of time from that moment we will find a correlation with the outcome of the first measurement.

For example, supposing we consider the following quantum states, known as Bell states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2.51}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{2.52}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{2.53}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{2.54}$$

These states form a particular basis in $\mathcal{H}^4$ as they are all entangled states and they are maximally entangled. If we have a system of two qubits in a Hilbert space in the mixed state $|\Phi^+\rangle$ and we measure the qubit A (by deciding the outcome 0 or 1 with a probability of 0.5 for each),we are automatically uncovering the value of the qubit B. In this state we know the second qubit will always yield the same value of the first measured qubit. The second value is correlated with the first one.

2.4 Game Theory

Game Theory did not exist as a field of own right before von Neumann published the article " On the Theory of Games of Strategy". This field deals mainly with the study of interactions between rational decision-makers [22]. Game Theory knows numerous applications in areas such as Economics, Political Science, Biology, and Artificial Intelligence.

2.4.1 Definition of a Game

A game (Γ), is a model of conflict between players characterized by [23] [24] [25]:

- A set of Players: $P = \{1, 2, .., N\}$.
- For each player there is set of Actions: X_i
- There are preferences , for each player, over the set of Actions.

The preferences in the previous definitions are defined with resort to the concept of utility, expected utility, or payoff derives from the theory that we can use real numbers (Equation 2.55), to model and represent the wants and needs of the players.

$$u : X \to \mathbb{R} \tag{2.55}$$

This simplified mathematical model allows to compare different states and rank preferred outcomes. If $u_i(x)$ is a utility function for player i, and $u_i(A) > u_i(B)$, that would mean that the player would strictly prefer A over B. The concept of expected utility is fundamental to analyse games, as a rational player would try to maximize her expected utility [22] [23] [26].

The options a player i has when choosing an action, when the outcome depends not only on their choice (but also on the actions taken by the other players), is referred as strategy s_i. The set of strategies available to a player is represented by the set S_i.

When a strategy gives strictly a higher expected utility in comparison to other strategies, we have a strategy that dominates, or a dominant strategy. The mathematical definition of dominance is represented in Equation (2.56); where, for a player i, in spite of the strategies chosen by the other players (denoted by s_{-i}), there is a strategy s^* which gives always an higher expected utility in comparison with other strategies $s^{'}$ available to player i.

$$\forall s_{-i} \in S_{-i}[u_i(s^*, s_{-i}) > u_i(s^{'}, s_{-i})] \tag{2.56}$$

The strategy that leads to the most favourable outcome for a player, taking into account other players strategies, is known as best response.

A pure strategy defines deterministically how the player will play the game. In the game represented in Table2.1 , in a pure strategy, the players either they choose “Cooperate” (C), or “Defect” (D).

If there is a probability distribution associated with probability of playing with a determined pure strategy, we have a mixed strategy.

There are two standard representations of games:

- Normal Form - lists what payoffs the players get as a function of their actions as if they all make their moves simultaneously. This games are usually represented by a matrix, for example 2.1.
- Extensive Form - extensive form games can be represented by a tree and represent sequential actions.

	Player 2: C	Player 2: D
Player 1: C	(2,2)	(0,3)
Player 1: D	(3,0)	(1,1)

Table 2.1: Example of a Normal Form game.

A finite game is a game that has a finite set of actions, a finite number or players, and it does not go on indefinitely.

A zero-sum game is a mathematical representation of a system where the gains of the players are completely evened out by the losses of the others; this means that the sum of the utilities of all players will always be zero.

2.4.2 Nash Equilibrium

If we observe the representation of the classic game “Prisoner’s Dilemma”, we can observe that each player has two strategies; either they choose “Cooperate” (C), or “Defect” (D). The “Defect” strategy, for both players (the game is symmetrical), always yield a higher payoff in spite of the other player's strategy, this means that it is a dominant strategy and also constitutes a best response to the game. If both players chose their best response the final outcome (D, D) becomes an equilibrium solution, more specifically a Nash Equilibrium.

When all players cannot improve their utility by changing their strategy unilaterally, we have an equilibrium point. This equilibrium point is named after John Nash, who proved that it exists at least one mixed strategy Nash equilibrium in a finite game [27] [28]. This concept is used to analyse game where several decision makers interact simultaneously and the final outcome depends on the players strategy [23].

In order to compute a mixed strategy Nash Equilibrium in a 2-player 2 actions game, we need to define the probability distribution $P(C) = q$, $P(D) = 1 - q$ that makes player 1 indifferent to the whether player 2 decides to Cooperate or to Defect, and the probability distribution $P(C) = p$, $P(D) = 1 - p$ that makes player 2 indifferent to whether player 1 decides to Cooperate or to Defect. In the game represented in Table 2.2 the mixed strategies $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$ are Nash Equilibria.

	Player 2: C	Player 2: D
Player 1: C	(2,1)	(0,0)
Player 1: D	(0,0)	(1,2)

Table 2.2: Example of a Normal Form game. The mixed strategies $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$ are Nash Equilibria for this game.

When there is at least a Nash Equilibrium when the players choose pure strategies we have a a strictly determined game [26]

In an extensive form game we have a concept of sub-game perfect Nash equilibrium, when a strategy is a Nash equilibrium for all sub-games in the original game [26].

2.4.3 Example: Prisoner's Dilemma

The Prisoner's Dilemma is a classic example of a game that can be represented in normal form [23]. This problem has received a great deal of attention because, in its simple form, rational individuals will seem to deviate from solutions that would represent the best social interest, the Pareto Optinal solution. In this game the Pareto Optimal solution is not a Nash Equilibrium. The Prisoner's Dilemma can be formulated as it follows:

Two suspects of being partners in a crime are arrested. The police needs more evidence in order to prosecute the prisoners. So each prisoner is locked in solitary confinement and has no means of communicating with the other suspect. The police will then try to extort a confession from the prisoners. A bargain will be proposed to the suspects:

- If the suspect testifies against the other suspect (Defects) and the other denies, he will go free and the second will get three years sentence.
- If they both testify against one another (both Defect), both will be convicted and they will get two years.
- In case both suspects deny the involvement (both Cooperate) of the other, they will get a one year sentence.

A matrix representation of the problem is in Table 2.3, here the payoff represented by the letter R would be the standard reward for the game, T would be the temptation to deviate from a cooperation profile, P represents the punishment when both entities do not cooperate, and finally S would be a sucker's payoff. A particular case for this problem is represented in Table 2.4. In each cell of the matrix we have a pair of the expected utility for the players for every outcome. A higher utility represents a more desirable state.

In this game the Pareto Optimal solution happens when both players chose to Cooperate (the pair (2,2) in Table 2.4). However both players have the incentive to Defect, because regardless what the opponent chooses they will have always a strictly higher payoff. Defecting becomes a dominant strategy in the Prisioner's Dillema and the outcome $(Defect, Defect)$ is a Nash Equilibrium to the game.

	Player 2: C	Player 2: D
Player 1: C	(R,R)	(S,T)
Player 1: D	(T,S)	(P,P)

Table 2.3: The canonical normal form representation for the Prisoner's Dilemma must respect $T > R > P > S$.

	Player 2: C	Player 2: D
Player 1: C	(2,2)	(0,3)
Player 1: D	(3,0)	(1,1)

Table 2.4: One possible normal form representation of Prisoner's Dilemma.

2.4.4 Pareto Optimal

From the point of view of an observer outside the game system some outcomes may seem better than others. For example on the game represented in Table 2.1 (Prisoners' Dilemma), the outcome (C, C) seems better than the outcome (D, D) because it provides a strictly higher utility to the players. However we know that the outcome (D, D) is the Nash Equilibrium of the game. If both players use their best response their outcome might not be the best outcome from the point of view of resource allocation.

When it is impossible to improve the payoff of a player without lowering another player's expected utility we have a Pareto Optimal solution. In the Prisoner's Dilemma every outcome but the Nash Equilibrium are Pareto Optimal because any attempt to improve the utility of one player ends up lowering the expected utility for the other.

For example if we have two players and 10 units of a finite resource to distribute among the players, a Pareto optimal solution is $(5, 5)$. Any other attempt to redistribute where the sum of the resource isn't 10 -$(1, 4)$, $(2, 3)$, etc...- could still be improved.

2.5 Quantum Game Theory

In the article “Quantum information approach to normal representation of extensive games” [29], the authors propose a representation for both normal and finite extensive form games [25]. This definition is based on the premise that any strategic game can be represented as an extensive form game where all the players have no knowledge about the actions taken by other players.

The representation assumes that for each action that a player can make in the game there are only two possible measurable outcomes. For example while voting on a referendum in which the question is ”Should the state declare war on country X?” despite the method using for voting the final answer is revealed as a ””yes”” or ””no””. In Quantum Game Theory we model an action as a qubit which is manipulated by the player.

A game in this form is represented by a six-tuple(2.57), where:

$$\Gamma = (\mathcal{H}^{2^a},\ N,\ |\psi_{in}\rangle,\ \xi,\ \{\mathcal{U}_j\},\ \{E_i\}) \tag{2.57}$$

- a is the number of actions (qubits), in the game;
- $\mathcal{H}^{2^a}$ is a 2^a-dimensional Hilbert space constructed as $\otimes_{j=1}^{a}\mathbb{C}^2$, with basis $\mathcal{B}$;
- N is the number of players;
- $|\psi_{in}\rangle$ is the initial state of the compound-system composed by a qubits: $|\varphi_1\rangle,\ |\varphi_2\rangle, ..., |\varphi_j\rangle, ..., |\varphi_a\rangle$;
- ξ is a mapping function that assigns each action to its respective player;
- For each qubit $j, \mathcal{U}_j$ is a subset of unitary operators from $\mathrm{SU}(2)$ (the general for the 2 dimensional Special Unitary Group is presented in Equation (2.58)). These operators can be used by the player to manipulate her qubit(s);

$$\begin{gathered}\mathcal{U}_j(w,x,y,z) = w.I + \mathbf{i}x.\sigma_x + \mathbf{i}y.\sigma_y + \mathbf{i}z.\sigma_z,\\ w,x,y,z \in \mathbb{R} \wedge w^2+x^2+y^2+z^2 = 1\end{gathered} \tag{2.58}$$

- Finally, for each player i, E_i is a utility functional that specifies her payoff. This is done by attributing a real number (representing a expected utility $u_i(b)$, in (2.59)), to the measurement for the projection of the final state ((2.60)), on a basis from the $\mathcal{B}$(2.59).

$$E_i = \sum_{b\in\mathcal{B}} u_i(b)|\langle b|\psi_{fin}\rangle|^2, u_i(b) \in \mathbb{R} \tag{2.59}$$

The strategy of a player i is a map τ_i which assigns a unitary operator U_j to every qubit j that is manipulated by the player ($j \in \xi^{-1}(i)$). The simultaneous move is represented in (2.60).

$$|\psi_{fin}\rangle = \otimes_{i=1}^{N} \otimes_{j\in\xi^{-1}(i)} \mathcal{U}_j |\psi_{in}\rangle \tag{2.60}$$

The tensor product of all the operators chosen by the players is referred as a super-operator, which act upon the game system.

2.5.1 Example: Quantum Prisoner's Dilemma

The importance of the Prisoner's Dilemma for the study of Game Theory made it a prime target for investigation in Quantum Game Theory. The problem has been modelled several times [5] [21]. Therefore we will use it in order to exemplify and consolidate the definition described above, in Section 2.5 [29].

Each player i in the quantum version of Prisoner's Dilemma will be able to manipulate one qubit (φ_1 and φ_2) in Equations (2.61) and (2.62), with the unitary operators (shown in Equation (2.63)). The classical strategies: Cooperate (C), and Defect (D). C is represented in the sub-set $\mathcal{U}_j$ (Equation (2.64)) . D, also known as the Bit-flip operator (σ_x), is not represented in the restricted space proposed in [5] [29]. D^y is an alternative to D in the sub-set $\mathcal{U}$.

$$|\varphi_1\rangle = a_0|0\rangle + a_1|1\rangle, \sum_{i=0}^{1} |a_i|^2 = 1 \tag{2.61}$$

$$|\varphi_2\rangle = b_0|0\rangle + b_1|1\rangle, \sum_{j=0}^{1} |b_j|^2 = 1 \tag{2.62}$$

$$\mathcal{U}_j(\theta,\phi) = \begin{bmatrix} cos(\frac{\theta}{2}) & e^{\mathbf{i}\phi} sin(\frac{\theta}{2}) \\ -e^{-\mathbf{i}\phi} sin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{bmatrix}, j \in \{1,2\}, \theta \in (0,\pi), \phi \in (0,\frac{\pi}{2}) \tag{2.63}$$

$$\begin{cases} C = U_j(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ D^y = U_j(\pi,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{cases}, j \in \{1,2\} \tag{2.64}$$

$\bigotimes$	C-$\|0\rangle$	D-$\|1\rangle$
C-$\|0\rangle$	$\|0,0\rangle$	$\|0,1\rangle$
D-$\|1\rangle$	$\|1,0\rangle$	$\|1,1\rangle$

Table 2.5: Construction of the basis for the game space; $\mathcal{H}^4$.

The system that holds the game is represented in a $\mathcal{H}^4$. Each basis ($|1\rangle, |2\rangle, |3\rangle, |4\rangle$), represents a final outcome as Table 2.5 suggests.

The fundamental difference from the classical version lies in the way the initial state is formulated in Equation and the strategies the players might use(2.66). We will entangle our state by applying the gate $\mathcal{J}$ [21] [5].

The parameter γ becomes a way to measure the entanglement in the system [5].

$$\mathcal{J} = exp\left\{\mathbf{i}\frac{\gamma}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\} \tag{2.65}$$

$$\begin{aligned} |\psi_{in}(\gamma)\rangle &= exp\left\{\mathbf{i}\frac{\gamma}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}|00\rangle \\ &= cos(\frac{\gamma}{2})|00\rangle + \mathbf{i}sin(\frac{\gamma}{2})|11\rangle, \gamma \in (0,\pi) \end{aligned} \tag{2.66}$$

$$|\psi_{fin}\rangle = \mathcal{J}^{\dagger} \otimes_{i=1}^{2} \mathcal{U}_i |\psi_{in}\rangle \tag{2.67}$$

The entanglement in quantum game theory can be viewed as an intrinsic unbreakable contract. Furthermore measuring an entangled state will cause the wave function that describes the state to collapse. Before measuring the final result we will de-entangle the system by applying the operator $\mathcal{J}^{\dagger}$, as shown in Equation (2.67).

The utility functions for each player is calculated by projecting the final state in each base and attributing a real number to each measurement, as in equation. In order to compare a classical version with this quantum model, the real numbers assigned are those in the classical example in Table 2.3, and the basis are presented in Table 2.5.

$$E_0(|\psi_{fin}\rangle) = 2 \times |\langle 00|\psi_{fin}\rangle|^2 + 3 \times |\langle 10|\psi_{fin}\rangle|^2 + 1 \times |\langle 11|\psi_{fin}\rangle|^2 \tag{2.68}$$

$$E_1(|\psi_{fin}\rangle) = 2 \times |\langle 00|\psi_{fin}\rangle|^2 + 3 \times |\langle 01|\psi_{fin}\rangle|^2 + 1 \times |\langle 11|\psi_{fin}\rangle|^2 \tag{2.69}$$

The gate $\mathcal{J}$ is chosen to be commutative with the super-operators created by the tensor product of the classical strategies C and D: $[\mathcal{J}, C \otimes C] = 0$, $[\mathcal{J}, C \otimes D] = 0$, $[\mathcal{J}, D \otimes C] = 0$, and $[\mathcal{J}, D \otimes D] = 0$.

This condition implies that any pair of strategies in the sub-set $S_0 = \{\mathcal{U}(\theta, 0), \theta \in (0, \pi)\}$ is the equivalent of a classical mixed strategy when $\gamma = 0$; the joint probability associated with measuring a outcome $(\delta_1, \delta_2) \in \{(C, C), (C, D), (D, C), (D, D)\}$ ($P(\delta_1, \delta_2) = |\langle \delta_1, \delta_2|\psi_{fin}\rangle|^2$), becomes $P(\delta_1, \delta_2) = P(\delta_1)P(\delta_2)$, with $P(C) = cos^2(\frac{\theta}{2})$, and $P(D) = 1 - P(C)$ [5].

If the parameter ϕ in the operator $U_j(\theta, \phi)$ and the entanglement coefficient differ from 0 we are able to explore quantum strategies that have no counterpart in the classical domain.

[5] finds that the pair of strategies $(\mathcal{U}(0, \frac{\pi}{2}), \mathcal{U}(0, \frac{\pi}{2}))$ for $\gamma = \frac{\pi}{2}$ yield a payoff of $(2, 2)$, which is a Nash equilibrium and it is Pareto Optimal when we have a restricted space described by $\mathcal{U}$(2.63). If we allow the operator D(2.64), however, according to [21] there is no quantum pure strategy Nash Equilibrium, when the entanglement is maximal, $\mathcal{U}(\pi, 0)$ is the optimal counter-strategy for C (represented by the identity matrix), $\mathcal{U}(0, \frac{\pi}{2})$ is the optimal counter strategy for $\mathcal{U}(\pi, 0)$, D becomes the optimal counter-strategy for $\mathcal{U}(0, \frac{\pi}{2})$, and C becomes an optimal counter strategy for D [30].

2.6 Overview

In this chapter we provided a theoretical background that can help the reader to grasp the contents presented in the subsequent chapters.

We started by laying down a comparison between the classical probability theory and the quantum probability theory. The von Neumann probability is the mathematical foundation behind Quantum Mechanics.

The von Neumann probability differs from the classical mainly because mutually exclusive events can interfere, this happens in the Double-slit Experiment, this means that the third axiom of Kolmogorov does not hold true in this probability theory. The concept of probability is deeply intertwined with Quantum Mechanics.

In the Quantum Computing we presented the fundamental concepts. The book "'Quantum Computing - A Gentle Introduction" [19] provides a more in-depth resource on this subject; including the Shor's algorithm that is used to find prime factorizations, and the Deutsch-Jozsa algorithm, that with a single query decides if an unknown function is constant or balanced. These algorithms fall outside the scope of this document.

The Game Theory section contains a brief description of some fundamental concepts from Game Theory needed in order to understand this work. A comprehensive reference such as [23] or [26] can be consulted for a deeper insight on the domain.

3

Related Work

Contents

In this section we present relevant work that it is related with the problem. This constitutes an in-depth analisys of some important problems but also tries to present a broad representative view on the path towards building our solution.

3.1 Quantum Walk on a Line

Quantum Walk is the quantum version of random walks, which are mathematical formalisms that describe a path composed of random steps. A Markov Chain might be used to describe these processes.

We can define the Discrete Quantum Walk on a Line as a series of Left/Right decisions. Understanding this algorithm is important towards being able to define and design more complex algorithms that make use of quantum properties.

We followed an approach suggested by [31] towards simulating n-steps of a Quantum Walk on a Line. The Matlab algorithm can be consulted on Appendix A. In a discrete quantum walk in a line we want to preserve the fact that the probability of turning left is equal to the probability of turning right. To represent a state in this algorithm we will need the number of the node and a direction (identified as L,R) (3.1).

$$|\psi\rangle = |n, L\rangle \tag{3.1}$$

With two equally possible direction choices in each step, we can use a coin metaphor [31] [32] to approach the decision. We toss a coin and go either Left or Right depending on the result.

In a quantum version we need to define a Coin Operator (Coin Matrix), which is responsible to imprint a direction to the current state. This operator is a unitary matrix in a 2-dimension Hilbert space. Some examples of Coin Operators are the Hadamard matrix(3.2) and a symmetric unitary matrix (3.3).

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{3.2}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\mathbf{i}}{\sqrt{2}} \\ \frac{\mathbf{i}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{3.3}$$

Taking the Hadamard matrix as an example(3.2), the coin matrix will operate on the state in the following way (3.4)(3.5) [31].

$$C|n, L\rangle = \frac{1}{\sqrt{2}}|n, L\rangle + \frac{1}{\sqrt{2}}|n, R\rangle \quad (3.4)$$

$$C|n, R\rangle = \frac{1}{\sqrt{2}}|n, L\rangle - \frac{1}{\sqrt{2}}|n, R\rangle \quad (3.5)$$

The Coin Matrix obtains its name by being the quantum equivalent of flipping a classic coin. After tossing a coin comes an operator that will move the node in the direction assigned. The operator responsible for this modification is commonly referred as Shift Operator (3.6)(3.7).

$$S|n, L\rangle = \frac{1}{\sqrt{2}}|n-1, L\rangle \quad (3.6)$$

$$S|n, R\rangle = \frac{1}{\sqrt{2}}|n+1, R\rangle \quad (3.7)$$

These matrices (Coin Matrix and Shift Operator) are used conceptually in various algorithms [19], therefore it is important to be familiar with them. A single step of the algorithm A is illustrated in Figure 3.1.

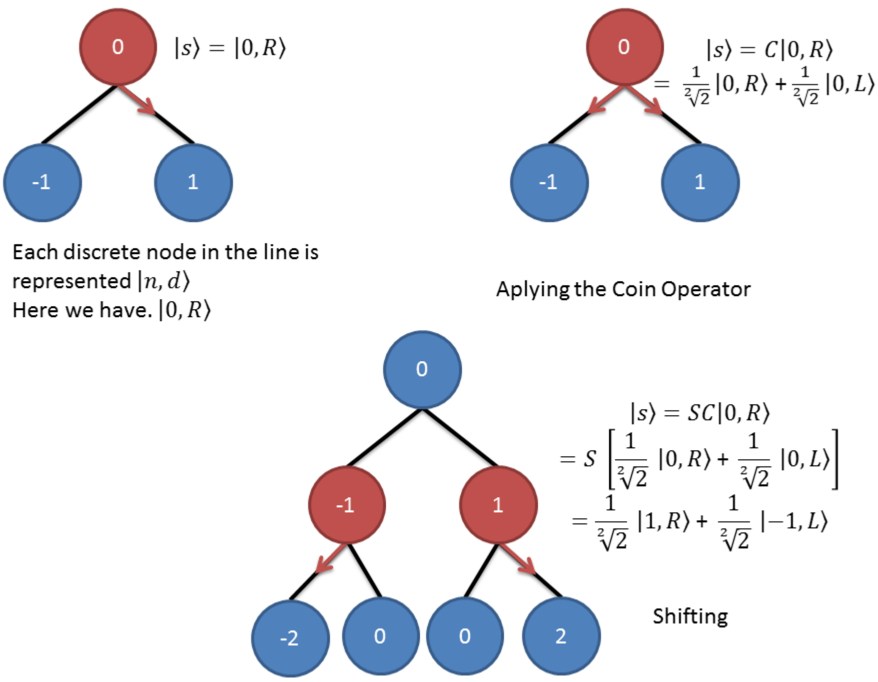

Figure 3.1: Simulating a step of a discrete quantum walk on a line. In the beginning we have a state characterized by the position (0) and a direction (either Left or Right).

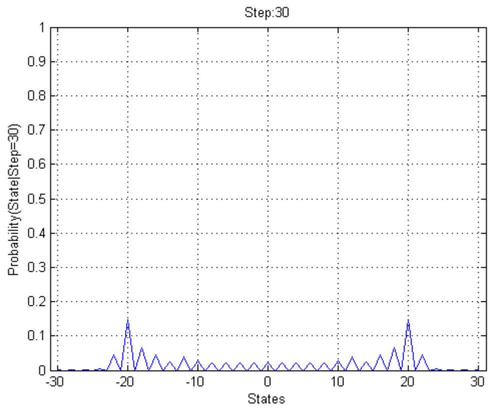

Figure 3.2: 30 Step of the Simulation A using Matrix (3.2) as a Coin Operator.

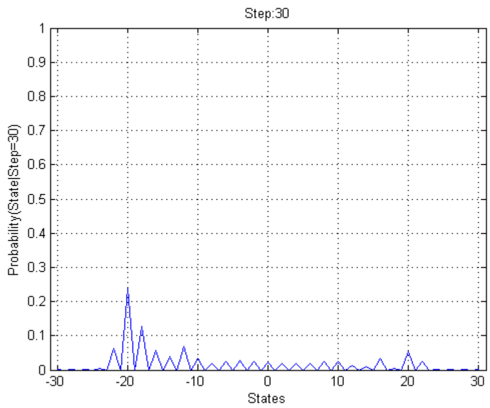

Figure 3.3: 30 Step of the Simulation A using a Hadamard Matrix (3.2) as a Coin Operator.

If we took the classical approach in which we tossed a fair coin to decide to go either left or right, and after n-steps we measured the final node repeatedly, by the Central Limit Theorem the final distribution would converge to a normal distribution. However, in the quantum approach, depending on the Coin Matrix we can get different distributions. In this simple problem we have the basis for some quantum algorithms.

In Figures 3.3 and 3.2, depending on the coin operator we get two different results. Despite that, on one moment the probability of shifting left, is always equal to the probability of shifting right. The main difference from the classical approach lies on the fact there is a quantum interference during the walk, when we measure the results after N steps, somewhat like the electrons interfere in the “double-slit experiment”, presented in Section 2.2.3.

3.2 Quantum Models

The rationale behind building a quantum version of a Game Theory and/or Statistics problem lays in bringing phenomena like quantum superposition, and entanglement into known frameworks. Converting known classical problems into quantum games is relevant to the familiarize with the potential differences these models bring.

3.2.1 Quantum Roulette

In the arbitrary n-State quantum roulette, [33] presented a n-State roulette model using permutation matrices.

This model is interesting because in captures the usage of permutation matrices to manipulate and change the state of the system.

To verify this model with two players we developed a Matlab simulation C, that followed the steps taken in [33].

The game in represented in a n-Dimensional Hilbert Space. There is a basis in the space that represents each of the equally probable entries as shown in (3.8). In a sense this is a generalization of a quantum coin flip that is also used in Section 3.1.

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; \ldots, \; |n\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{3.8}$$

Each state transition is obtained using a permutation matrix denoted by P^i. There are $n!$ permutation matrices, so in the particular case of having a 3-State roulette, there are 6 possible transition choices. The classical strategy considered will rely on choosing an arbitrary probability distribution, that verifies(3.10), and that maps the usage of the permutation matrices. This step will not affect the density matrix (ρ) of the roulette(3.9).

$$\rho = \frac{1}{n!} \sum_{i=0}^{n!-1} P^i \tag{3.9}$$

$$\sum_{i=0}^{n!-1} p_i = 1 \tag{3.10}$$

The density matrix is diagonalizable by a Discrete Fourier Transform because it is a kind of circulant matrix [34], as we can see in (3.11). In (3.11) λ_k are eigenvalues of ρ. $\lambda_1 = 1$ while $\lambda_2 = \lambda_3 = \lambda_k = \lambda_{n-1} = 0$. Each column i of the Fourier matrix will represent a eigenvector $|\lambda_i\rangle$. If we construct the diagonalizing matrix by rotating the columns of the Fourier Matrix we can obtain the projection states as in (3.12).

$$F^\dagger \rho F = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{n-1} \end{bmatrix} \tag{3.11}$$

$$|1\rangle\langle 1| = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} = F^\dagger \rho F \tag{3.12}$$

The quantum strategy advantage in this case is that the first player will not alter the density matrix(3.13).

$$\rho = \sum_{i=0}^{n!-1} p_i P^i \rho P^{i\dagger}, \; \sum_{i=0}^{n!-1} p_i = 1 \tag{3.13}$$

This means that if the second player knows the initial state and the first player plays with a classical strategy, thus never modifying the system density matrix, the second player will be able to manipulate the game under optimal conditions. This result is confirms the demonstration done in [35]; on Quantum Strategies, where in a classical 2 player zero-sum game, if one player adopts a quantum strategy, she increases her chances of winning the game.

3.2.2 Ultimatum Game

The ultimatum game is an example of an extensive form game where two players interact in order to divide a sum of money.

A finite amount of money (or other finite resource), is given to the players, and player 1 must propose how the money will be divided between the two players. If the second player agrees with the proposal, the resource will be split accordingly. When the player 2 rejects the proposal, neither player will receive the money.

If we consider that we have 100 coins, the number of coins received can be considered the expected utility associated with the proposal. The first player can either present a fair division (F), where the coins are split evenly, or an unfair division (U) (defined by a parameter $\theta > 50$), game tree that represents the Ultimatum Game is shown in Figure 3.4. To each definition of θ corresponds a game.

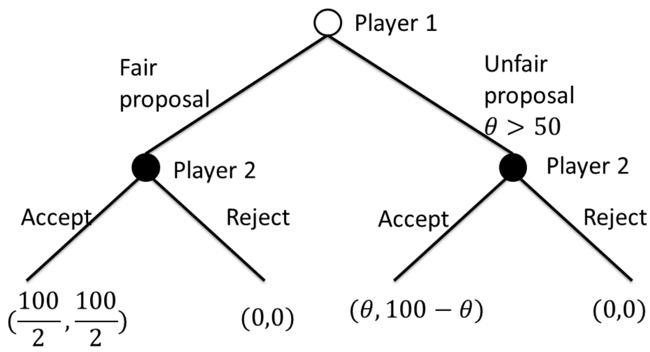

Figure 3.4: Ultimatum Game representation in the extensive form.

3.2.2.A Quantum Model

In a "Quantum information approach to the ultimatum game" [25] we are presented with a quantization scheme for the ultimatum game that uses the definition of quantum game in Section 2.5.

If we present the game in Figure 3.4 in the normal form we get the matrix represented in Table 3.1. The player 2 has 4 possible strategies. The strategy $A_F R_U$ means that the player 2 will accept a fair division proposed by player 1 but will reject a unfair division, when ($\theta > 50$).

The quantum game representation for this game is $\Gamma_{Ultimatum} = (\mathcal{H}^{2^3},\ 2,\ |\psi_{in}\rangle,\ \xi,\ \{\mathcal{U}_j\},\ \{E_i\})$. The game system will consist of 3 qubit, which correspond to the number of actions in the game. Player 1

will be able to manipulate the qubit 1, the player 2 can manipulate the remaining qubits.

	Player 2: $A_F A_U$	Player 2: $A_F R_U$	Player 2: $R_F A_U$	Player 2: $R_F R_U$
Player 1: F	$(50, 50)$	$(50, 50)$	$(0, 0)$	$(0, 0)$
Player 1: U	$(\theta, 100 - \theta)$	$(0, 0)$	$(\theta, 100 - \theta)$	$(0, 0)$

Table 3.1: Normal form representation of the ultimatum game.

The extensive form approach in [25] allows the differentiation between simultaneous moves and sequential moves. This is accomplished by measuring the game state in order to separate game stages. This ”Sequential procedure” uses the Lüders Rule, a quantum analogous of conditional probability.

The main points discussed in [25] Quantum Ultimatum approach are that the game definition presented in Section 2.5 makes the game more convenient to analyse than the extensive form approach.

3.3 Overview

There are more examples of games that have attracted interest in the Quantum domain. In this overview we are going to present a general picture of the work already done in this field.

For example various Models have been proposed to describe a quantum version of the Monty Hall problem [36] [37]. This popular problem [38] is known for its counter-intuitiveness. The problem can be posed as a contest where the player must choose a door (from a set of 3), and has $\frac{1}{3}$ of probability of getting a prize. After the player has chosen the door, one of the remaining 2 doors which does not have a prize is opened. The contestant is asked whether is to her advantage to switch her initial choice.

As the host reveals information, the initial set-up is modified. This is an interesting property. Despite being a counter-intuitive problem, a quantum approach to this problem allows an in-depth comparison between the classical measurement and the quantum measurement. The classic Monty Hall problem is modelled using conditional probability and Baye's Rule to learn that it is to the advantage of the contestant to switch doors. In the quantum version, measuring the outcome of the final state yields the result, instead of taking into account the intermediate actions [25]. In [39] we can observe the attempt to stick as closely to the classical formulation as possible, the host has a system that is correlated to the game system.

The principal information taken from this problems is that there is not a unique way to model a classical problem [39]. Therefore, when modelling a classical problem, we need to select properties that could potentially benefit from a quantum approach.

From the point of view of Quantum Cognition (a domain that seeks to introduce Quantum Mechanics Concepts in the field of Cognitive Sciences), these games are approached from the perspective of trying to model the mental state of the players in a quantum manner. The Prisoner's Dilemma is an example of a problem that has been modelled in order to explain discrepancies from the theoretical results of the classical Game Theory approach and the way humans play the game [40].

One last example worth mention is the quantum approach of the Stackelberg Duopoly problem [41] [42]. This is an Economics Game Theory model that seeks to represent the interactions of two companies, a market leader and a follower which play sequentially; the leader makes a decision and the follower responds.

4

Quantum Pirate Game

Contents

In this chapter we describe the Pirate Game and the steps to model a quantum approach to the problem.

4.1 Pirate Game

4.1.1 Problem Description

The original Pirate Game is a multi-player version of the Ultimatum game that was first published as a mathematical problem in the Scientific American as a mathematical problem posed by Omohundro [43]. The main objective of the Pirate Game was to present a fully explainable problem with a non-obvious solution. The problem can be formulated as it follows:

> Suppose there are 5 rational pirates: A; B; C; D; E. The pirates have a loot of 100 indivisible gold coins to divide among themselves.
>
> As the pirates have a strict hierarchy, in which pirate A is the captain and E has the lowest rank, the highest ranking pirate alive will propose a division. Then each pirate will cast a vote on whether or not to accept the proposal.
>
> If a majority or a tie is reached the goods will be allocated according to the proposal. Otherwise the proposer will be thrown overboard and the next pirate in the hierarchy assumes the place of the captain.
>
> We consider that each pirate privileges her survival, and then will want to maximize the number of coins received. When the result is indifferent the pirates prefer to throw another pirate overboard and thus climbing in the hierarchy.

4.1.2 Analysis

We can arrive at the sub-game perfect Nash equilibrium in this problem by using backward induction. At the end of the problem, supposing there are two pirates left, the equilibrium is very straight forward. This sub-game is represented in Table 4.1, and its Nash Equilibrium is (C, D).

	Player 2: C	Player 2: D
Player 1: C	(100, 0)	(100, 0)
Player 1: D	(100, 0)	(-200, 100.5)

Table 4.1: Representation of the 2 player sub-game in normal form.

As the highest ranking pirate can pass the proposal in spite of the other's decision, her self-interest dictates that she will get the 100 gold coins. Knowing this, pirate E knows that any bribe other higher

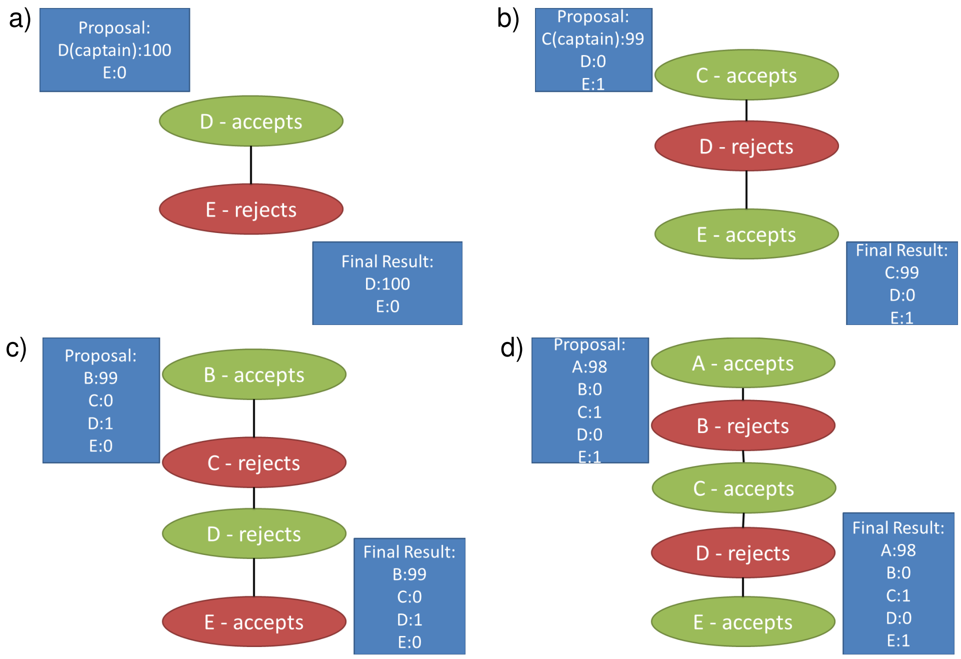

Table 4.2: The equilibrium for the Pirate Game can be found through backward induction. From a), where there's only two pirates left, to d), that corresponds to the initial problem, we define the best response.

ranking pirate offers her will leave her better than if the game arrives to the last proposal.

When applying this reasoning to the three pirate move, as pirate C knows she needs one more vote to pass her proposal and avoiding death, she will offer the minimum amount of coins that will make pirate E better off than if it comes to the last stage with two pirates. This means that pirate C will offer 1 gold coin to pirate E, and keep the remaining 99 coins.

With 4 pirates, B would rather bribe pirate D with 1 gold coin, because E would rather like climb on the hierarchy and getting the same payoff. Finally, with 5 pirates the captain (A), will keep 98 gold coins and rely on pirate C and E to vote in favour of the proposal, by giving 1 gold coin each. This means that this game is a strictly determined game.

4.1.2.A Analysis of the Pirate Game for 3 Players

In Figure 4.1 we have an extensive form representation of the classic Pirate Game for 3 players without taking into account particular values for the proposals. Each node in the game tree has the number of the player who will make the decision, either to Cooperate (vote yes to the proposal), or Defect. The dashed arrows represent states where the player does not have information of the current state (simultaneous move).

The green accent, in Figure 4.1, shown in the nodes represent a state where the first captain (player

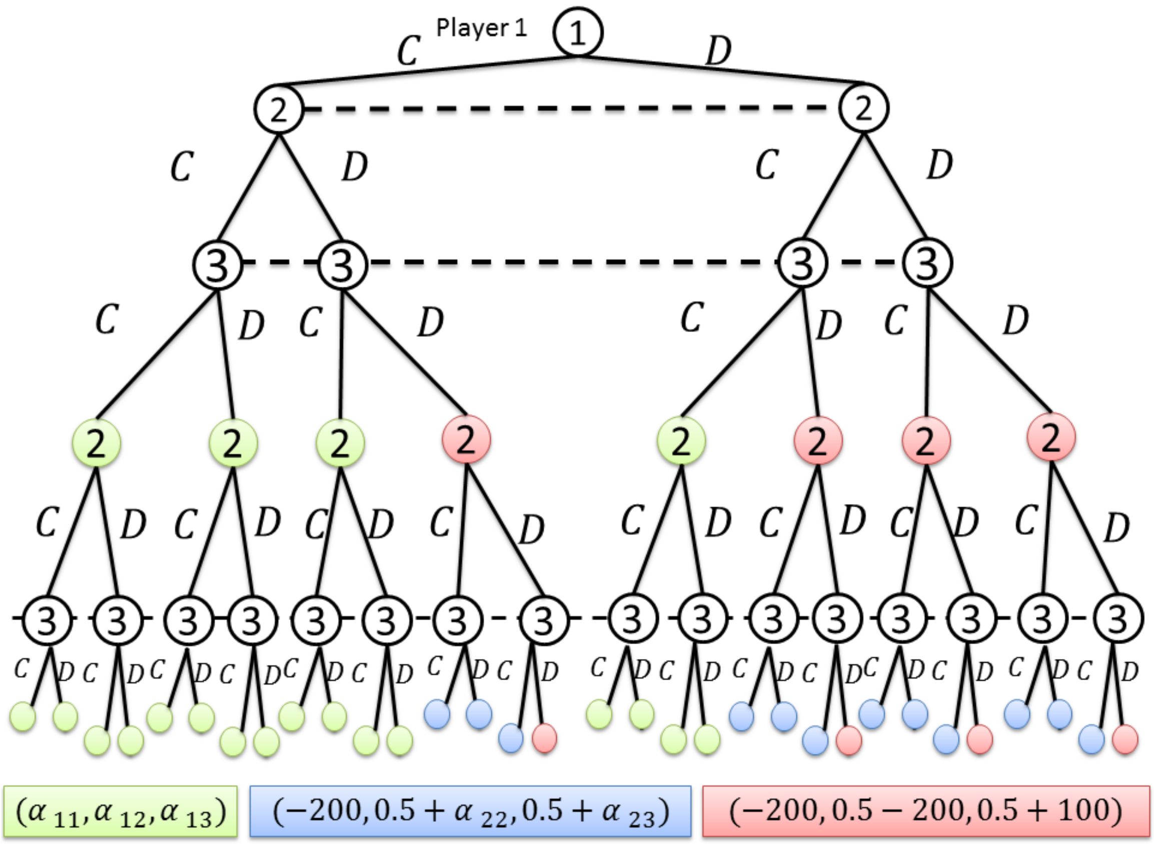

Figure 4.1: Extensive form representation of the classic Pirate Game for 3 players.

1), will see her proposal accepted, the utility associated . The blue accent denotes the outcomes where the second captain makes a proposal and has seen it accepted. The red accent color represents the outcomes where the player 3 will be the remaining pirate.

The number of coins will translate directly the utility associated with getting those coins. For example if a pirate receives 5 gold coins and the proposal is accepted he will get a utility of 5. The highest ranking pirate in the hierarchy is responsible to make a proposal to divide the 100 gold coins. This proposal means choosing the number of coins each player gets if the proposal is accepted. A captain i chooses the amount of coins a player j get; this amount will be represented by α_{ij}.

In the initial stage of the game the captain will define $\alpha_{11}, \alpha_{12}, \alpha_{13}$, and they will obey to the Equation 4.1, that imposes the rule that the captain i must allocate all the 100 coins, to the players that are still alive. N the number of pirates in the game.

$$\forall i \in \{1, 2, ..., N-1\} : \sum_{j=i}^{N} \alpha_{ij} = 100, \forall i, j : \alpha_{ij} \in \mathbb{N}_0 \tag{4.1}$$

The values for $(\alpha_{11}, \alpha_{12}, \alpha_{13}) = (99, 0, 1)$ will be the allocation that results in an equilibrium for the 3

player game . To account for all possible proposals would be overwhelming from the point of the game tree. However from the formulation of the problem, the pirates know the share proposal before voting. This means we will concentrate our analysis on the voting aspect. For each specification of the values α_{ij} we will have a different game.

The proposed goods allocation will be executed if there is a majority (or a tie), in the voting step. A step in the game consists on the highest ranking pirate defining a proposal and the subsequent vote, where all players choose simultaneously an operator.

If the proposal is rejected the captain will be thrown off board, to account for the fact that this situation is very undesirable for the captain he will receive a negative payoff of -200 (that can be seen in the blue and red outcome in Figure 4.1). This value was derived from the fact that a pirate values her integrity more than any number of coins she might receive.

> "When the result is indifferent the pirates prefer to throw another pirate overboard and thus climbing in the hierarchy."

This means that the pirates have a small incentive to climb the hierarchy. For example in the three player classical game, the third player, who has the lowest rank, will prefer to defect the initial proposal if the player 1 doesn't give her a coin, even knowing that in the second round the player 2 will be able to keep the 100 coins. We will account for this preference by assigning an expected value of half a coin (0.5), to the payoff of the players that will climb on the hierarchy if the voting fails. This tie breaker is shown in the blue and red outcomes in Figure 4.1.

4.1.2.B Consideration on the generalization for N players

We can generalize this problem for N pirates. If we assign a number to each pirate, where the captain is number 1 and the lower the number the higher the rank. If the number of coins is superior to the number of pirates, the equilibrium will have the captain (highest ranking pirate), giving a gold coin to each odd pirate, in case the number of players alive is odd, while keeping the rest to herself. When we have a even number of players the captain will assign a gold piece to each pirate with a even number, and the the remaining coins to herself.

If the number of pirates is greater than two times the amount of coins $N > 2C$, a new situation arises. If we have 100 coins and 201 pirates, the captain will not get any coin. By the same reasoning with 202 pirates the captain will still be able to survive by bribing the majority of the pirates and keeping no coins for herself. With 203 pirates the first captain will die. However with 204 pirates, the first captain will be able to survive even though he won't be able to bribe the majority, because her second in command knows that when she makes a proposal, she'll be thrown off board. In the game with 205 pirates, however the captain is not able to secure the vote from the second in command on the 204 pirate game, because the second captain is safe and she is able to make a have her proposal accepted and have the third pirate safe.

We can generalize this problem for $N > 2C$ as [43], as the games with a number of pirates equal to $2C$ plus a power of two will have an equilibrium in the first round, in the others every captain until a sub-game with a number of pirates equal to $2C$ plus a power of two will be thrown off board.

4.2 Quantum Pirate Game

The original Pirate Game is posed from the point of view of the captain. How should she allocate the treasure to the crew in order to maximize her payoff? We can find the a pure strategy Nash equilibrium in the original Pirates Game and, while the solution may seem unexpected at first sight, it is fully described using backwards induction.

When modelling this problem from a quantum theory perspective we are faced with some questions, such as:

- Will the initial conditions provide different equilibria?
- What are the similarities with the classical problem?
- Will the quantum version of the problem be equally strictly determined?
- Is it possible for a captain, in a situation where we have more than two pirates left, to acquire all the coins?

The main difference from the original problem will rely on how the system is set up and the fact that we will allow quantum strategies. We propose to study this problem for a 3 player game and trying to extrapolate for N players.

We will analyse the role of entanglement and quantum strategies in the game system.

Another aspect worth studying is the variation in the coin distribution on the payoff functions for the players. We are particularly interested in studying the classical equilibrium where the captain retains 99 coins and gives a single coin to the player with the lowest rank. Moreover we want to study what happens when the captain tries to get all the coins.

4.2.1 Quantum Model

In order to model the problem we will start by defining it using the definition of quantum game (Γ), referred in 2.57, Section 2.5 [29].

We want to keep the problem as close to the original as possible in order to better compare the results. Thus we will analyse the game from the point of view of the captain. Will her best response change?

For the purpose of demonstration this problem could be described using 3 players; the lowest number of players that has an equilibrium in which the captain has to bribe another pirate.

We begin by assigning an offset to each pirate (in order to identify her), as in the Section 4.1.1. The captain is number 1 and the lower the number the higher the rank.

4.2.1.A Game system: Setting up the Initial State

A game Γ can be viewed as a system composed by qubits manipulated by players. We will use the definition of quantum game discussed in Section 2.5 ($\Gamma = (\mathcal{H}^{2^a},\ N,\ |\psi_{in}\rangle,\ \xi,\ \{\mathcal{U}_j\},\ \{E_i\})$), to model our game system. Akin to the Quantum Ultimatum game desbribed in [25], our objective is to apply that quantization scheme to the normal form representation of the game tree in Figure 4.1.

In this 3 player game there will be 5 qubits representing the actions or the players decision; three qubits will represent the first voting round, the other two will portray the actions of the second voting round. The number of qubits needed to represent the game grows exponentially with the number of players. For N players we need $\sum_{i=2}^{N} i$ qubits. With 8 players, this game would already be impractical to simulate in a classical computer. In this regard a quantum computer may enhance our power to simulate this kinds of experiments [19].

The mapping function ξ that assigns each action/qubit labled φ_j (with $j = \{1, 2, 3, 4, 5\}$), to a player is represented on Equation 4.2.

$$\xi(j) = \begin{cases} 1 & ,\ if\ j = 1; \\ 2 & ,\ if\ j \in \{2, 4\}; \\ 3 & ,\ if\ j \in \{3, 5\}. \end{cases} \tag{4.2}$$

With 3 players and 5 actions our system with be represented in a $\mathcal{H}^{32}$ using a state ψ. This means that to represent our system we will need $2^5 \times 1$ vectors, our system grows exponentially with the number of players/qubits. Each pure basis of $\mathcal{H}^{32}$, shown in Equation 4.3, will represent a possible outcome in the game. We assign a pure basis as $|0\rangle = |C\rangle$ ("C" from "Cooperate"), and $|1\rangle = |D\rangle$ ("D" from "Defect").

$$\begin{aligned}\mathcal{B} = \{&|00000\rangle, |00001\rangle, |00010\rangle, |00011\rangle, |00100\rangle, |00101\rangle, |00110\rangle, |00111\rangle,\\ &|01000\rangle, |01001\rangle, |01010\rangle, |01011\rangle, |01100\rangle, |01101\rangle, |01110\rangle, |01111\rangle,\\ &|10000\rangle, |10001\rangle, |10010\rangle, |10011\rangle, |10100\rangle, |10101\rangle, |10110\rangle, |10111\rangle,\\ &|11000\rangle, |11001\rangle, |11010\rangle, |11011\rangle, |11100\rangle, |11101\rangle, |11110\rangle, |11111\rangle\}\end{aligned} \tag{4.3}$$

The initial system ($|\psi_0(\gamma)\rangle$), will be set up by defining an entanglement coefficient γ, that affect the way the five qubits (belonging to the three pirate players), are related; this is shown in Equation 4.5. We will entangle our state by applying the gate $\mathcal{J}$ [21]. The parameter γ becomes a way of measuring the entanglement in the system [5].

The concept of entanglement is crucial to explain some phenomena in Quantum Mechanics (Section 2.3.4).We analysed the role of the entanglement of the system since other examples researched pointed to it being the prominent factor regarding behaviour changes from the classsical perspective [29] [25] [21] [41] [44].

We can interpret the existence (or non-existence), of entanglement or superposition in the initial system as an unbreakable contract between the players [45]. The initial state starts by revealing a group of pirates that cooperate by default. We chose this initial set-up because it is prevalent in the literature [5] [29] [25] [21], and we want to test if there is any equilibrium situation where the first captain can pass her proposal while taking all the 100 coins.

Due to the nature of quantum mechanics we have to pay attention of how we set-up our architecture; we cannot copy or clone unknown quantum states (No-cloning Theorem) [19].

$$\mathcal{J} = exp\left\{\mathbf{i}\frac{\gamma}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\} \tag{4.4}$$

$$\begin{aligned}|\psi_{ini}(\gamma)\rangle = exp&\left\{\mathbf{i}\frac{\gamma}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\}|00000\rangle\\ &= cos(\frac{\gamma}{2})|00000\rangle + \mathbf{i}sin(\frac{\gamma}{2})|11111\rangle, \gamma \in (0, \frac{\pi}{2})\end{aligned} \tag{4.5}$$

4.2.1.B Strategic Space

In Equation 2.57 ($\Gamma = (\mathcal{H}^{2^a}, N, |\psi_{in}\rangle, \xi, \{\mathcal{U}_j\}, \{E_i\})$), there is the notion of a subset of unitary operators that the players can use to manipulate their assigned qubits.

Each player will be able to manipulate at least one qubit in the system. Those qubits (of the form 4.6) are $|\varphi_1\rangle$, $|\varphi_2\rangle$, $|\varphi_3\rangle$, $|\varphi_4\rangle$, and $|\varphi_5\rangle$. The Equation 4.2 assigns the qubit $|\varphi_1\rangle$ to player 1, qubits $|\varphi_2\rangle$ and $|\varphi_4\rangle$ to player 2, the remaining qubits are assigned to player 3.

$$\varphi_j = a.|C\rangle + b.|D\rangle, j \in \{1,2,3,4,5\}, \{a,b\} \in \mathbb{C} : a^2 + b^2 = 1 \tag{4.6}$$

Each player will be able to manipulate her assigned qubits j with an unitary operator of the form shown in Equation 2.58 (shown in Section 2.5 $\mathcal{U}_j(w,x,y,z) = w.I + \mathbf{i}x.\sigma_x + \mathbf{i}y.\sigma_y + \mathbf{i}z.\sigma_z$, with $w,x,y,z \in \mathbb{R} \wedge w^2+x^2+y^2+z^2 = 1$). However in order to explore the potential of quantum strategies, [5] proposes that it is sufficient to restrict the strategic space span by to the 2-parameter (polar coordinates), set of matrices in Equation 4.7, with $\theta \in (0,\pi)$, and $\phi \in (0, \frac{\pi}{2})$. We will try to use the strategic space $\mathcal{U}_j(\theta,\phi)$ to represent our player's actions.

$$\mathcal{U}_j(\theta,\phi) = \begin{bmatrix} cos(\frac{\theta}{2}) & e^{\mathbf{i}\phi}sin(\frac{\theta}{2}) \\ -e^{-\mathbf{i}\phi}sin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{bmatrix}, j \in \{1,2,3,4,5\}, \theta \in (0,\pi), \phi \in (0,\frac{\pi}{2}) \tag{4.7}$$

However the two operators that correspond to the original classical actions of voting "Yes" or to Cooperate, and voting "No" (Defect) are not entirely characterized by the subset $\mathcal{U}_j(\theta,\phi)$. These classical actions belong to a subset S_j described in Equation 4.8. The classical cooperation operator will be represented by the Identity operator (o_{j0}, where j identifies the qubit that the respective player will act upon). When assigned to a qubit this operator will leave it unchanged. This operator is described by Equation 4.7 when $\mathcal{U}(0,0)$.

The defection operator (D), is represented by one of Pauli's Operators - the Bit-flip operator. This operator was chosen because it performs the classical operation NOT on a qubit. Within the restricted space $\mathcal{U}_j$, approximate alternative for the defect operator in the set $\mathcal{U}_j$ is $\mathcal{U}_j(\pi,0)$; they are interchangeable when $\gamma = 0$. For $\gamma > 0$ we will include the pure strategy D represented by the Bit-flip operator into our restricted strategic scape $\mathcal{U}_j$.

$$S_j = \begin{cases} C_j = o_{j0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ D_j = o_{j1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{cases}, j \in \{1, 2, 3, 4, 5\} \tag{4.8}$$

Each player will have a strategy τ_i which assigns a unitary operator U_j to every qubit j that is manipulated by the player ($j \in \xi^{-1}(i)$). $\tau_2 = \{D_2, C_4\}$ represents the strategy where the player 2 votes D in the first stage and C in the second stage.

In Quantization schemas of the Prisoner's Dilemma [21] [5], and Quantum Ultimatum Game [25], the strategic space, described in Equation 4.7, was analysed allowed a infinity of mixed quantum strategies. In the Quantum Roulette Game [33] and [35] we have a demonstration that in a classical two-person zero-sum strategic game, if only one player is allowed to adopt a quantum strategy, she has a better chance of winning the game.

The notation for the pure basis as $|C\rangle$ refers to a quantum state and should not be confused as a Cooperate operator C that is a matrix (the identity matrix). A original player quantum state (the qubits of the form φ_j) is defined by Equation 4.6.

4.2.1.C Final State

We can play the Pirate Game by considering a succession of steps or voting rounds. In each step we have a simultaneous move(the players sellect their strategies at the same time), however, considering the potential rounds the game has, we have a sequential game.

With three players, the first move will correspond to the player 1 (or the captain), if the proposal fails we will proceed to the second step in the game, where the remaining two players will vote on a new proposal made by player 2 (who will be the new captain).

This final state is calculated by constructing a super-operator, by performing the tensor product of each player chosen strategy, from Equation 4.7 or 4.8. The super-operator, containing each player's strategy, will then be applied to the initial state,as shown in Equation4.9.

$$|\psi_{fin}\rangle = \otimes_{i=1}^{3} \otimes_{j \in \xi^{-1}(i)} \mathcal{U}_j |\psi_{ini}(\gamma)\rangle \tag{4.9}$$

In the Figure 4.2 we have a representation of the game. When the players select their strategies, a super operator is constructed by performing a tensor product of the selected operators.

In order to calculate the expected payoff functions we need to de-entangle the system, before measuring. The act of measuring, in quantum computing, gives an expected value that can be understood as the probability of the system collapsing into that state.

We can de-entangle the our $\mathcal{H}^{32}$ system by applying $\mathcal{J}^\dagger$ (Equation 4.9), this will produce a final state that we will be able to measure. If we do not apply the inverse transformation $\mathcal{J}^\dagger$ we are introducing errors in the system (when the entanglement parameter γ is different than 0), because we are introducing a correlation between the qubits in the system. In the Figure 4.2 we have represented the way we entangle and de-entangle the system.

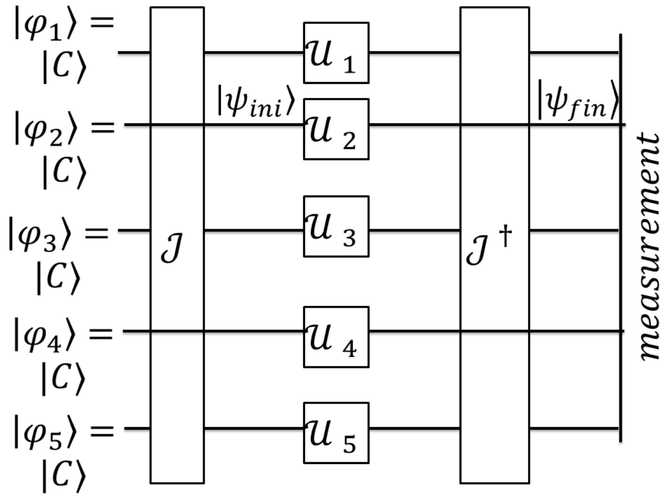

Figure 4.2: Scheme that represents the set-up of the 3-player Pirate Game. Before we measure the final result we need to apply the transpose operator $\mathcal{J}^\dagger$.

4.2.1.D Utility

To build the expected payoff functionals for the three player situation we must take into account the sub-games created when the proposal is rejected. In Figure 4.1 we can see an extensive form representation of the game.

As defined on Equation 2.59 ($E_i = \sum_{b\in\mathcal{B}} u_i(b)|\langle b|\psi_{fin}\rangle|^2, u_i(b) \in \mathbb{R}$),for each player we must specify a utility functional that attributes a real number to the measurement of the projection of a basis in the quantum state that we get after the game.

This measurement can be understood as a probability of the system collapsing into that state (that derives from the Born Rule detailed in Section 2.2.2).

These utility functions will represent the degree of satisfaction for each pirate after game by attributing a real number to a measurement performed to the system (as in Equation 2.59, Section 2.5). The real numbers used convey the logical relations of utility posed by the original problem description. Those numbers will represent the utility associated with the number of coins that a pirate gets (α_{ij}), a death penalty (-200), and a small incentive to climb the hierarchy (0.5), and were derived from the classical analysis of the problem in Section 4.1.2.A. As each pirate wants to maximize her utility, the Nash equilibrium will be thoroughly used to find the strategies that the pirates will adopt [27] [28].

We can observe in Figure 4.1 that in that we have three separate groups (denoted by the colour accents), of outcomes that share the same payoff, in the original problem. In our quantum scheme we can aggregate the pure-basis quantum states ($\mathcal{B}$), associates with a payoff in the following manner:

- States where the first proposal is accepted - "Accepted 1' (or A_1), with a green colour accent in Figure 4.1:
  - $|C, C, C, x_4, x5\rangle$ or $|0, 0, 0, x_4, x5\rangle$, with $x_4 \in \{0, 1\}$ and $x5 \in \{0, 1\}$;
  - $|D, C, C, x_4, x5\rangle$ or $|1, 0, 0, x_4, x5\rangle$, with $x_4 \in \{0, 1\}$ and $x5 \in \{0, 1\}$;
  - $|C, D, C, x_4, x5\rangle$ or $|0, 1, 0, x_4, x5\rangle$, with $x_4 \in \{0, 1\}$ and $x5 \in \{0, 1\}$;
  - $|C, C, D, x_4, x5\rangle$ or $|0, 0, 1, x_4, x5\rangle$, with $x_4 \in \{0, 1\}$ and $x5 \in \{0, 1\}$.
- States where the first captain will be eliminated and the second player gets her proposal accepted - "Accepted 2" (or A_2) with a blue colour accent in Figure 4.1:
  - $|D, D, D, C, x5\rangle$ or $|1, 1, 1, 0, x5\rangle$, with $x5 \in \{0, 1\}$;
  - $|D, D, D, D, C\rangle$ or $|1, 1, 1, 1, 0\rangle$;
  - $|D, D, C, C, x5\rangle$ or $|1, 1, 0, 0, x5\rangle$, with $x5 \in \{0, 1\}$;
  - $|D, D, C, D, C\rangle$ or $|1, 1, 0, 1, 0\rangle$;
  - $|C, D, D, C, x5\rangle$ or $|0, 1, 1, 0, x5\rangle$, with $x5 \in \{0, 1\}$;
  - $|C, D, D, D, C\rangle$ or $|0, 1, 1, 1, 0\rangle$;
  - $|D, C, D, C, x5\rangle$ or $|1, 0, 1, 0, x5\rangle$, with $x5 \in \{0, 1\}$;
  - $|D, C, D, D, C\rangle$ or $|1, 0, 1, 1, 0\rangle$.
- States where all proposals are rejected - "Rejected 2' (or R_2)' with a red colour accent in Figure 4.1:
  - $|D, D, D, D, D\rangle$ or $|1, 1, 1, 1, 1\rangle$;
  - $|D, D, C, D, D\rangle$ or $|1, 1, 0, 1, 1\rangle$;
  - $|C, D, D, D, D\rangle$ or $|0, 1, 1, 1, 1\rangle$;
  - $|D, C, D, D, D\rangle$ or $|1, 0, 1, 1, 1\rangle$.

In order to calculate the probability of the final state collapsing onto a basis state $b \in \mathcal{B}$ we perform a projection of the state in the chosen basis and we measure the squared length of the projection, $P(b) = |\langle b|\psi_{fin}\rangle|^2$ [16].

$$P(A_1) = \sum_{x_3}\sum_{x_4} |\langle 0,0,0,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_3}\sum_{x_4} |\langle 1,0,0,x_4,x5|\psi_{fin}\rangle|^2 + \\ + \sum_{x_3}\sum_{x_4} |\langle 0,1,0,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_3}\sum_{x_4} |\langle 0,0,1,x_4,x5|\psi_{fin}\rangle|^2 \tag{4.10}$$

$$P(A_2) = \sum_{x_5} |\langle 1,1,1,0,x5|\psi_{fin}\rangle|^2 + |\langle 1,1,1,1,0|\psi_{fin}\rangle|^2 + \\ + \sum_{x_5} |\langle 1,1,0,0,x5|\psi_{fin}\rangle|^2 + |\langle 1,1,0,1,0|\psi_{fin}\rangle|^2 + \\ + \sum_{x_5} |\langle 1,0,1,0,x5|\psi_{fin}\rangle|^2 + |\langle 1,0,1,1,0|\psi_{fin}\rangle|^2 + \\ + \sum_{x_5} |\langle 0,1,1,0,x5|\psi_{fin}\rangle|^2 + |\langle 0,1,1,1,0|\psi_{fin}\rangle|^2 \tag{4.11}$$

$$P(R_2) = |\langle 1,1,1,1,1|\psi_{fin}\rangle|^2 + |\langle 1,1,0,1,1|\psi_{fin}\rangle|^2 + \\ + |\langle 1,0,1,1,1|\psi_{fin}\rangle|^2 + |\langle 0,1,1,1,1|\psi_{fin}\rangle|^2 \tag{4.12}$$

If the first proposal is accepted ($P(A_1) = 1$) the players can expect to receive the following playoff (a_{11}, a_{12}, a_{13}). If $P(A_2) = 1$, then the expected payoff is $(-200, a_{22} + 0.5, a_{23} + 0.5)$. If both proposals are rejected ($P(R_2) = 1$), the players might expect to receive $(-200, -200 + 0.5, 100 + 0.5)$. The expected utility function for each player will be a weighted average of all possible outcomes. Each player has an expected utility functionals. The expected utility for player 1, shown in Equation 4.13, give a real number that represents the payoff associated with a final state. The same goes to player 2 that has her expected utility functional specified in Equation 4.14, and the player 3 in Equation 4.15.

$$E_1(|\psi_{fin}\rangle) = \alpha_{11} \times P(A_1) - \\ -200 \times (P(A_2) + P(R_2))) \tag{4.13}$$

$$E_2(|\psi_{fin}\rangle) = \alpha_{12} \times P(A_1) - \\ +(0.5 + \alpha_{22}) \times P(A_2) + \\ +(0.5 - 200) \times P(R_2) \tag{4.14}$$

$$E_3(|\psi_{fin}\rangle) = \alpha_{13} \times P(A_1) + \\ +(0.5 + \alpha_{23}) \times P(A_2) + \\ +(100.5) \times P(R_2) \tag{4.15}$$

The gate $\mathcal{J}$ is chosen to be commutative with the super-operators created by the tensor product of the classical actions C (cooperate, indicated by the identity matrix and $\mathcal{U}(0,0)$)), and D (defect, indicated by the Bit-flip operator matrix, and $\mathcal{U}(\pi,0)$ when $\gamma=0$)). For example $[\mathcal{J}, C\otimes D\otimes C\otimes C\otimes D]=0$.

This condition implies that choosing any operator from the sub-set $O=\{\mathcal{U}(\theta,0),\theta\in(0,\pi)\}$ with $\gamma=0$ is the equivalent of a classical mixed action. A strategy τ_i is a classical pure strategy iff all operators in the strategy belong to the subset S (defined in 4.8). A strategy τ_i is a classical mixed strategy iff all operators in the strategy belong to O. If the parameter ϕ in the operator $U_j(\theta,\phi)$ differs from 0, and the entanglement parameter $\gamma>0$ we are able to explore quantum strategies that have no counterpart in the classical domain [5].

5

Analysis and Results

Contents

In this chapter we will analyse and discuss the results obtained by simulating our quantization scheme for the Pirate Game.

5.1 Analysis and Results

In order to test our model we developed a Matlab simulation for the 2-player version of the game and for the 3-player version, the simulations are available in the Appendix D. When defining our game space Γ in Section 4.2.1.A we defined a number of variable parameters, such as: the entanglement coefficient γ; the coin distributions α_{ij}; the strategies $\mathcal{U}_j$ that the players might use (restricting the operator space $\mathsf{SU}(2)$).

According to [30] when the entanglement is maximal, $\mathcal{U}(\pi, 0)$ is the optimal counter-strategy for C (represented by the identity matrix), $\mathcal{U}(0, \frac{\pi}{2})$ is the optimal counter strategy for $\mathcal{U}(\pi, 0)$, D becomes the optimal counter-strategy for $\mathcal{U}(0, \frac{\pi}{2})$, and C becomes an optimal counter strategy for D.

5.1.1 2 Player Game

We simulated the Pirate Game for 2-players according to the rules defined for our quantum model in Section . The simulation can be consulted for reference in Appendix D. The classical sub-game with 2-players can be represented in Table 5.1. It is a strictly determined game, that means that there is at least a Nash Equilibrium when the players choose pure strategies [26]. In this case the outcome of the game is $(100, 0)$, when the players, use the strategies C_1 and D_2. In the quantum game we can find the following 4 pure quantum strategies :

- I
- D (also a Pauli operator σ_x)
- $\mathcal{U}(\pi, 0)$ - a quantum equivalent of a defection operator, (also $\mathbf{i}\sigma_y$).
- $\mathcal{U}(0, \frac{\pi}{2})$ - the quantum equivalent of a cooperation operator, (also $\mathbf{i}\sigma_z$).

The expected utility when players can choose only pure quantum strategies and the game is maximally entangled is shown in Figures 5.1 and 5.2. When each player has access to the 4 quantum strategies and the entanglement is maximum ($\gamma = \frac{\pi}{2}$), the game is not strictly determined anymore. The mixed strategy will make the players choose with equal probability between the 4 pure quantum strategies described above giving the players an expected payoff of $(25, 25.125)$, this outcome constitutes a Pareto optimal outcome because it is impossible to improve one player's expected utility without lowering the other's.

As there were initially 100 coins we decided to study what would happen if the captain decided to share the treasure fairly: 50 coins for player 1, 50 for player 2. A Nash equilibrium of the game would also consist in a mixed strategy where the players would choose indifferently which strategy to use, but in this case $\frac{3}{4}$ of the time player 1 would get 50 coins and $\frac{1}{4}$ he would die and receive a payoff of -200. This means that the meek captain would rather have a division "supervised" by a contract (entanglement), than to trust her crew and propose a fair division from the start.

	Player 2: C	Player 2: D
Player 1: C	(100, 0)	(100, 0)
Player 1: D	(100, 0)	(-200, 100.5)

Table 5.1: Representation of the 2 player sub-game in normal form.

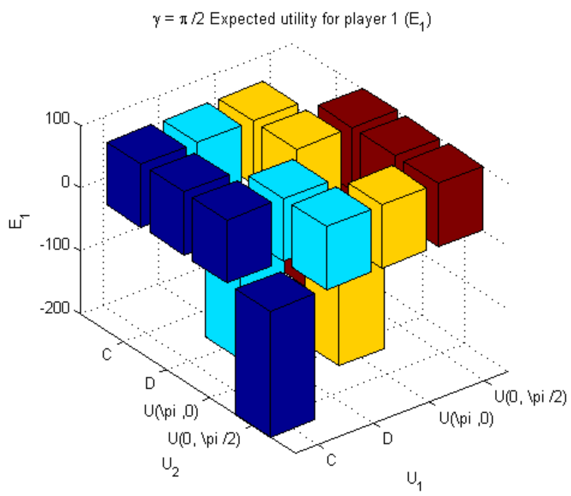

Figure 5.1: Expected utility for player 1, when both players have access to pure quantum strategies and the entanglement is maximum.

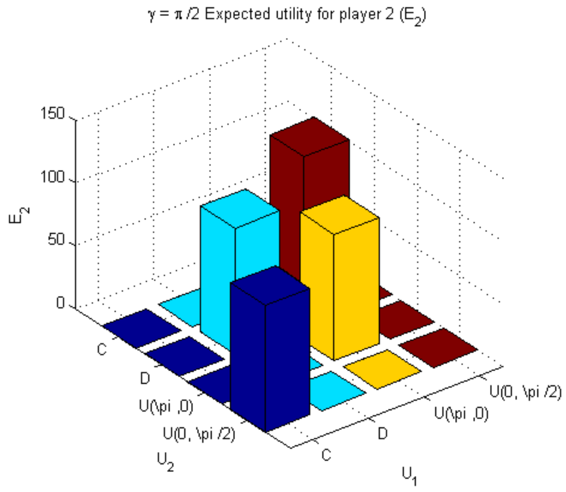

Figure 5.2: Expected utility for player 2, when both players have access to pure quantum strategies and the entanglement is maximum.

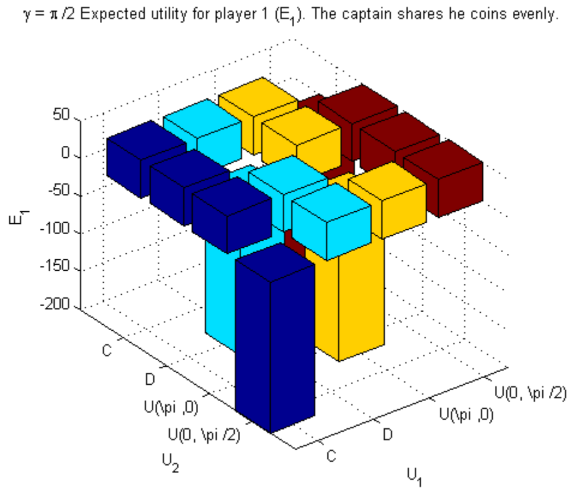

Figure 5.3: Expected utility for player 1 (the captain), when the system is maximaly entangled and he proposes a fair division of the treasure $(50, 50)$.

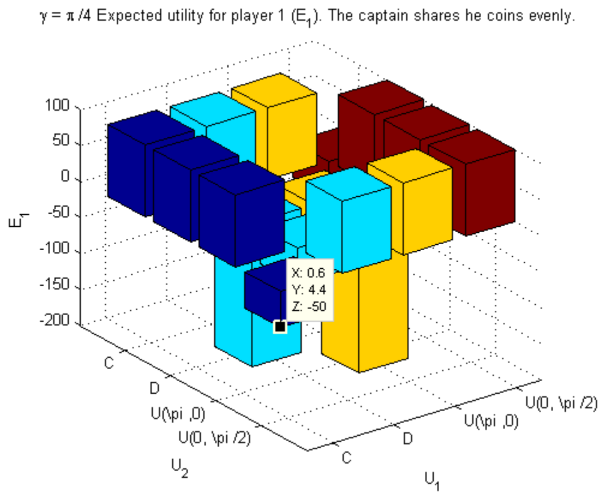

Figure 5.4: Expected utility for player 1 (the captain), when the entanglement coefficient is $\gamma = \frac{\pi}{4}$ and he proposes a division of the treasure of $(100, 0)$.

5.1.2 3 Player Game

5.1.2.A The captain proposes: $(99, 0, 1)$

The move $(C_1, D_2, C_3, C_4, D_5)$ with a proposal of $(\alpha_1, \alpha_2, \alpha_3) = (99, 0, 1)$ represents Nash Equilibrium of the classic Pirate Game (for 3 players). In the classical version, when the players chose at least 2 operators $Cooperate$ on the initial proposal the game ends right away, and the first proposal, made by player 1, is accepted ($Accepted1$ or A_1). Once again the original version of this game game is a strictly determined game because it has a pure-strategy Nash Equilibrium.

After the players make their strategic moves and the disentangle operator $\mathcal{J}^\dagger$ is applied, and the payoff functionals are calculated given the final state. The final state will be calculated as shown in Equation 5.1.

$$|\psi_{fin}\rangle = \otimes_{i=1}^{3} \otimes_{j\in\xi^{-1}(i)} \mathcal{U}_j |\psi_{ini}(\gamma)\rangle \tag{5.1}$$

As expected from the problem definition when players use only classical strategies the entanglement coefficient will not affect the final expected utilities ($[\mathcal{J}, C \otimes D \otimes C \otimes C \otimes D] = 0$). An example of this behaviour is shown in Figure 5.5, where we measure the probability A_1 (first proposal is accepted), A_2, and R_2, with different entanglement levels ($\gamma = \{0, \frac{\pi}{4}, \frac{\pi}{2}\}$).

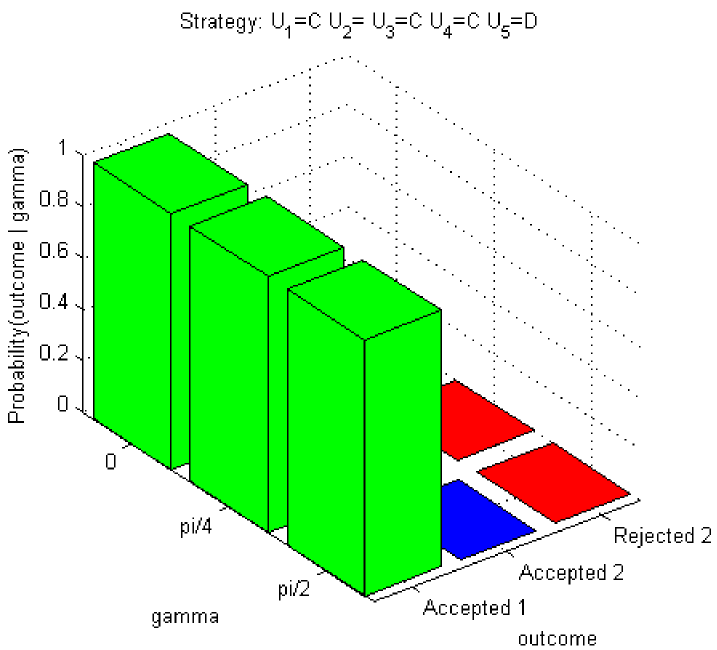

Figure 5.5: When players play classical strategies the entanglement will not affect the final result.

When each player chooses an operator from the set $O = \{\mathcal{U}(\theta, 0), \theta \in (0, \pi)\}$ and $\gamma = 0$ we have a separable game. This is an equivalent situation to the classical game, but contrarily to the classical game where we have only a Nash Equilibrium (C, D, C, C, D), we have more Equilibria, all with an outcome of $(99, 0, 1)$. This is verifiable because in the separable game the players act only in their qubits φ_j, and $D|0\rangle = |1\rangle$ and $\mathcal{U}(\pi, 0)|0\rangle = -|1\rangle$, when measuring the probability of the operators D and $\mathcal{U}(\pi, 0)$ transforming a pure state $|0\rangle$ into state $|1\rangle$ we square the probability amplitude associated with the final state. When $\gamma = 0$ C is interchangeable with $\mathcal{U}(0, \frac{\pi}{2})$, and D is interchangeable with $\mathcal{U}(\pi, 0)$.

However for $\gamma > 0$ the $\mathcal{U}(\theta, 0)$ may present an interference behaviour that is inherently quantum. This effect is shown in Figures 5.12 and 5.12, allowing us to understand the difference between the Bit-flip operator D and $\mathcal{U}(\pi, 0)$.

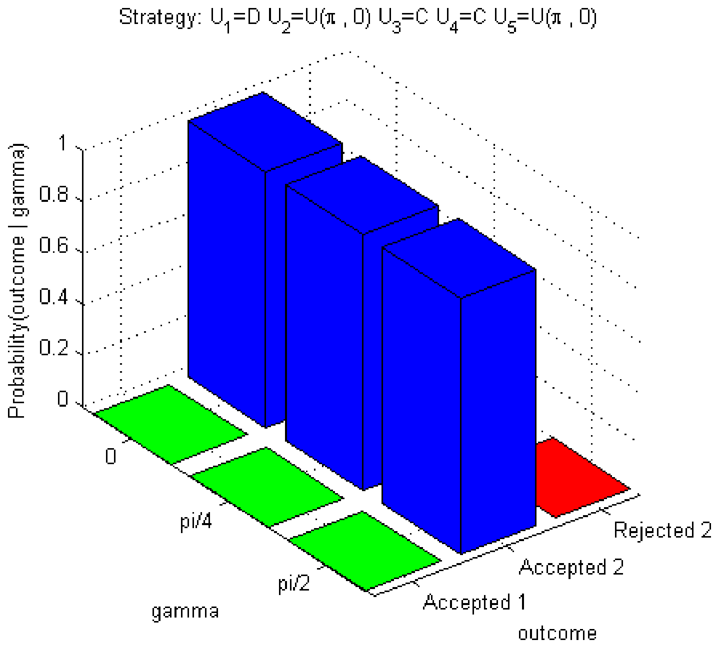

Figure 5.6: Players use the operators .

The Tables 5.2 and 5.3 represent a 3-player pirate game where given the classical equilibrium $(CDCCD)$ we test what are the outcomes if the players can use quantum strategies. By considering the quantum operators that constitute an alternative to the classical cooperation ($\mathcal{U}(0, \frac{\pi}{2})$), or defection ($\mathcal{U}(\pi, 0)$), like in the 2-player analysis when the entanglement is maximum we found that there is only a mixed strategy Nash Equilibrium because we detect a restricted cycled as suggested by [30].

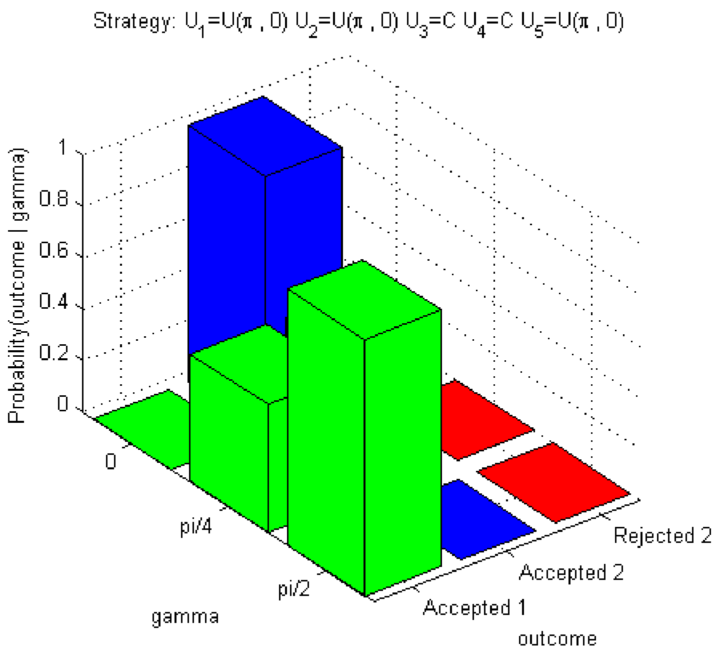

Figure 5.7: Unlike with the Bit-flip operator $D, \mathcal{U}(\pi,0)$ is affected by the entanglement coefficient γ in this case.

	Player 3: $\tau_3 = \{C, D\}$	$\tau_3 = \{C, \mathcal{U}(\pi,0)\}$
Player 2: $\tau_2 = \{D, C\}$	(99,0,1)	(-200,100.5,0.5)
$\tau_2 = \{D, \mathcal{U}(0,\frac{\pi}{2})\}$	(-200,100.5,0.5)	(99,0,1)
$\tau_2 = \{\mathcal{U}(\pi,0), C\}$	(-200,100.5,0.5)	(99,0,1)
$\tau_2 = \{\mathcal{U}(\pi,0), \mathcal{U}(0,\frac{\pi}{2})\}$	(99,0,1)	(-200,100.5,0.5)
	$\tau_3 = \{\mathcal{U}(0,\frac{\pi}{2}), D\}$	$\tau_3 = \{\mathcal{U}(0,\frac{\pi}{2}), \mathcal{U}(\pi,0)\}$
Player 2: $\tau_2 = \{D, C\}$	(99,0,1)	(-200,100.5,0.5)
$\tau_2 = \{D, \mathcal{U}(0,\frac{\pi}{2})\}$	(-200,-199.5,100.5)	(99,0,1)
$\tau_2 = \{\mathcal{U}(\pi,0), C\}$	(-200,100.5,0.5)	(99,0,1)
$\tau_2 = \{\mathcal{U}(\pi,0), \mathcal{U}(0,\frac{\pi}{2})\}$	(99,0,1)	(-200,-199.5,100.5)

Table 5.2: Player 1 plays C. τ_i represents the player i's strategy, by assigning an operator to the qubits that she controls. $\gamma = \frac{\pi}{2}$. The first captain proposes the following distribution $(99,0,1)$; the second captain proposes $(100,0)$.

	Player 3: $\tau_3 = \{C, D\}$	$\tau_3 = \{C, \mathcal{U}(\pi,0)\}$
Player 2: $\tau_2 = \{D, C\}$	(99,0,1)	(-200,100.5,0.5)
$\tau_2 = \{D, \mathcal{U}(0,\frac{\pi}{2})\}$	(-200,-199.5,100.5)	(99,0,1)
$\tau_2 = \{\mathcal{U}(\pi,0), C\}$	(-200,100.5,0.5)	(99,0,1)
$\tau_2 = \{\mathcal{U}(\pi,0), \mathcal{U}(0,\frac{\pi}{2})\}$	(99,0,1)	(-200,-199.5,100.5)
	$\tau_3 = \{\mathcal{U}(0,\frac{\pi}{2}), D\}$	$\tau_3 = \{\mathcal{U}(0,\frac{\pi}{2}), \mathcal{U}(\pi,0)\}$
Player 2: $\tau_2 = \{D, C\}$	(-200,100.5,0.5)	(99,0,1)
$\tau_2 = \{D, \mathcal{U}(0,\frac{\pi}{2})\}$	(99,0,1)	(-200,-199.5,100.5)
$\tau_2 = \{\mathcal{U}(\pi,0), C\}$	(99,0,1)	(-200,100.5,0.5)
$\tau_2 = \{\mathcal{U}(\pi,0), \mathcal{U}(0,\frac{\pi}{2})\}$	(-200,-199.5,100.5)	(99,0,1)

Table 5.3: Player 1 plays $\mathcal{U}(0,\frac{\pi}{2})$.$\gamma = \frac{\pi}{2}$. The first captain proposes the following distribution $(99,0,1)$; the second captain proposes $(100,0)$.

5.1.2.B The captain proposes: $(100, 0, 0)$

Suppose captain is greedy and proposes to get the 100 coins. In the classical Pirate Game this would pose a conflict with his self-preserving needs. A pertinent question would be if this Quantum Model of the Pirate Game would allow the first captain to approve that allocation proposal.

If the captain is the only one with access to quantum strategies, and the other players do not know it and play the classical strategies C (identity matrix) and D (Bit-flip operator), there is a dominant quantum strategy that allows her to pass her proposal and get all the 100 coins, for $\gamma = \frac{\pi}{2}$. Figure 5.9 provides an example that corroborates where we verify that player 1 has a set of dominant strategies (for $\tau_1 = \mathcal{U}_1(\pi, \phi), \phi \in (0, \frac{\pi}{2})$ or $\tau_1 = \mathcal{U}_1(\theta, \frac{\pi}{2}), \theta \in (0, \pi)$). We verified experimentally that the sub-games with 2 players (when $(C_4, C_5), (C_4, D_5), (D_4, C_5), (D_4, D_5)$), did not affect this dominant strategy. If fact if we analyse the expected utility function for player 1 (E_1.), and the Figure 4.1 representing the extensive form game, we verify that the the first captain is indifferent to the results of the sub-game with 2-players, when both player play a classical strategy, because she is already dead.

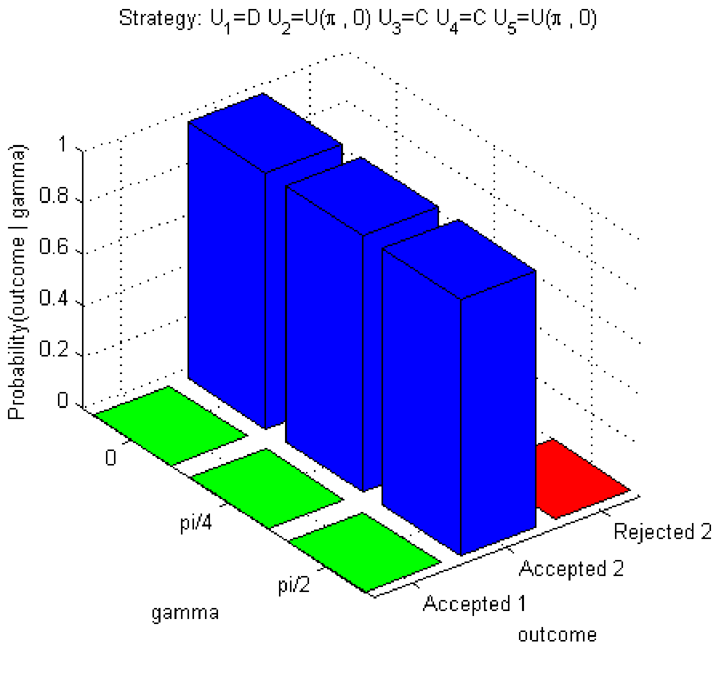

Figure 5.8: Player 1 uses the classical operator D. The probabilities don't change with the entanglement.

If players 2 and 3 know that the captain has access to quantum strategies, with a maximally entangled game, and they have the restricted sub-set of pure classical strategies C_j and D_j, if they both play C, the dominant strategy for the captain becomes $\tau_1 = \mathcal{U}_1(0, 0)$. Their best response becomes playing a mixed strategy where half the time they will play C and the other half D. The best response for player 1 in this case is to play a mixed quantum strategy where half the time she plays $\mathcal{U}(0, 0)$, half $\mathcal{U}(0, pi/2)$.

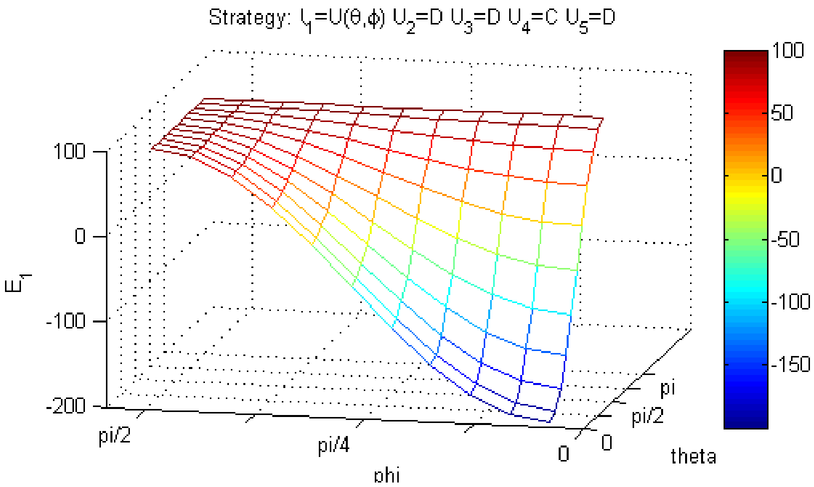

Figure 5.9: Expected utility for player 1 when she adopts a strategic move of the form $U_1(\theta, \phi)$ and the other players play D_2, D_3, C_4, D_5.

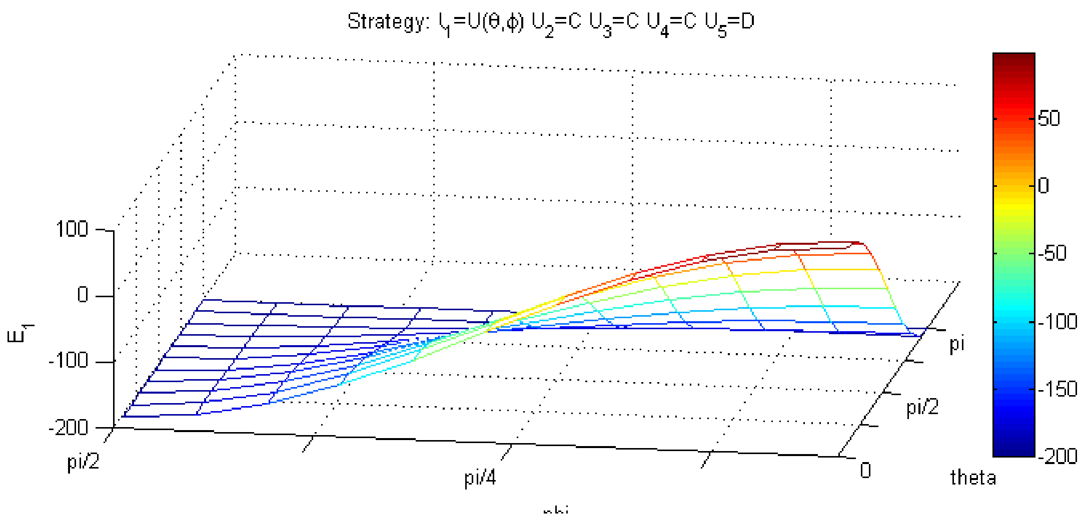

Figure 5.10: Players 2 and 3 use a mixed classical strategy to choose the operators O_2 and O_3, where half the time they will play C and the other half D.

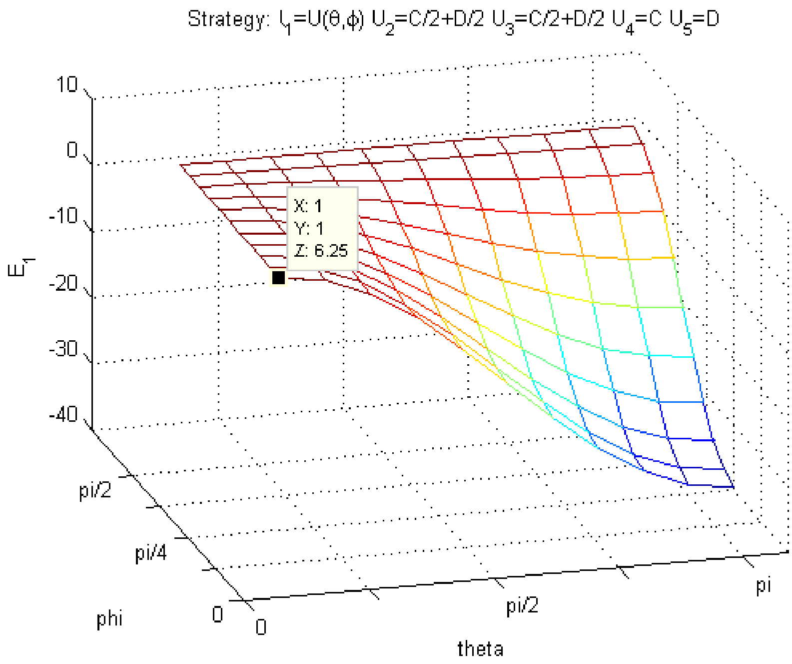

Figure 5.11: Players 2 and 3 use the operators C_2 and C_3.

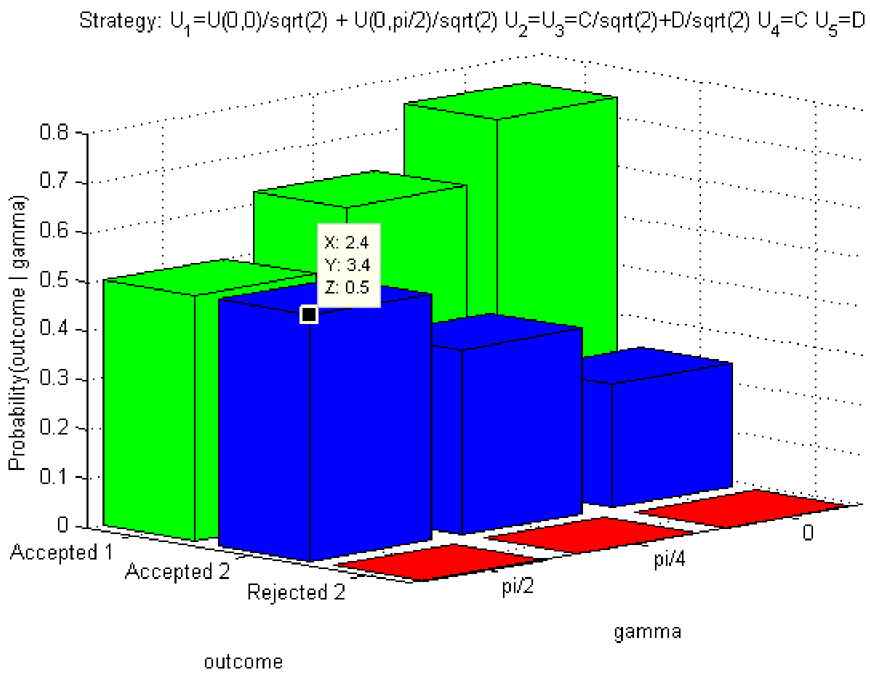

Figure 5.12: Players 2 and 3 use the operators C_2 and C_3.

5.2 Discussion

We tried to get our Quantum Model of the Pirate Game as close as possible in order to compare the the original game. However we did not incorporate the concept of measuring the results between rounds of voting. In the original problem after each stage the results are accounted. Our approach was deemed more adequate to analyse in [25].

In works such as the Prisoner's Dilemma [21] [5], and Quantum Ultimatum Game [25], the strategic space was analysed allowing a infinity of mixed quantum strategies. Our game space was much more vast than the works previously mentioned, we used 5 qubits to model our system, their models had no more than 3 qubits. This imposed that in order to analyse our system while conceptually, our strategic space is also infinite, we restricted in order to analyse. In particular we payed special interest in the strategies: C, D, $\mathcal{U}(\pi, 0)$, and $\mathcal{U}(0, \frac{\pi}{2})$. The first two mentioned strategies are the equivalent of the classical strategies "Cooperate" and "Defect", the other two are quantum strategies. Together these 4 strategies when paired with the coefficients (w, x, y, z) are known to span a class that contains all unitary operators, the $\mathrm{SU}(2)$. According to [30] these 4 operators when the entanglement is maximum provided a cycle where each operator was an optimal counter strategy to other.

The set-up of the initial system was crucial to introduce the phenomenon of entanglement. We concluded that without entanglement the game has a strictly determined solution and behaves as the original problem, however there are more pure strategy Nash Equilibria.

For the 2-player game we concluded that the expected utility when the players use a mixed strategy Nash equilibrium renders an expected utility of $(25, 25.125)$ which is also a Pareto optimal solution because it is not possible to improve one player's expected utility without harm the other. The classical setting is more beneficial to the captain in this game, because she will receive the 100 coins. We also found that distributing 50 coins would give a lower expected utility to the captain. This result is interesting because it is a Pareto optimal solution in the classical version, though not a Nash Equilibrium, and it renders an higher payoff than the Nash Equilibrium and Pareto Optimal solution when the system is maximally entangled and the players have access to the 4 pure quantum strategies discussed in [30] and [21].

The 3-player game also has a mixed quantum strategy Nash Equilibrium, like the 2-player game, when the entanglement is maximum, however its calculus fell beyond the scope of this dissertation.

When trying to find if it is possible for the captain in the first stage of the game to acquire all gold coins we found that if the other players have a restricted set of strategies, they can only use the classical Cooperate or Defect operators, and if they don't know that the captain has access to quantum strategies, the captain will be able to get the 100 gold coins. However if players 2 and 3 have a restricted set where they can only use the classical Cooperation operator, or the classical Defection operator, the player 1 will no longer be able to acquire the 100 coins with certainty. Instead she may be able to acquire the 100 coins with probability $\frac{1}{2}$ or end up thrown off board with equal probability.

These results corroborate the literature in the Chapter 3 and with the Quantum Prisoner's Dilemma, while contributing with another case study.

6

Conclusions and Future Work

Contents

This section closes this document. It provides an overview on the work, presents a summary of the main results and relevant contributions. Moreover we compiled a list of interesting points worth pursuing in the future.

From an historic point of view there is a character who is active on the fields of Quantum Mechanics, Game Theory, and Computer Science (among others), John von Neumman. von Neumman died at the age of 53 years, so we can only speculate if he would ever try to apply the principles of Quantum Mechanics to Game Theory.

With this dissertation we developed an approach to a novel Quantum Game - the Pirate Game. While defining the quantum model for the game we found that the most unique features of the problem, that appear when the number of coins does not allow for the captain to bribe the necessary votes in order for him to be spared, would pose an hard problem to analyse because of the exponential growth the system experiences when we increase the number of qubits. Possibly with the advent of the first commercial quantum computers these we will have resources to simulate these kinds of systems in a more spacial optimized way - the physical model for a qubit can be the polarization of a photon.

Nevertheless we developed a model for the Quantum Pirate Game with 3 players, and we also studied the system for 2 players. Both games are strictly determined in the original problem, however when we allow the system to be entangled we start noticing that there are quantum operators that can cause interference. When the game is maximally entangled, we could not find a pure strategy Nash Equilibrium. These results are similar to the findings in models such as the Prisoner's Dilemma [5] [21], and Stackelberg Duopoly [41].

We also found that, when the system is maximally entangled, if we restrict the strategic space for some players we break the cycle of optimal quantum counter-strategies suggested by [30]: when the entanglement is maximal, $\mathcal{U}(\pi, 0)$ is the optimal counter-strategy for C (represented by the identity matrix), $\mathcal{U}(0, \frac{\pi}{2})$ is the optimal counter strategy for $\mathcal{U}(\pi, 0)$, D becomes the optimal counter-strategy for $\mathcal{U}(0, \frac{\pi}{2})$, and C becomes an optimal counter strategy for D. This means that it is possible for a player to explore this weakness if other players are unaware. If we had a quantum gambling system, or a voting system such as in the Pirate Game, the strategic space and the entanglement are information that needs to be available for all players equally in order for them to choose their best-response.

Another contribution from this thesis is the simulation code in Matlab provided in the appendices, these examples might be used in order to experiment the quantum models of the Prisoner's Dilemma, a Quantum Roulette, and finally the Quantum Pirate Game. We also developed a Matlab simulation of the Quantum algorithms Walk in a Line. These simulations were built while analysing related work on the area.

The implications of Quantum Mechanics are still very detached from the way we think. The Quantum theory is deeply rooted in the concept of probability. Robert Laughlin uses the concept of emergence to explain the way classical phenomena arises from quantum mechanics [3]. It is almost poetic that in our apparently deterministic world arises with all its patterns surges from the chaos.

6.1 Future Work

The principles of quantum computing provide an extremely rich source of ideas to extend other fields of knowledge. Some suggestions for possible extensions for this work would be:

1. **To make conditional measurements by applying the Lüder's Rule** We mentioned a proposal for an extensive form game that uses measurements to separate sequential stages in the game in Section 3.2.2.A, while discussing a quantum approach to the Ultimate Game. The objective would be to separate each stage of the game with a measurement operation. After a stage where the players made their strategic moves, the resulting quantum state would be de-entangled, projected in order to separate the sub-game where the actual proposal was accepted and the complementary state where the proposal was not accepted. The game would be then re-entangled with a new entanglement coefficient.
2. **Optimize this Quantum Pirate Game Model.** To model this game as a Quantum Bayesian Network. The Bayesian Networks (also referred to as Belief Networks, Probabilistic networks or causal networks) consist of an Directed Acyclic Graph to represent a set of conditional dependencies between random variables, that provide a compact description that allows to calculate a joint probability distribution, having in mind those dependencies into account. [46] proposes a model for Quantum Bayesian Networks, so designing a Quantum Bayesian Network approximation for the Quantum Pirate Game, having the description of this model as a starting point, could be a relevant research topic.
3. **To implement and test this quantum model with human subjects.** The field of Quantum Cognition tries to explain the Human reasoning process by using principles from quantum game theory, namely quantum probabilities. To expose subjects to a quantum model and having them try to take advantage of the rules of the system might provide valuable insights to the way we reason.
4. **Increase the number of players.** Studying the quantum system with 8 players and 1 gold coin, would be an interesting extensions for this work. This would allow to experiment the bizarre survival situations that happens when there are not enough coins for the captain to bribe the other pirates. However this particular case of the pirate game would need a 35-qubit system in order to be studied (8 qubits for the first stage, 7 for the second stage, until reaching a 2-player sub-game).

In a classical computer and full joint probability this would mean working with vectors in the scale 10^{10}.

5. **A graphical user interface (GUI) for the Pirate Game.** This would provide a more approachable way to tweak parameters.

6. **To create and structure a platform to promote scientific dissemination of quantum computing.** While investigating and developing this solution, there was often the thought that it would be important for undergraduate Computer Science students to come with contact with this paradigm. This field is still shrouded with mystery to many people, and this makes it a priority topic for scientific promulgation. In this work we tried to present examples that someone with a basic understanding of Algebra may be able to follow.

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A

Matlab Simulation: Discrete Quantum Walk on a Line

```
%— number of steps in the simulation
steps= 30;

%— hadamard matrix
H = [1/sqrt(2) 1/sqrt(2); 1/sqrt(2) −1/sqrt(2) ];
%— symetric matrix
M = [1/sqrt(2) 1i/sqrt(2); 1i/sqrt(2) 1/sqrt(2) ];
%— coin flip unitary operator
C = H;
C=M;

%— coin flip matrix
CM= zeros((steps*2+1),2);
%— shift matrix
SM= zeros(steps*2+1,2);

%— middle index
i0= steps+1;

%— initialize flip probability amplitudes (coin has 1/2 chance of going +1 or −1)
CM(i0,1)=1/sqrt(2);
CM(i0,2)=1/sqrt(2);

for i=1:steps
   %— clean SM
   SM= zeros(steps*2+1,2);

   for j=1:(steps*2+1)
       if CM(j, 1)≠0
           SM(j−1,1)=CM(j, 1);
       end
       if CM(j, 2)≠0
           SM(j+1,2)=CM(j, 2);
       end
   end
   SM;
   %disp('−−−−−−');
   %— clean CM
   CM= zeros(steps*2+1,2);
   for j=1:(steps*2+1)
       if SM(j, 1)≠0
           CM(j, 1)= CM(j, 1)+C(1,1)*SM(j, 1);
           CM(j, 2)= CM(j, 2)+C(1,2)*SM(j, 1);
       end
       if SM(j, 2)≠0
           CM(j, 1)= CM(j, 1)+C(2,1)*SM(j, 2);
           CM(j, 2)= CM(j, 2)+C(2,2)*SM(j, 2);
       end
   end
   CM;

   %Display

   figure

   probability=zeros(steps*2+1);

   for j=1:steps*2+1
      probability(j)=abs(SM(j,1)).^2+ abs(SM(j,2)).^2;
   end
   axisP = −steps : steps;
   plot(axisP,probability)
   title(strcat('Step: ' ,num2str(i)))
   axis([−steps−1 , steps+1, 0, 1]);
   grid on
   ylabel(strcat('Probability(State|Step=',num2str(i), ')'));
   xlabel('States');

   %Display
end
```

B

Quantum Prisioner's Dillema

```
% Quantum Prisioner's Dillema
function out = quantumprisionersdillema(g,T,R,P,S,U1,U2)
%quantumprisionersdillema(g)
%
%                Simulates the payoff of 2-player prisioner's dillema game
%
%        IN:
%                g   : entanglement coefitient, by defaut g=0 (no
%                entanglement)
%                T   : temptation, by defaut T=3
%                R   : reward, by defaut R=2
%                P   : punishment, by defaut P=1
%                S   : suckers, by defaut S=0
%                U1   : Player 1 strategy
%                U2   : Player 2 strategy
%
% To have a prisoner's dilemma game according to the cannonical form,
% it must respect:
% T > R > P > S
%            P      Player 2
%            l        C       D
%            a  C: (R, R)   (S, T)
%            y  D: (T, S)   (P, P)
%            e
%            r
%            1
%
%        OUT:
%                out: 5x2 matrix, column 1 has the expected utility for
%                     player 1, column 2 has the expected utility for
%                     player 2. A possible outcome corresponds to each
%                     line like [CC;CD;DC;DD;U1U2]
%
% All parameters are optional
%-------------------------%

%-- entanglement coeficient (checks if exists)
%-- Check variables and set to defaults
if exist('g','var')≠1, g=0; end

%-- Utility
%-- Check variables and set to defaults
if exist('T','var')≠1, T=3; end
if exist('R','var')≠1, R=2; end
if exist('P','var')≠1, P=1; end
if exist('S','var')≠1, S=0; end

%-- Strategies
if exist('U1','var')≠1, U1=[0 1i;1i 0]; end
if exist('U2','var')≠1, U2=[0 1i;1i 0]; end

%-- Actions
%   cooperate= [1 0;0 1]
C= eye(2);
%   defect= [0 1;1 0]
D= ones(2)-eye(2);

%-- Building the initial state
ini= cos(g)*kron([1 0]',[1 0]') + 1i*sin(g)*kron([0 1]',[0 1]');
%-- Deentangles to produce a final state
Jt = ctranspose( expm(1i*(g)*kron(D,D)));
out= zeros(5,2);
%-- Simulation a outcome where player1 (C)ooperates
%   and player 2 (C)ooperates
   finCC= Jt*kron(C,C)*ini;
   out(1,1)=payofffunc_player1(finCC, T, R, P, S);
   out(1,2)=payofffunc_player2(finCC, T, R, P, S);
%-- Simulation a outcome where player1 (C)ooperates
%   and player 2 (D)efects
   finCD= Jt*kron(C,D)*ini;
   out(2,1)=payofffunc_player1(finCD, T, R, P, S);
   out(2,2)=payofffunc_player2(finCD, T, R, P, S);
%-- Simulation a outcome where player1 (D)efects
```

```
%   and player 2 (C)ooperates
   finDC= Jt*kron(D,C)*ini;
   out(3,1)=payofffunc_player1(finDC, T, R, P, S);
   out(3,2)=payofffunc_player2(finDC, T, R, P, S);
%-- Simulation a outcome where player1 (D)efects
%   and player 2 (D)efects
   finDD= Jt*kron(D,D)*ini;
   out(4,1)=payofffunc_player1(finDD, T, R, P, S);
   out(4,2)=payofffunc_player2(finDD, T, R, P, S);
%-- Simulation a outcome where player1 plays U1
%   and player 2 U2
   finUU= Jt*kron(U1,U2)*ini;
   out(5,1)=payofffunc_player1(finUU, T, R, P, S);
   out(5,2)=payofffunc_player2(finUU, T, R, P, S);
end

function u = payofffunc_player1(fin, T, R, P, S)
%payofffunc_player1(fin)
%
%                 Calculates the payoff for player 1
%               IN
%                    fin: final state
%                    T, R, P, S: payoff numbers
%               OUT
%                    u: expected utility for player 1
%(norm(conj([0 1 1 1 0 1 1 1]').*fin2CC))^2
    u= R*(norm(conj([1 0 0 0]').*fin))^2 + T*(norm(conj([0 0 1 0]').*fin))^2+ ...
        P*(norm(conj([0 0 0 1]').*fin))^2;
end

function u = payofffunc_player2(fin, T, R, P, S)
%payofffunc_player1(fin)
%
%                 Calculates the payoff for player 2
%               IN
%                    fin: final state
%                    T, R, P, S: payoff numbers
%               OUT
%                    u: expected utility for player 2
    u= R*(norm(conj([1 0 0 0]').*fin))^2 + T*(norm(conj([0 1 0 0]').*fin))^2+ ...
        P*(norm(conj([0 0 0 1]').*fin))^2;
end
```

C

Quantum Roullete

```
function []= quantum_roullete3()
%%
% Simulation based on
% S. Salimi and M. M. Soltanzadeh, "Investigation of quantum roulette arXiv : 0807 . ...
    3142v3 [ quant-ph ] 30 Apr 2009," 2009.

%%

N=3
%% make states
I = eye(N);
D = (1/N)*ones(N);


%% permutation matrices change between states
 X0 = circshift(I, 0);
%X0 = I;
 X1 = circshift(I, 1);
%X1 = [0 1 0; 1 0 0; 0 0 1];
 X2 = circshift(I, 2);
%X2 = [0 0 1; 0 1 0; 1 0 0];
 X3 = circshift(I, 3);
%X3 = [1 0 0; 0 0 1; 0 1 0];
X4 = circshift(I, 4);
%X4 = [0 0 1 ; 1 0 0; 0 1 0];
 X5 = circshift(I, 5);
%X5 = [0 1 0; 0 0 1; 1 0 0];


%%Fourier Matrix
F= (1/sqrt(N))*fft(I)

T0 = circshift(F,[0 0]);
T1 = circshift(F,[0 1]);
T2 = circshift(F,[0 2]);
%%

%%Step1

%Assuming Alice places the roullete in state 2 were both player can see
Ro0= [0; 1; 0]*[0; 1; 0]'

%%Step2

%As Alice chose state 2, Bob will select T1 to rotate the state
Ro1 = T1 * Ro0 * T1'

%step 3 -
Ro1 = T2 * Ro1 * T2'

%%Step3
%Alice will play again
%

Ro2 = (2/6)*X0*Ro1*X0' + (1/12)*X1*Ro1*X1' +(1/12)*X2*Ro1*X2' +(1/6)*X3*Ro1*X3' ...
    +(1/6)*X4*Ro1*X4' +(1/6)*X5*Ro1*X5'

%%Step4
%Bob can choose which state he wants

Ro3 = T0'*Ro2*T0

Ro3 = T1'*Ro2*T1

Ro3 = T2'*Ro2*T2

Ro4 = (2/6)*X0*Ro3*X0' + (1/12)*X1*Ro3*X1' +(1/12)*X2*Ro3*X2' +(1/6)*X3*Ro3*X3' ...
    +(1/6)*X4*Ro3*X4' +(1/6)*X5*Ro3*X5'

Ro3 = T0'*Ro3*T0
end
```

D

Results: Pirate Game

D.1 2 Player Game

D.1.1 Simulation

```
% Quantum Pirate Game
function out = quantum2piratesanalise()
    C= eye(2);
%   defect= [0 1;1 0]
D= ones(2)—eye(2);

Dy= D;
Dy(2,1)=—1;
Dz= [1i 0;0 —1i];

g=pi/2;

u1=zeros(4,4);
u2=zeros(4,4);

for s1=1:1:4


    switch s1
    case 1
        U1=C;
    case 2
        U1=D;
    case 3
        U1=Dy;
    otherwise
        U1=Dz;
end
for s2=1:1:4

   switch s1
    case 1
        U2=C;
    case 2
        U2=D;
    case 3
        U2=Dy;
    otherwise
        U2=Dz;

   end

   out=quantum2pirates(g,U1,U2);
    u1(s1,s2)=out(1);
    u2(s1,s2)=out(2);

end

end

figure


h=bar3(u1);

% set(h(1),'facecolor','green');
% set(h(2),'facecolor','blue');
% set(h(3),'facecolor','red');
%mesh(prob);
%

set(gca,'yTickLabel',{'C', 'D',  'U(\pi ,0)', 'U(0, \pi /2)'})
% set(gca,'xTick',0:length(t)/4:length(t))
set(gca,'xTickLabel',{'C', 'D',  'U(\pi ,0)', 'U(0, \pi /2)'})
```

```
ylabel('U_2');
xlabel('U_1');
zlabel('E_1');

title('\gamma = \pi /2 Expected utility for player 1 (E_1)')

figure


h=bar3(u2);

% set(h(1),'facecolor','green');
% set(h(2),'facecolor','blue');
% set(h(3),'facecolor','red');
%mesh(prob);
%

set(gca,'yTickLabel',{'C', 'D',  'U(\pi ,0)', 'U(0, \pi /2)'})
% set(gca,'xTick',0:length(t)/4:length(t))
set(gca,'xTickLabel',{'C', 'D',  'U(\pi ,0)', 'U(0, \pi /2)'})
ylabel('U_2');
xlabel('U_1');
zlabel('E_2');

title('\gamma = \pi /2 Expected utility for player 2 (E_2)')

end

function out = quantum2pirates(g,U1,U2)
%quantumprisionersdillema(g)
%
%                 Simulates the payoff of 2-player prisioner's dillema game
%
%         IN:
%                 g   : entanglement coefitient, by defaut g=0 (no
%                 entanglement)
%
%                 U1   : Player 1 strategy
%                 U2   : Player 2 strategy
%
% All parameters are optional
%-------------------------%

%-- entanglement coeficient (checks if exists)
%-- Check variables and set to defaults
if exist('g','var')≠1, g=0; end

%-- Utility
%-- Check variables and set to defaults
a22=100;
a23=0;
death=-200;
incentive=0.5;


%-- Strategies
if exist('U1','var')≠1, U1=[0 1i;1i 0]; end
if exist('U2','var')≠1, U2=[0 1i;1i 0]; end

%-- Actions
%   cooperate= [1 0;0 1]
C= eye(2);
%   defect= [0 1;1 0]
D= ones(2)-eye(2);

%-- Building the initial state
ini= cos(g/2)*kron([1 0]',[1 0]') + 1i*sin(g/2)*kron([0 1]',[0 1]');
%-- Deentangles to produce a final state
Jt = ctranspose( expm(1i*(g/2)*kron(D,D)));

   fin= Jt*kron(U1,U2)*ini;
   u1=payofffunc_player1(fin, a22, a23, death, incentive);
   u2=payofffunc_player2(fin, a22, a23, death, incentive);
```

```
    out= [u1 u2];
end

function u = payofffunc_player1(fin, a22, a23, death, incentive)
%payofffunc_player1(fin)
%
%                 Calculates the payoff for player 1
%               IN
%                    fin: final state
%                    a22, a23, death, incentive: payoff coefficients
%               OUT
%                    u: expected utility for player 1
%(norm(conj([0 1 1 1 0 1 1 1]').*fin2CC))^2
    u= a22*(measure2([1 0 0 0]',fin)+measure2([0 1 0 0]',fin)+measure2([0 0 1 0]',fin) ...
        )-200*measure2([0 0 0 1]',fin) ;
end

function u = payofffunc_player2(fin, a22, a23, death, incentive)
%payofffunc_player1(fin)
%
%                 Calculates the payoff for player 2
%               IN
%                    fin: final state
%                    a22, a23, death, incentive: payoff coefficients
%               OUT
%                    u: expected utility for player 2
    u= a23*(measure2([1 0 0 0]',fin)+measure2([0 1 0 0]',fin)+measure2([0 0 1 0]',fin) ...
        )+(a22+a23+incentive)*measure2([0 0 0 1]',fin) ;
end

function m = measure2(b,fin)
m=abs( sum(b.*conj(fin)))^2;
end
```

D.2 3 Player Game

D.2.1 Simulation

```


function out = simulationQuantumPirateGame3Players(a_11,a_12,a_13,a_22,a_23, gamma, ...
    U1,U2,U3,U4,U5)
%
%         IN:
%-- alocation proposal a_ij -i number of the player that proposes a_ij coins
%   to player j
%-- gamma entanglemen coefitient
%-- U1, U2, U3, U4, U5 player's strategies; default CDCCD
%
%        OUT:
%-- expected utility for players 1 ,2 ,3:  [u1 u2 u3]

%-- Check variables and set to defaults
if exist('a_11','var')≠1, a_11=99; end
if exist('a_12','var')≠1, a_12=0; end
if exist('a_13','var')≠1, a_13=1; end
if exist('a_22','var')≠1, a_22=100; end
if exist('a_23','var')≠1, a_23=0; end
if exist('gamma','var')≠1, gamma=0; end

%-- Classic Pure Strategy Operators
%   cooperate= [1 0;0 1]
C= eye(2);
%   defect= [0 1;1 0]
```

```
D= ones(2)-eye(2);
D(2,1)=1;

if exist('U1','var')≠1, U1=C; end
if exist('U2','var')≠1, U2=D; end
if exist('U3','var')≠1, U3=C; end
if exist('U4','var')≠1, U4=C; end
if exist('U5','var')≠1, U5=D; end

%-- Matrix containing all pure-states for hilbert space 32
H32B=eye(32);

step=0.1;
t=0:step:pi;

i=gamma;

%-- varying the entanglement parameter

%-- Building the initial state, for the entanglement parameter i
ini= cos(i/2)*kron([1 0]', kron([1 0]',[1 0]'))+1i*sin(i/2)*kron([0 1]', kron([0 1]',[0 ...
    1]'));
%-- Entanglement Gate J
J= expm(1i*(i/2)*kron(D,kron(D,kron(D,kron(D,D)))));
%-- Alternative way to build the initial state
ini= J*H32B(:,1);
%-- Deentangles to produce a final state
Jd = ctranspose( J);

H= fft(eye(2))/sqrt(2);
fin = kron(U1,kron(U2,kron(U3,kron(U4,U5))))*ini;
fin= Jd*fin;
out = expectedUtility(fin, a_11,a_12,a_13,a_22,a_23);

end


function out = expectedUtility(fin, a_11,a_12,a_13,a_22,a_23)
%payofffunc_player1(fin)
%
%                Calculates the payoff for player 1
%              IN
%                    fin: final state
%

%-- Check variables and set to defaults
if exist('a_11','var')≠1, a_11=99; end
if exist('a_12','var')≠1, a_12=0; end
if exist('a_13','var')≠1, a_13=1; end
if exist('a_22','var')≠1, a_22=100; end
if exist('a_23','var')≠1, a_23=0; end

%-- Matrix containing all pure-states for hilbert space 32
H32B=eye(32);

prob_proposal_1_accepted=0;
for accepted=[1 2 3 4 5 6 7 8 9 10 11 12 17 18 19 20]
    prob_proposal_1_accepted= prob_proposal_1_accepted + measure(H32B(:,accepted),fin);
end
prob_proposal_1_accepted;

prob_proposal_1_rejected=0;
for rejected=[13 14 15 16 21 22 23 24 25 26 27 28 29 30 31 32]
    prob_proposal_1_rejected= prob_proposal_1_rejected + measure(H32B(:,rejected),fin);
end
prob_proposal_1_rejected;

prob_proposal_2_accepted=0;
for accepted=[13 14 15 21 22 23 25 26 27 29 30 31]
    prob_proposal_1_accepted= prob_proposal_1_accepted + measure(H32B(:,accepted),fin);
end
prob_proposal_2_accepted;

```

```
prob_proposal_2_rejected=0;
for rejected=[16 24 28 32]
    prob_proposal_1_rejected= prob_proposal_1_rejected + measure(H32B(:,rejected),fin);
end
prob_proposal_2_rejected;

u1 = a_11*prob_proposal_1_accepted -200* prob_proposal_1_rejected;

u2 = a_12*prob_proposal_1_accepted +(0.5 + a_22)*prob_proposal_2_accepted ...
    -199.5*prob_proposal_2_rejected;

u3 = a_13*prob_proposal_1_accepted +(0.5 + a_23)*prob_proposal_2_accepted + ...
    100.5*prob_proposal_2_rejected;
out =[ u1 u2 u3];
end

function m = measure(b,fin)
m= (norm(conj(b).*fin))^2;
end


function u= U_thetha_phi(theta,phi)
u= [exp(1i*phi)cos(theta/2) *sin(theta/2);sin(theta/2) -exp(-i*phi)*cos(theta/2)];
end
```