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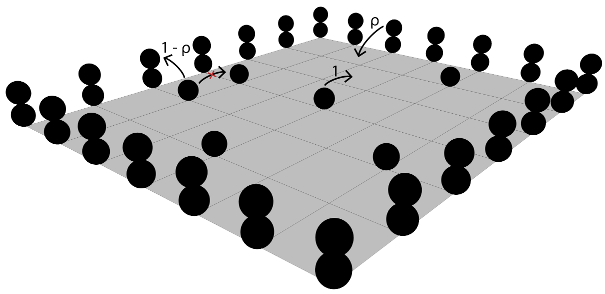

Convergence to stationary states of interacting particle systems

Rodrigo Marinho de Souza

Supervisor: Doctor Ana Patrícia Carvalho Gonçalves
Co-Supervisor: Doctor Milton David Jara Valenzuela

Thesis approved in public session to obtain the PhD Degree in

Mathematics

Jury final classification: **Pass with Distinction**

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Resumo

Nessa tese de doutoramento estudamos a convergência aos estados estacionários de sistemas de partículas interagentes com fronteira aberta. A tese é dividida em duas partes, sendo a primeira um estudo quantitativo dessa convergência quando o sistema está em equilíbrio e a segunda, um estudo qualitativo dessa convergência quando o sistema está fora do equilíbrio.

No caso de modelos com presença de dinâmica de Glauber em equilíbrio, estudamos o tempo necessário para que a distribuição do processo esteja próxima da distribuição estacionária na métrica de variação total, obtendo resultados bem finos. Além disso, mostramos que essa convergência é abrupta e que pode ser descrita através de um perfil Gaussiano. Para explicarmos nossa abordagem, utilizamos o processo de exclusão em contato com reservatórios.

No caso de sistemas fora do equilíbrio, estudamos os seus estados estacionários e provamos um Teorema Central do Limite para o seu campo de flutuações. Como esse resultado já é conhecido para o processo de exclusão com reservatórios, utilizamos como exemplo um modelo de reação-difusão d-dimensional, sendo o resultado válido para dimensões menores que quatro. A nossa grande contribuição vem do fato de que o método funciona para modelos dirigidos por equações diferenciais parciais não lineares. Além disso, é a primeira vez em que se obtém este tipo de resultado em um modelo de dimensão alta.

Ambos os problemas são abordados com o método da entropia relativa de Yau, o qual combinamos com uma desigualdade log-Sobolev bastante geral e que é válida para medidas produto do tipo Bernoulli associadas a perfis definidos em cubos ou toros d-dimensionais. Essa desigualdade, que pode ser considerada a nossa maior ferramenta em ambos os problemas solucionados, possui uma prova que pode ser adaptada para modelos gradientes com fronteira aberta gerais, em qualquer dimensão. Utilizando a força da desigualdade log-Sobolev mencionada, é possível obter cotas superiores bem precisas para as entropias relativas entre algumas medidas de probabilidade e assim provar os resultados supracitados.

Palavras-chave: Convergência fina, desigualdade de Yau, entropia relativa, tempos de mistura, teorema central do limite.

Abstract

In this PhD thesis we study the convergence to stationary states of interacting particle systems with open boundary. The thesis is divided into two parts, being the first one a quantitative study of that convergence when the system is in equilibrium and the second one, a qualitative study of that convergence when the system is out of equilibrium.

In the case of models with the presence of Glauber dynamics in equilibrium, we study the time required so that the distribution of the process is close to the stationary one in total variation distance, obtaining very precise results. Furthermore, we show that this convergence is abrupt and that it can be described by a Gaussian profile. In order to explain our approach, we use the exclusion process in contact with reservoirs.

In the case of a non-equilibrium system, we study its stationary states and prove a Central Limit Theorem for its fluctuation field. Since this result is already known for the exclusion process in contact with reservoirs, we use a d-dimensional reaction-diffusion model as an example, being the result valid for dimensions smaller than four. Our great contribution comes from the fact that the method works for models that are driven by non-linear partial differential equations. Moreover, this is the first time that one obtains this kind of result from a high-dimensional model.

Both problems are approached with Yau's relative entropy method, which we combine with a very general logarithmic-Sobolev inequality that is valid for Bernoulli product measures associated with profiles defined on d-dimensional cubes or tori. This inequality, which may be considered our strongest tool in both solved problems, has a proof that can be adapted to general gradient models with open boundary, in any dimension. Using the strength of the above logarithmic-Sobolev inequality, it is possible to obtain very sharp upper bounds on the relative entropy between some probability measures and thus to prove the aforementioned results.

Keywords: Central limit theorem, mixing times, relative entropy, sharp convergence, Yau's inequality.

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Chapter 1

Introduction

Understanding the convergence to stationary states of interacting particle systems with Glauber dynamics has called a lot of attention of researchers of probability theory in the last decades. Depending on the action of the boundary dynamics, the process can be either in or out of equilibrium. Namely, the process is said to be out of equilibrium if there exists a non-null current driving the particles in some direction.

One of the most famous particle systems with Glauber dynamics is the simple exclusion process in contact with reservoirs, see for instance [1, 6, 7, 8, 9] and [14]. The dynamics of this system consists of: particles performing nearest-neighbor random walks on the discrete interval (a path graph) of length $n-1$, with the exclusion rule which forbids more than one particle at the same vertex; and Glauber dynamics at the end-vertices of the discrete interval. This Glauber dynamics creates a particle at a boundary vertex $x \in \{1, n-1\}$ with rate $c_x \in (0,1)$ if that vertex is empty, and it annihilates a particle at x with rate $1-c_x$ if it is occupied. The Glauber dynamics can be seen as the action of reservoirs on the left and on the right of the graph. When $c_x = \rho$ for every $x \in \{1, n-1\}$ the process is said to be in equilibrium, otherwise it is out of equilibrium.

The simple exclusion process in contact with reservoirs in equilibrium has the Bernoulli product measure with parameter ρ as its unique stationary measure, that is, if at each vertex one tosses a coin with probability ρ for heads and puts a particle in that vertex if and only if that coin lands heads up, then these actions induce the above measure which is invariant under the action of the infinitesimal generator of the particle system. Although it is simple to understand the stationary measure of this process in equilibrium, it is difficult to understand how fast the distribution of the process converges to the stationary one. In [8], the authors studied this problem and they obtained a sharp result for the case where there is only one reservoir in the system, for instance at the right boundary vertex. In the first part of this thesis, we study the equilibrium scenario, with both reservoirs, for any $\rho \in (0,1)$. We start the process from probability measures that are associated with profiles defined on the open unit interval and we show that there exists a sequence of times t_n and windows w_n (with $w_n/t_n \to 0$ as $n \to \infty$) for which the total variation distance between the distribution of the process at time $t_n + bw_n$ and the stationary one converges to a Gaussian profile that depends only on b. Furthermore, we show that depending on the choice of the initial profile, the times t_n may change.

The idea of the proof is very simple to understand. It is well known that the hydrodynamic behavior of the simple exclusion process in contact with reservoirs in equilibrium is driven by the unique solution of the heat equation with Dirichlet boundary conditions that has value ρ at the boundary of the unit interval. Thus, instead of computing the aforementioned total variation distance, we replace the distribution of the process at time t by the Bernoulli product measure associated with the solution of this partial differential equation. This replacement makes sense due to the law known as the *conservation of local equilibrium* [13, Chp. 9] and it has a cost. To make this idea work, we have to show that the relative entropy between the replaced and substitute measures converges to zero for small order n-dependent times, as $n \to \infty$. We prove that by using Yau's relative entropy method [19] which we briefly explain below.

Yau's relative entropy inequality allows one to upper bound the time derivative of the relative entropy between the law of the process at time t and a reference measure μ by two terms: the first involves the *carré du champ operator* and the second one involves correlations. The first term is negative and therefore it can be simply discarded from the inequality. However, if we do so we may lose the possibility of sharply bound the relative entropy. In order to take advantage of this term, we prove a very general logarithmic-Sobolev inequality which allows replacing this term by a negative multiple of the relative entropy. Thus, using the integrating factor and sharp estimates on the correlations, we obtain a very sharp bound on the relative entropy.

The second part of this thesis is devoted to qualitatively understanding the non-equilibrium stationary states of particle systems with Glauber dynamics. However, since this has already been done for the simple exclusion process in contact with reservoirs (see [14]), we illustrate our method with the d-dimensional reaction-diffusion model studied in [11], with $d \leq 3$. In this model, particles perform nearest-neighbor random walks of rate n^2, with the exclusion rule, on the d-dimensional discrete torus (representing the diffusion), while Glauber dynamics occurs at every vertex of the discrete torus with rate 1 (representing the reactions).

The strategy used in this second problem is also based on Yau's relative entropy method: the particle system chosen for our study depends on a parameter $\lambda \geq 0$ where the case $\lambda = 0$ treats the equilibrium scenario. Let ρ be the unique real solution of $\rho = (1-\rho)(1+\lambda d \rho^2)$. Starting from the Bernoulli product measure associated with the stationary profile, which in this case is the constant profile equal to ρ, and combining an adaptation of the Sobolev logarithmic inequality mentioned in the first problem with Yau's inequality, we obtain an upper bound on the relative entropy between the distribution of the process at time t and the initial probability measure itself, independent of time, provided λ belongs to a sufficiently small interval $[0, \lambda^*]$. Since the distribution of the process, which is irreducible, converges to the stationary probability measure, which is unknown, and the estimate obtained for the relative entropy does not depend on time, we obtain an estimate for the relative entropy between the stationary probability measure and the Bernoulli product measure associated with the stationary profile.

The estimate mentioned above immediately implies a result known as the *hydro-*

static limit, which is a kind of *Law of Large Numbers*. Using this upper bound obtained for the entropy, we prove a result known as the *Boltzmann-Gibbs Principle*, which allows us to deal with functionals that present long-range correlations. After we prove these results, we show that, starting from the stationary states, the density fluctuation field (observable of great interest in this area of knowledge) converges to a Gaussian random variable with a very interesting variance – A kind of *Central Limit Theorem*. The idea developed here to prove this type of result, which as said above is known for the simple exclusion process, contributes significantly to the development of the area due to the fact that it is applicable to particle systems driven by non-linear partial differential equations, and also because it works for high-dimensional models, which had not been rigorously proven in the literature yet.

Although it is possible that the main result remains valid for larger values of λ, there are examples in real life where the solution of a problem in a non-equilibrium environment may change depending on how far it is from equilibrium. Let us consider an analogy. Assume first that a sailor sails a small boat on a lake and he wants to go from a point A to a point B. Clearly, the simplest and fastest way to do that is to follow the line that passes through A and B. Now let us change the situation to a very large river (for instance the Tejo). There are small waves in the river but the solution for the sailor remains the same. Last, let us consider the sailor with his boat on the sea. Depending on the hight of the tide and of the waves, the sailor must perform zigzags so that he does not confront the breaking waves. This is a well known strategy which is used even in sailing championships. The example of the lake suggests a process in equilibrium, since there are no currents in the space. In the case of the river, the process is out of equilibrium because there exists a current in the river, but this current is not too strong. When we replace the river by the sea, the process can be so perturbed that if the sailor uses the same strategy, he may sink the boat. In this analogy, the relative entropy method represents the boat.

All in all, the thesis is dived into two parts that are placed in Chapters 2 and 3, respectively. In Chapter 2 we do a quantitative analysis of the convergence to stationary states of particle systems with Glauber dynamics in equilibrium, where we use the simple exclusion process in contact with reservoirs as an example. In Chapter 3, we do a qualitative analysis of the stationary states of non-equilibrium particle systems with Glauber dynamics. Our main tool is a logarithmic-Sobolev inequality, which when combined with Yau's relative entropy inequality, implies very sharp upper bounds on the relative entropy of these Markov processes, as long as they start from adequate probability measures. The study of the speed of convergence of non-equilibrium particle systems to their stationary states remains open and we pretend to do it in a future work.

Chapter 2

Sharp convergence to equilibrium

In this chapter we introduce a method developed to prove the abrupt convergence, from one to zero, of the total variation distance between the law of an interacting particle system and its stationary measure. The idea relies on the conservation of local equilibrium [13, Chapter 9] and on Yau's relative entropy method [19]. We explain our method using the *symmetric simple exclusion process* (SSEP) in contact with reservoirs.

2.1 SSEP with reservoirs

Let $n \in \{2, 3, \dots, \}$ be a scaling parameter. Let $\Lambda_n := \{1, \dots, n-1\}$ be the discrete interval with $n-1$ points. We will call the set $\{1, n-1\}$ the *boundary* of Λ_n. We give to Λ_n a graph structure by taking $E_n := \{\{x, x+1\} \; ; \; x \in \{1, \dots, n-2\}\}$ as the edge set in $G_n = (\Lambda_n, E_n)$. We call the vertex set Λ_n the *bulk* and we say that $x, y \in \Lambda_n$ are *neighbors* if $\{x, y\} \in E_n$. In that case we write $x \sim y$.

Let us define $\Omega_n := \{0, 1\}^{\Lambda_n}$. The elements $\eta = \{\eta(x) \; ; \; x \in \Lambda_n\}$ of Ω_n are called *configurations of particles*. We say that a vertex $x \in \Lambda_n$ is occupied by a particle (resp. empty) in configuration $\eta \in \Omega_n$ if $\eta(x) = 1$ (resp. $\eta(x) = 0$). Given a configuration $\eta \in \Omega_n$ and two vertices $x, y \in \Lambda_n$, we denote by $\eta^{x,y}$ the configuration of particles obtained from η by exchanging the occupations at x and y, that is,

$$\eta^{x,y}(z) = \begin{cases} \eta(x) & \text{if } \; z = y, \\ \eta(y) & \text{if } \; z = x, \\ \eta(z) & \text{if } \; z \neq x, y. \end{cases}$$

Given a configuration $\eta \in \Omega_n$ and a vertex $x \in \Lambda_n$, we denote by η^x the configuration of particles obtained from η by changing the value of $\eta(x)$ to $1 - \eta(x)$, that is,

$$\eta^{x}(z) = \begin{cases} 1 - \eta(x) & \text{if } \; z = x, \\ \eta(z) & \text{if } \; z \neq x. \end{cases}$$

Given a function $f : \Omega_n \to \mathbb{R}$ and $x, y \in \Lambda_n$, let $\nabla_{x,y} f, \nabla_x f : \Omega_n \to \mathbb{R}$ be defined as

$$\nabla_{x,y} f(\eta) = f(\eta^{x,y}) - f(\eta), \quad \nabla_x f(\eta) = f(\eta^x) - f(\eta) \tag{2.1.1}$$

for any $\eta \in \Omega_n$.

The SSEP with reservoir densities α and β is the continuous-time Markov chain $\{\eta_t \,;\, t \geq 0\}$ with state space Ω_n, generated by the operator $\mathfrak{L}_n$ given by

$$\mathfrak{L}_n f(\eta) := n^2 \sum_{x=1}^{n-2} \nabla_{x,x+1} f(\eta) + n^2 \sum_{x \in \{1, n-1\}} \big(c_x (1 - \eta(x)) + (1 - c_x) \eta(x) \big) \nabla_x f(\eta)$$

for any function $f : \Omega_n \to \mathbb{R}$ and any $\eta \in \Omega_n$, where $c_1 = \alpha$ and $c_{n-1} = \beta$ for $\alpha, \beta \in (0, 1)$.

The dynamics generated by $\mathfrak{L}_n$ can be informally described as follows: in the bulk, particles perform nearest-neighbor random walks with rate n^2 under the *exclusion* rule which forbids more than one particle at any vertex and at any time. At the boundary, the Glauber dynamics injects (resp. annihilates) particles independently at each empty (resp. occupied) vertex x in $\{1, n-1\}$ with rate $c_x n^2$ (resp. $(1 - c_x) n^2$). The factor n^2 speeds up time so that the process is observed in a diffusive time scale.

2.1.1 Stationary measure

For each function $u : \Lambda_n \to [0, 1]$, let $\nu^n_{u(\cdot)}$ the Bernoulli product measure in Ω_n with density $u(\cdot)$, that is,

$$\nu^n_{u(\cdot)}(\eta) := \prod_{x \in \Lambda_n} \big\{ \eta(x) u(x) + (1 - \eta(x))(1 - u(x)) \big\}$$

for any $\eta \in \Omega_n$. Notice that if the values of u belong to the *open* interval $(0, 1)$, then the measures $\nu^n_{u(\cdot)}$ have full support.

Since the process $\{\eta_t \,;\, t \geq 0\}$ is irreducible, it has a unique stationary measure. When α and β are equal (let us differ this case writing $\rho := \alpha = \beta$), the reservoirs induce a null current in the bulk. In this case, the process is said to be in equilibrium and the Bernoulli product measure $\bar{\nu}^n_\rho$, associated with the constant function equal to ρ, is the stationary one. From now on, we fix a parameter $\rho \in (0, 1)$ and we deal with the SSEP with reservoir density ρ. Observe that the process $\{\eta_t \,;\, t \geq 0\}$ depends on n and ρ. In order to simplify the notation, we do not make this dependence explicit in the notation. The same observation applies to the dependence in ρ of $\mathfrak{L}_n$, as well as to various other objects we define later.

2.1.2 Initial measure and law of the process

For each function $u : [0, 1] \to [0, 1]$, let $u^n : \Lambda_n \to [0, 1]$ be defined by $u^n(x) := u\big(\frac{x}{n}\big)$ for every $x \in \Lambda_n$. For $\varepsilon_0 \in (0, \min\{\rho, 1 - \rho\}]$ and $\kappa \geq 0$, let $\mathcal{U}_{\varepsilon_0, \kappa}$ be the family of differentiable functions $u : [0, 1] \to [0, 1]$ such that:

- $\varepsilon_0 \leq u(q) \leq 1-\varepsilon_0$ for every $q \in [0,1]$;
- $u(0) = u(1) = \rho$;
- $|u'(q)| \leq \kappa$ for every $q \in [0,1]$.

Fix $\varepsilon_0 \in (0, \min\{\rho, 1-\rho\}]$ and $\kappa > 0$ and let $u_0 \in \mathcal{U}_{\varepsilon_0,\kappa}$. We call u_0 a *profile* and we call the measures $\{\nu^n_{u_0^n(\cdot)} \; ; \, n \in \{2,3,\dots\}\}$ the *profile measures*. From now on, we consider the process $\{\eta_t \; ; \, t \geq 0\}$ with initial distribution $\nu^n_{u_0^n(\cdot)}$. In order to simplify the notation, let us define $\nu_0^n := \nu^n_{u_0^n(\cdot)}$.

Let $D([0,\infty),\Omega_n)$ be the space of càdlàg trajectories in Ω_n. We denote by $\mathbb{P}_{\nu_0^n}$ the probability measure in $D([0,\infty),\Omega_n)$ induced by the Markov process $\{\eta_t \; ; \, t \geq 0\}$ with initial measure ν_0^n. We denote by $\mathbb{E}_{\nu_0^n}$ the expectation with respect to $\mathbb{P}_{\nu_0^n}$. We denote the law of η_t with respect to $\mathbb{P}_{\nu_0^n}$ by μ_t^n.

2.1.3 Sharp convergence

The distance to equilibrium of the process $\{\eta_t \; ; t \geq 0\}$ with initial measure ν_0^n is defined as

$$D_n(t; \nu_0^n) := \|\mu_t^n - \bar{\nu}_\rho^n\|_{TV},$$

where $\|\mu - \nu\|_{TV}$ stands for the total variation distance between the probability measures μ and ν in Ω_n, that is,

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{\eta \in \Omega_n} |\mu(\eta) - \nu(\eta)| = \max_{A \subset \Omega_n} |\mu(A) - \nu(A)|. \tag{2.1.2}$$

Let us introduce the Fourier coefficients of the initial profile u_0. For $u_0 : [0,1] \to [0,1]$ and $\ell \in \mathbb{N}$, define

$$c_\ell(u_0) := \sqrt{2} \int (u_0(q) - \rho) \sin(\pi \ell q) dq. \tag{2.1.3}$$

Let us define the *Gaussian profile* $\mathcal{G} : \mathbb{R} \to [0,1]$ by

$$\mathcal{G}(m) := \|\mathcal{N}(m,1) - \mathcal{N}(0,1)\|_{TV}, \tag{2.1.4}$$

for any $m \in \mathbb{R}$.

The goal of this chapter is to prove the following result:

Theorem 2.1.1. *Let $u_0 : [0,1] \to [0,1]$ be differentiable. Assume that $u_0(0) = u_0(1) = \rho$ and that $u_0(q) \in (0,1)$ for every $q \in [0,1]$. Let $\ell_0 \in \mathbb{N}$ be the smallest integer such that $c_{\ell_0}(u_0) \neq 0$. For every $b \in \mathbb{R}$,*

$$\lim_{n \to \infty} D_n\Big(\tfrac{1}{2\pi^2 \ell_0^2} \log n + \tfrac{1}{\pi^2 \ell_0^2} b; \nu_0^n\Big) = \mathcal{G}(\gamma e^{-b}),$$

where $\gamma := \frac{|c_{\ell_0}(u_0)|}{\sqrt{\rho(1-\rho)}}$.

In order to explain this result, let us recall the Gauss error function

$$\text{erf}(q) = \frac{1}{\sqrt{\pi}} \int_{-q}^{q} e^{-t^2} dt. \tag{2.1.5}$$

By Theorem 2.1.1 and Proposition A.1.1, at times $t^n(b) := \frac{1}{2\pi^2\ell_0^2}\log n + \frac{1}{\pi^2\ell_0^2} b$ the distance to equilibrium of the SSEP with reservoir density ρ converges, as $n \to \infty$, to the Gaussian profile

$$\mathcal{G}(\gamma e^{-b}) = \text{erf}\left(\frac{\gamma e^{-b}}{2\sqrt{2}}\right).$$

Thus, we can see that $\mathcal{G}(\gamma e^{-b})$ converges to one as $b \to -\infty$, and to zero as $b \to +\infty$. Observe that the parameter b is inside the parcel of small order in $t^n(b)$, and that we are passing the parameter b to the limit just after we pass $n \to \infty$. This shows that the distance to equilibrium converges sharply (or abruptly) from one to zero at times $t_n(b)$ with a window of order $w_n = 1$. More rigorously, for any $\varepsilon \in (0, 1)$ there exists $b > 0$ such that

$$\limsup_{n\to\infty} D_n(t^n(b); \nu_0^n) \leq \varepsilon$$

and

$$\liminf_{n\to\infty} D_n(t^n(-b); \nu_0^n) \geq 1 - \varepsilon.$$

2.2 The strategy

In this section we explain the strategy that we use to prove Theorem 2.1.1. We start recalling the definition of relative entropy. Indeed, let ν be a probability measure in Ω_n and let f be a density with respect to ν. The *relative entropy* of f with respect to ν is defined as

$$H_\nu(f) := \int f \log f d\nu.$$

Relative entropy and total variation are related by Pinsker's inequality:

Proposition 2.2.1. *(Pinsker's inequality) Let μ and ν be two probability measures in Ω_n. Let f be the Radon-Nikodym derivative of μ with respect to ν. It holds that*

$$2\|\mu - \nu\|_{TV}^2 \leq H_\nu(f).$$

Proof. This proof can be found in [13, page 341], with a small modification. From (2.1.2),

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{\eta\in\Omega_n} \nu(\eta)|f(\eta) - 1|.$$

Thus, applying the elementary inequality $3(a-1)^2 \leq (2a+4)(a\log a - a + 1)$, which is valid for any $a \geq 0$, we obtain

$$\frac{1}{2}\sum_{\eta\in\Omega_n}\nu(\eta)|f(\eta)-1| \leq \frac{1}{2\sqrt{3}}\sum_{\eta\in\Omega_n}\nu(\eta)\sqrt{2f(\eta)+4}\sqrt{f(\eta)\log f(\eta)-f(\eta)+1}.$$

Last, by Cauchy-Schwarz's inequality, we bound the last expression from above by

$$\frac{1}{2\sqrt{3}}\bigg(\sum_{\eta\in\Omega_n}(2f(\eta)+4)\nu(\eta)\bigg)^{1/2}\bigg(\sum_{\eta\in\Omega_n}(f(\eta)\log f(\eta)-f(\eta)+1)\nu(\eta)\bigg)^{1/2} = \frac{\sqrt{2}}{2}\sqrt{H_\nu(f)}.$$

□

We say that a probability measure ν in Ω_n is a *reference measure* if $\nu(\eta) > 0$ for every $\eta \in \Omega_n$. Let ν be a reference measure, which may depend on time, and let f_t^n be the Radon-Nikodym derivative of μ_t^n with respect to ν. By the triangle's inequality,

$$\big|D_n(t;\nu_0^n) - \|\nu - \bar{\nu}_\rho^n\|_{TV}\big| \leq \|\mu_t^n - \nu\|_{TV} \leq \sqrt{\frac{H_\nu(f_t^n)}{2}}. \tag{2.2.1}$$

Therefore, if we are able to find some reference measures ν_t^n for which $H_{\nu_t^n}(f_t^n)^{1/2}$ converges to 0 faster than $D_n(t;\nu_0^n)$, then the proof of Theorem 2.1.1 is reduced to the computation of the distance $\|\nu_t^n - \bar{\nu}_\rho^n\|_{TV}$.

Now we explain our choice for the reference measures ν_t^n. Recall that we fixed constants $\varepsilon_0 \in (0, \min\{\rho, 1-\rho\}]$, $\kappa > 0$ and a profile $u_0 \in \mathcal{U}_{\varepsilon_0,\kappa}$. For each $t \geq 0$ let us define $u_t^n : \{0,1,\dots,n\} \to [0,1]$ as

$$u_t^n(x) := \begin{cases} \mathbb{E}_{\nu_0^n}[\eta_t(x)] & \text{if } x \in \Lambda_n, \\ \rho & \text{if } x \in \{0,n\}. \end{cases}$$

By Dynkin's formula (Lemma B.2.1), we see that $\{u_t^n\ ;\ t \geq 0\}$ is the unique solution of the boundary-value problem

$$\begin{cases} \frac{d}{dt}u_t^n(x) = \Delta_n u_t^n(x) & \text{for } t \geq 0 \text{ and } x \in \Lambda_n, \\ u_t^n(x) = \rho & \text{for } t \geq 0 \text{ and } x \in \{0,n\}, \\ u_0^n(x) = u_0\big(\frac{x}{n}\big) & \text{for } x \in \Lambda_n. \end{cases} \tag{2.2.2}$$

Above, Δ_n stands for the discrete Laplacian operator defined on functions $f : \{0,1,\dots,n\} \to \mathbb{R}$ as

$$\Delta_n f(x) = n^2\big(f(x+1) + f(x-1) - 2f(x)\big)$$

for any $x \in \Lambda_n$.

Let us define $\nu_t^n := \nu_{u_t^n(\cdot)}$. The property known as *conservation of local equilibrium* [13, Chp. 9] states that for any fixed $t \geq 0$, the measures μ_t^n and ν_t^n are close as $n \to \infty$, if observed on an subset of Λ_n of fixed size. Therefore, it is reasonable to use the measures $\{\nu_t^n\ ;\ t \geq 0\}$ as the reference measures to be plugged into (2.2.1).

Observe, however, that this conservation of local equilibrium is too weak to be useful in our situation: first, it only holds over a *finite* time interval (while we need to go to times of order $\log n$), and second, it only holds on a *finite* spatial interval (we need to go to the whole bulk Λ_n). In Section 2.5 we prove the following bound on the aforementioned relative entropy:

Theorem 2.2.2. *Let $\varepsilon_0 \in (0, \min\{\rho, 1-\rho\}]$ and $\kappa > 0$ be given. Let $u_0 \in \mathcal{U}_{\varepsilon_0,\kappa}$ and let f_t^n be the Radon-Nikodym derivative of the measure μ_t^n with respect to ν_t^n. Define $H_n(t) := H_{\nu_t^n}(f_t^n)$. There exist constants $C_0 = C_0(\varepsilon_0, \kappa)$, $\delta_0 = \delta_0(\varepsilon_0, \kappa) > 0$ such that*

$$H_n(t) \leq C_0 e^{-\delta_0 t}$$

for every $n \in \{2, 3, \dots\}$, every $u_0 \in \mathcal{U}_{\varepsilon_0,\kappa}$ and every $t \geq 0$.

A version of this estimate was obtained in [12] in the context of non-equilibrium fluctuations from the hydrodynamic limit. Our contribution is the exponential decay as a function of t.

Using Theorem 2.2.2, we show that the relative entropy in (2.2.1) converges to zero in times of order $\log n$. To accomplish our goal, we still need to identify the time window at which the convergence, in total variation, of ν_t^n to $\bar{\nu}_\rho^n$ happens. Indeed, in Section 2.6 we prove the following result:

Theorem 2.2.3. *Let $u_0 : [0,1] \to [0,1]$ be a differentiable function. Assume that $u_0(0) = u_0(1) = \rho$ and that $u_0(x) \in (0,1)$ for every $x \in [0,1]$. Let ℓ_0 be the smallest positive integer such that (2.1.3) is non-null. Recall that $\gamma := \frac{|c_{\ell_0}(u_0)|}{\sqrt{\rho(1-\rho)}}$. For every $b \in \mathbb{R}$,*

$$\lim_{n\to\infty} \left\| \nu^n_{\frac{1}{2\pi^2\ell_0^2}\log n + \frac{b}{\pi^2\ell_0^2}} - \nu_\rho \right\|_{TV} = \mathcal{G}(\gamma e^{-b}).$$

For each $\kappa > 0$, $n \in \{2, 3, \dots\}$ and $\varepsilon_0 \in (0, \min\{\rho, 1-\rho\}]$, let $\mathcal{U}^n_{\kappa,\varepsilon_0}$ be the class of functions $u : \Lambda_n \to [0,1]$ such that:

- $u(x) \in [\varepsilon_0, 1-\varepsilon_0]$ for any $x \in \Lambda_n$;
- $n|u(x+1) - u(x)| \leq \kappa$ for any $x \in \{1, \dots, n-2\}$.

Now let Γ_n be the *carré du champ* operator associated to $\mathfrak{L}_n$: for every $f : \Omega_n \to \mathbb{R}$, $\Gamma_n f := \mathfrak{L}_n f^2 - 2f\mathfrak{L}_n f$. The following inequality, which we prove in Section 2.3, is the main ingredient of the proof of Theorem 2.2.2:

Theorem 2.2.4 (Logarithmic Sobolev inequality for inhomogeneous product measures)**.** *Let $\rho \in (0,1)$, $\varepsilon_0 \in (0, \min\{\rho, 1-\rho\}]$ and $\kappa > 0$ be fixed. There exists a positive constant $K_0 = K_0(\rho, \varepsilon_0, \kappa)$ such that*

$$H_{\nu^n_{u(\cdot)}}(f) \leq \frac{1}{K_0} \int \Gamma_n \sqrt{f} d\nu^n_{u(\cdot)} \tag{2.2.3}$$

for every $u \in \mathcal{U}^n_{\varepsilon_0,\kappa}$ and every density f with respect to $\nu^n_{u(\cdot)}$.

Estimates of this kind are known in the literature as *log-Sobolev inequalities*. The log-Sobolev constant K_{LS} is defined as the largest constant K_0 that satisfies (2.2.3). Theorem 2.2.4 shows that K_{LS}^{-1} is uniformly bounded in n.

2.3 The log-Sobolev inequality

In this section we prove Theorem 2.2.4. First we prove a simpler version of that log-Sobolev inequality, from which Theorem 2.2.4 follows by comparison of quadratic forms.

Fix $\gamma > 0$. Recall (2.1.1). For each $f : \Omega_n \to \mathbb{R}$ and each $x \in \{1, \dots, n-2\}$, let us define

$$\mathcal{D}_x(f) := \int \big(\nabla_x f(\eta)\big)^2 d\nu^n_{u(\cdot)},$$

$$\mathcal{D}_{x,x+1}(f) := \int \big(\nabla_{x,x+1} f(\eta)\big)^2 d\nu^n_{u(\cdot)},$$

$$\mathcal{D}(f) := \frac{\gamma}{n}\mathcal{D}_1(f) + \sum_{x=1}^{n-2} \mathcal{D}_{x,x+1}(f). \tag{2.3.1}$$

We prove the following:

Theorem 2.3.1. *Let $\gamma, \kappa > 0$ and $\varepsilon_0 \in (0, \frac{1}{2}]$. There exists a positive constant $K = K(\varepsilon_0, \kappa, \gamma)$ such that*

$$H_{\nu^n_{u(\cdot)}}(f) \le \frac{n^2}{K}\mathcal{D}(\sqrt{f}) \tag{2.3.2}$$

for every $u \in \mathcal{U}^n_{\kappa,\varepsilon_0}$ and every density f with respect to $\nu^n_{u(\cdot)}$.

To prove Theorem 2.3.1, we need the following result:

Lemma 2.3.2 (Comparison of quadratic forms)**.** *There exists a finite constant $C = C(\varepsilon_0, \kappa, \gamma)$ such that for every $n \in \{2, 3, \dots\}$, every $\ell \in \{2, \dots, n-1\}$, every $f : \Omega_n \to \mathbb{R}$ and every $u \in \mathcal{U}^n_{\kappa,\varepsilon_0}$,*

$$\mathcal{D}_\ell(f) \le Cn\mathcal{D}(f).$$

Proof. Observe that for each $x \in \{2, \dots, n-1\}$,

$$\nabla_x f(\eta) = \nabla_{x-1,x} f(\eta) + \nabla_{x-1,x} f\big((\eta^{x-1,x})^{x-1}\big) + \nabla_{x-1} f(\eta^{x-1,x}).$$

Using the inequality

$$(a+b+c)^2 \le 2(1+\beta)(a^2+b^2) + \big(1+\tfrac{1}{\beta}\big)c^2,$$

which is valid for every $a, b, c \in \mathbb{R}$ and any $\beta > 0$, we obtain

$$\mathcal{D}_x(f) \le 2(1+\beta)\int \Big(\big(\nabla_{x-1,x}(f)\big)^2 + \big(\nabla_{x-1,x} f\big((\eta^{x-1,x})^{x-1}\big)\big)^2\Big) d\nu^n_{u(\cdot)}$$
$$+ \big(1+\tfrac{1}{\beta}\big)\int \big(\nabla_{x-1} f(\eta^{x-1,x})\big)^2 d\nu^n_{u(\cdot)}.$$

Now we perform some changes of variables. Indeed,

$$\int \big(\nabla_{x-1,x} f\big((\eta^{x-1,x})^{x-1}\big)\big)^2 d\nu^n_{u(\cdot)} = \int \big(\nabla_{x-1,x} f(\eta)\big)^2 \frac{\nu^n_{u(\cdot)}\big((\eta^{x-1})^{x-1,x}\big)}{\nu^n_{u(\cdot)}(\eta)} d\nu^n_{u(\cdot)},$$

$$\int \big(\nabla_{x-1} f(\eta^{x-1,x})\big)^2 d\nu^n_{u(\cdot)} = \int \big(\nabla_{x-1} f(\eta)\big)^2 \frac{\nu^n_{u(\cdot)}(\eta^{x-1,x})}{\nu^n_{u(\cdot)}(\eta)} d\nu^n_{u(\cdot)}.$$

Observe that the Jacobian factors satisfy

$$\frac{\nu^n_{u(\cdot)}\big((\eta^{x-1})^{x-1,x}\big)}{\nu^n_{u(\cdot)}(\eta)} \leq \frac{1}{\varepsilon_0} - 1,$$

$$\frac{\nu^n_{u(\cdot)}(\eta^{x-1,x})}{\nu^n_{u(\cdot)}(\eta)} \leq 1+\alpha,$$

where $\alpha = \frac{\kappa}{\varepsilon_0^2 n}$. Therefore,

$$\mathcal{D}_x(f) \leq \frac{2(1+\beta)}{\varepsilon_0}\mathcal{D}_{x-1,x}(f) + \Big(1+\frac{1}{\beta}\Big)(1+\alpha)\mathcal{D}_{x-1}(f). \tag{2.3.3}$$

The idea now is to use this estimate to transport $\mathcal{D}_\ell(f)$ to the boundary. Using (2.3.3) successively for $x = \ell, \dots, 2$ we conclude that

$$\begin{aligned}
\mathcal{D}_\ell(f) &\leq \frac{2(1+\beta)}{\varepsilon_0}\sum_{x=1}^{\ell-1}\Big[\Big(1+\frac{1}{\beta}\Big)(1+\alpha)\Big]^{\ell-1-x}\mathcal{D}_{x,x+1}(f) + \Big[\Big(1+\frac{1}{\beta}\Big)(1+\alpha)\Big]^{\ell-1}\mathcal{D}_1(f)\\
&\leq \Big[\Big(1+\frac{1}{\beta}\Big)(1+\alpha)\Big]^{\ell-1}\Big(\frac{2(1+\beta)}{\varepsilon_0}\sum_{x=1}^{\ell-1}\mathcal{D}_{x,x+1}(f) + \mathcal{D}_1(f)\Big)\\
&\leq \Big[\Big(1+\frac{1}{\beta}\Big)(1+\alpha)\Big]^{\ell-1}\max\Big\{\frac{2(1+\beta)}{\varepsilon_0}, \frac{n}{\gamma}\Big\}\mathcal{D}(f).
\end{aligned}$$

Taking $\beta = n-1$ and using the bound $(1+a)^b \leq e^{ab}$ we obtain the estimate

$$\mathcal{D}_\ell(f) \leq n\max\Big\{\frac{2}{\varepsilon_0}, \frac{1}{\gamma}\Big\}e^{\alpha n+1}\mathcal{D}(f).$$

The lemma follows from the fact that $\alpha n = \frac{\kappa}{\varepsilon_0^2}$. $\square$

Proof of Theorem 2.3.1. We use the Yau's martingale method [16]. We follow the approach underlined in [18, Chapter 3]. In what follows, all conditional expectations are taken with respect to $\nu^n_{u(\cdot)}$. For every $u \geq 0$, define $\varphi(u) := u\log u$. The key observation is that for every $f : \Omega_n \to \mathbb{R}$ and every σ-algebra $\mathcal{G}$, if $g = E[f|\mathcal{G}]$ and $h = f/g$, then

$$\int \varphi(f)d\nu^n_{u(\cdot)} = \int \varphi(g)d\nu^n_{u(\cdot)} + \int \varphi(h)g d\nu^n_{u(\cdot)}. \tag{2.3.4}$$

Furthermore, since $\nu^n_{u(\cdot)}$ is a product measure, we can use this relation to recursively estimate the log-Sobolev constant K_{LS}. For each $\ell \in \{2, \dots, n-1\}$ let

$$\mathcal{G}_\ell := \sigma(\eta(x); x \in \{1, \dots, \ell\})$$

be the σ-algebra generated by the first ℓ coordinates of the configuration η. Let us define

$$\mathcal{D}^\ell(f) := \sum_{x=1}^{\ell-1} \mathcal{D}_{x,x+1}(f) + \frac{\gamma}{n}\mathcal{D}_1(f)$$

and

$$K_\ell = K_\ell(\gamma, \varepsilon_0, \kappa) := \inf_{f,u} \frac{\mathcal{D}^\ell(\sqrt{f})}{\int \varphi(f) d\nu^n_{u(\cdot)}},$$

where the infimum runs over all densities f with respect to $\nu^n_{u(\cdot)}$ which are measurable with respect to $\mathcal{G}_\ell$ and all profiles $u \in \mathcal{U}^n_{\varepsilon_0,\kappa}$. The reader must not confuse the quadratic operators $\mathcal{D}_x$ and $\mathcal{D}^x$. Observe that $K_{LS} = K_{n-1}(\varepsilon_0, \kappa, \gamma)$.

Fix $u \in \mathcal{U}^n_{\varepsilon_0,\kappa}$. Let f be a density with respect to $\nu^n_{u(\cdot)}$ that is measurable with respect to $\mathcal{G}_{n-1}$. Define $g : \{0,1\} \to \mathbb{R}$ as $g(\theta) := E[f|\eta(n-1) = \theta]$ for each $\theta \in \{0,1\}$. Let $\nu^{n-2}_{u(\cdot)}$ be the law of $\xi := (\eta(x); x \in \{1, \dots, n-2\})$ and let $\hat{\nu}^{n-1}_{u(\cdot)}$ be the law of $\eta(n-1)$ with respect to $\nu^n_{u(\cdot)}$. Observe that f can be thought as a function of (ξ, θ).

By (2.3.4),

$$\int \varphi(f) d\nu^n_{u(\cdot)} = \int \varphi(g) d\hat{\nu}^\ell_{u(\cdot)} + \int \Big(\int \varphi\Big(\frac{f(\xi,\theta)}{g(\theta)}\Big) \nu^{n-2}_{u(\cdot)}(d\xi) \Big) g(\theta) \hat{\nu}^{n-1}_{u(\cdot)}(d\theta).$$

Observe that for each $\theta \in \{0,1\}$, the function $\xi \mapsto \frac{f(\xi,\theta)}{g(\theta)}$ is $\mathcal{G}_{n-2}$-measurable and it is a density with respect to $\nu^n_{u(\cdot)}$. From the definition of K_ℓ,

$$\int \varphi\Big(\frac{f(\xi,\theta)}{g(\theta)}\Big) \nu^{n-2}_{u(\cdot)}(d\xi) \leq \frac{1}{K_{n-2} g(\theta)} \mathcal{D}^{n-2}(\sqrt{f(\cdot,\theta)}).$$

Therefore, we have that

$$\int \varphi(f) d\nu^n_{u(\cdot)} \leq \int \varphi(g) \hat{\nu}^{n-1}_{u(\cdot)}(d\theta) + \frac{1}{K_{n-2}} \mathcal{D}^{n-2}(f).$$

Notice that $\hat{\nu}^{n-1}_{u(\cdot)}$ is a Bernoulli law of parameter $u(n-1)$. By [15, Lemma 2] (first proven in [5, Example 3.1]), there exists a constant B independent of ξ and $u(n-1)$ such that

$$\int \varphi(g) \hat{\nu}^{n-1}_{u(\cdot)}(d\theta) \leq 2B\Big(\sqrt{g(1)} - \sqrt{g(0)}\Big)^2.$$

Let X and Y be non-negative random variables. By Cauchy-Schwarz inequality

$$\big(\sqrt{\mathbb{E}[X]} - \sqrt{\mathbb{E}[Y]}\big)^2 \leq \mathbb{E}\big[\big(\sqrt{X} - \sqrt{Y}\big)^2\big].$$

Taking $X(\xi) = f(\xi, 1)$ and $Y(\xi) = f(\xi, 0)$, we see that

$$\big(\sqrt{g(1)} - \sqrt{g(0)}\big)^2 \le \int \big(\sqrt{f(\xi,1)} - \sqrt{f(\xi,0)}\big)^2 \nu^{n-2}_{u(\cdot)}(d\xi) = \mathcal{D}_{n-1}(\sqrt{f}).$$

Using Lemma 2.3.2 with $\ell = n-1$, we see that

$$\begin{aligned}\int \varphi(f) d\nu^n_{u(\cdot)} &\le 2B\mathcal{D}_{n-1}(\sqrt{f}) + \frac{1}{K_{n-2}}\mathcal{D}^{n-2}(\sqrt{f}) \\ &\le \Big(2BCn + \frac{1}{K_{n-2}}\Big)\mathcal{D}^{n-1}(\sqrt{f}).\end{aligned}$$

Therefore, by definition of K_{n-1}, we have

$$\frac{1}{K_{n-1}} \le \tilde{C}n + \frac{1}{K_{n-2}}$$

for some finite constant $\tilde{C} = \tilde{C}(\varepsilon_0, \kappa, \gamma)$. Iterating this strategy, we conclude the proof. □

We finally prove Theorem 2.2.4:

Proof of Theorem 2.2.4. By Proposition B.1.1, for any $f : \Omega_n \to \mathbb{R}$,

$$\int \Gamma_n f d\nu^n_{u(\cdot)} = n^2 \sum_{x=1}^{n-2} \mathcal{D}_{x,x+1}(f) + n^2 \sum_{x\in\{1,n-1\}} \int \big(\rho(1-\eta(x)) + (1-\rho)\eta(x)\big)\big(\nabla_x f(\eta)\big)^2 d\nu^n_{u(\cdot)}.$$

Therefore, for $\gamma = \min\{\rho, 1-\rho\}$,

$$n^2\mathcal{D}(f) \le \int \Gamma_n f d\nu^n_{u(\cdot)}$$

for every $n \in \{2, 3, \dots\}$ and every $f : \Omega_n \to \mathbb{R}$. Theorem 2.2.4 follows from this bounds and Theorem 2.3.1, with $K_0(\rho, \varepsilon_0, \kappa) = K(\varepsilon_0, \kappa, \min\{\rho, 1-\rho\})$. □

2.4 Estimates on the solution of the discrete heat equation

Before we estimate the relative entropy and compute the total variation distance between the profile measures, we collect and prove various facts about solutions of (2.2.2). Using discrete Fourier series, (2.2.2) can be solved in terms of trigonometric functions: for $n \in \{2, 3, \dots\}$ and $\ell \in \{1, \dots, n-1\}$, define $\varphi^n_\ell : \Lambda_n \to \mathbb{R}$ as

$$\varphi^n_\ell(x) := \sqrt{2}\sin\left(\tfrac{\pi\ell x}{n}\right)$$

for every $x \in \Lambda_n$ and define

$$\lambda_\ell^n := 2n^2\big(1 - \cos\big(\tfrac{\pi \ell}{n}\big)\big) = 4n^2 \sin^2\big(\tfrac{\pi\ell}{2n}\big).$$

Observe that $\Delta_n \varphi_\ell^n(x) = -\lambda_\ell^n \varphi_\ell^n(x)$ for every $x \in \Lambda_n$, that is, for any $\ell \in \{1, \dots, n-1\}$ φ_ℓ^n is an eigenfunction of the discrete Laplacian operator associated with the eigenvalue $-\lambda_\ell^n$. The solution $\{u_t^n(x); x \in \Lambda_n, t \geq 0\}$ has the following representation:

$$u_t^n(x) = \rho + \sum_{\ell=1}^{n-1} c_\ell^n e^{-\lambda_\ell^n t} \varphi_\ell^n(x) \tag{2.4.1}$$

for every $x \in \Lambda_n$ and every $t \geq 0$, where

$$c_\ell^n = c_\ell^n(u_0) := \frac{1}{n} \sum_{x \in \Lambda_n} (u_0(x) - \rho)\varphi_\ell^n(x), \ell \in \{1, \dots, n-1\}$$

are the *Fourier coefficients* of $u_0 - \rho$.

Our first lemma gives a very useful estimate on the eigenvalues $-\lambda_\ell^n$. Below we use the notation $f_n \sim g_n$ if $\lim_{n\to\infty} f_n/g_n = 1$.

Lemma 2.4.1. *For each* $n \in \{2, 3, \cdots\}$ *and each* $\ell_0, \ell \in \{1, \dots n-1\}$ *such that* $\ell_0 \leq \min\{\ell, \frac{n}{2}\}$,

$$\frac{\lambda_\ell^n}{\lambda_{\ell_0}^n} \geq \frac{\ell}{\ell_0}.$$

Proof. Observe that as $n \to \infty$, $\lambda_\ell^n \sim \pi^2 \ell^2$ for $\ell = o(\sqrt{n})$. Therefore, in fact the ratio on the left-hand side of the inequality in the lemma approaches $\frac{\ell^2}{\ell_0^2}$. The point on this lemma is that the estimate is uniform in ℓ_0, ℓ and n. Let us consider $f(x) := 1 - \cos x$ and fix $x_0 \in (0, \frac{\pi}{2}]$. Thus, $f'(x) = \sin x \geq \sin x_0$ for every $x \in [x_0, \pi - x_0]$. Integrating in x, we see that

$$f(x) \geq f(x_0) + (x - x_0)\sin x_0,$$

from where

$$\frac{f(x)}{f(x_0)} \geq 1 + \frac{\sin x_0}{1 - \cos x_0}(x - x_0) = \frac{x}{x_0} + (x - x_0)\Big(\frac{\sin x_0}{1 - \cos x_0} - \frac{1}{x_0}\Big)$$

for every $x_0 \in (0, \frac{\pi}{2}]$ and every $x \in [x_0, \pi - x_0]$. Therefore, the lemma will be proved if we show that

$$\frac{\sin x_0}{1 - \cos x_0} - \frac{1}{x_0} \geq 0, \tag{2.4.2}$$

because we can take $x_0 = \frac{\pi \ell_0}{n}$ and $x = \frac{\pi \ell}{n}$. Observe that

$$\frac{\sin x_0}{1 - \cos x_0} = \cot\big(\tfrac{x_0}{2}\big).$$

Therefore, the difference on the left-hand side of (2.4.2) is asymptotically equivalent to $\frac{1}{x_0}$ for $x_0 \ll 1$ and it is decreasing in x_0. For $x_0 = \frac{\pi}{2}$, the difference on the left-hand side of (2.4.2) is equal to $1 - \frac{2}{\pi} > 0$. Hence, the lemma is proved. $\square$

The following lemma is useful whenever we need a rough estimate on λ_ℓ^n:

Lemma 2.4.2. *For every* $\theta \geq 0$*,*

$$1 - \cos\theta \geq \tfrac{1}{2}\theta^2\big(1 - \tfrac{1}{12}\theta^2\big). \tag{2.4.3}$$

In particular, for every $\ell \in \{1, \dots, n-1\}$*,*

$$\lambda_\ell^n \geq \pi^2\ell^2\big(1 - \tfrac{\pi^2\ell^2}{12n^2}\big) \tag{2.4.4}$$

and for every $n \in \{2, 3, \dots\}$*,*

$$\lambda_1^n \geq \pi^2\big(1 - \tfrac{\pi^2}{12n^2}\big) \geq \pi^2\big(1 - \tfrac{\pi^2}{48}\big) \geq \tfrac{3\pi^2}{4}. \tag{2.4.5}$$

Proof. Let us define $F : [0, +\infty) \to \mathbb{R}$ as

$$F(\theta) = 1 - \cos\theta - \tfrac{1}{2}\theta^2\big(1 - \tfrac{1}{12}\theta^2\big).$$

Let us denote the k-th derivative of F by $F^{(k)}$. Computing the first six derivatives of F we obtain

$$F^{(1)}(\theta) = \sin\theta + \frac{1}{6}\theta(\theta^2 - 6)\ ,\ \ F^{(2)}(\theta) = \frac{\theta^2}{2} - (1 - \cos\theta)\ \ ,\ \ F^{(3)}(\theta) = \theta - \sin\theta$$

$$F^{(4)}(\theta) = 1 - \cos\theta\ ,\ \ F^{(5)}(\theta) = \sin\theta\ \ \text{and}\ \ F^{(6)}(\theta) = \cos\theta.$$

Since $F'(\theta) = 0$ if and only if $x = 0$, since $F^{(k)}(0) = 0$ for every $k \in \{1, \dots, 5\}$ and since $F^{(6)}(0) > 0$, $\theta = 0$ is a global minimizer. Therefore, for any $\theta \geq 0$

$$F(\theta) \geq F(0) = 0,$$

which implies (2.4.3). Inequality (2.4.4) follows from (2.4.3) and from the definition of λ_ℓ^n. Inequality (2.4.5) follows from (2.4.4) and from the facts that $n \geq 2$ and $\pi^2/(48) < 1/4$. $\square$

Our next estimate establishes the exponential decay of the ℓ_∞-norm of the discrete gradient of u_t^n:

Lemma 2.4.3. *For every* $n \in \{2, 3, \dots\}$*, every* $u_0 : \Lambda_n \to [0, 1]$*, every* $x \in \Lambda_n$ *and every* $t \geq \frac{1}{\lambda_1^n}\log 2$*, the solution of* (2.2.2) *satisfies*

$$n\big|u_t^n(x+1) - u_t^n(x)\big| \leq 8\pi e^{-\lambda_1^n t}. \tag{2.4.6}$$

Proof. Observe that $|c_\ell^n| \leq 2$ for every $\ell \in \{1, \dots, n-1\}$. Thus, using the bound

$$|\sin x - \sin y| \leq |x - y|,$$

which is valid for every $x, y \in \mathbb{R}$, we see that

$$n|u_t^n(x+1) - u_t^n(x)| \le \sum_{\ell=1}^{n-1} 2ne^{-\lambda_\ell^n t}\left|\sin\left(\tfrac{\pi \ell x}{n}\right) - \sin\left(\tfrac{\pi\ell(x+1)}{n}\right)\right| \le \sum_{\ell=1}^{n-1} 2\pi\ell e^{-\lambda_\ell^n t}. \quad (2.4.7)$$

From Lemma 2.4.1,

$$\sum_{\ell=1}^{n-1} 2\pi\ell e^{-\lambda_\ell^n t} \le \sum_{\ell=1}^{\infty} 2\pi\ell e^{-\lambda_1^n \ell t} = \frac{2\pi e^{-\lambda_1^n t}}{(1-e^{-\lambda_1^n t})^2}.$$

Putting this estimate into (2.4.7), we obtain the estimate

$$n|u_t^n(x+1) - u_t^n(x)| \le \sum_{\ell=1}^{n-1} 2\pi\ell e^{-\lambda_1^n \ell t} \le \frac{2\pi e^{-\lambda_1^n t}}{(1-e^{-\lambda_1^n t})^2}. \quad (2.4.8)$$

Last, observe that for every $t \ge \frac{1}{\lambda_1^n}\log 2$, the denominator of this expression is bounded below by $\frac{1}{4}$, which proves the lemma. □

Remark 2.4.4. *Observe that in this lemma we are not assuming any condition on the Lipschitz constant of u_0. Therefore, a lower bound on the time t at which* (2.4.6) *holds is needed. In particular the restriction $t \ge \frac{1}{\lambda_1^n}\log 2$ is sharp up to a constant.*

Remark 2.4.5. *Being more careful on the computations, it is possible to replace λ_1^n by π^2, at the cost of taking n large enough and $t \le n^2$. Since we only need an exponential decay in this lemma, we did not pursue a more refined bound.*

Define $\varphi_\ell : [0,1] \to \mathbb{R}$ as $\varphi_\ell(x) = \sqrt{2}\sin(\pi\ell x)$ for every $x \in [0,1]$. Observe that $\varphi_\ell^n(x) = \varphi_\ell\big(\frac{x}{n}\big)$. Observe as well that the Fourier coefficients c_ℓ^n are Riemann sums of the Fourier coefficients of u_0, $c_\ell(u_0)$, in the continuous interval. We have the following lemma:

Lemma 2.4.6. *Let $\kappa > 0$ and $\ell_0 \in \{2, 3. \dots\}$. There exists a constant $C = C(\kappa, \ell_0)$ such that*

$$\big|c_\ell^n(u_0) - c_\ell(u_0)\big| \le \frac{C}{n}$$

for every $n \in \{\ell_0+1, \ell_0+2, \dots\}$, every $\ell \in \{1, \dots, \ell_0\}$ and every $u_0 : [0,1] \to [0,1]$ such that $u_0(0) = u_0(1) = \rho$ and $\|u'\|_\infty \le \kappa$.

Proof. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function satisfying $f(0) = f(1) = 0$. Thus,

$$\begin{aligned}\left|\frac{1}{n}\sum_{x\in\Lambda_n} f\big(\tfrac{x}{n}\big) - \int_0^1 f(x)dx\right| &= \left|\sum_{x=1}^{n-1}\left\{\frac{1}{n}f\big(\tfrac{x}{n}\big) - \int_{\frac{x}{n}}^{\frac{x+1}{n}} f(y)dy\right\} - \int_0^{1/n} f(y)dy\right| \\ &= \left|\left\{\sum_{x=1}^{n-1}\int_{\frac{x}{n}}^{\frac{x+1}{n}} \left\{f\big(\tfrac{x}{n}\big) - f(y)\right\}dy\right\} - \int_0^{1/n}(f(y)-f(0))dy\right| \\ &\le \left\{\sum_{x=1}^{n-1}\int_{\frac{x}{n}}^{\frac{x+1}{n}} \left|f\big(\tfrac{x}{n}\big) - f(y)\right|dy\right\} + \int_0^{1/n}|f(y)-f(0)|dy.\end{aligned}$$

By the mean-value Theorem, the right-hand side of the above equation is bounded from above by

$$\|f'\|_\infty \sum_{x=1}^{n-1} \int_{\frac{x}{n}}^{\frac{x+1}{n}} \left| y - \frac{x}{n} \right| dy + \|f'\|_\infty \int_0^{1/n} y\, dy.$$

Therefore,

$$\left| \frac{1}{n} \sum_{x \in \Lambda_n} f\left(\tfrac{x}{n}\right) - \int_0^1 f(x) dx \right| \leq \frac{2\|f'\|_\infty}{n}. \tag{2.4.9}$$

The lemma follows by computing the derivatives of the functions $2(u_0(x) - \rho)\varphi_\ell(x)$ for $\ell \leq \ell_0$. $\square$

Remark 2.4.7. *The error in* (2.4.9) *is bounded by* $\frac{\|f''\|_\infty}{24n^2}$ *if* f *is twice differentiable, but we do not need that precision here.*

Our next lemma derives the asymptotic behaviour of u_t^n on the relevant time window:

Lemma 2.4.8. *Let* $\ell_0 \in \mathbb{N}$ *and let* $u_0 : [0,1] \to [0,1]$ *be differentiable, such that* $u_0(0) = u_0(1) = \rho$ *and such that* $c_\ell(u_0) = 0$ *for every* $\ell \in \{1, \dots, \ell_0 - 1\}$*. For every* $B > 0$*,*

$$\lim_{n \to \infty} \sup_{|b| \leq B} \sup_{x \in \Lambda_n} \left| \sqrt{n}(u_{t^n(b)}^n(x) - \rho) - c_{\ell_0}(u_0) e^{-b} \varphi_\ell^n(x) \right| = 0,$$

where

$$t^n(b) := \frac{1}{2\lambda_{\ell_0}^n} \log n + \frac{1}{\lambda_{\ell_0}^n} b.$$

Proof. The idea is to divide the sum in (2.4.1) into two parts:

$$\sqrt{n}\left| u_t^n(x) - \rho - c_{\ell_0}(u_0) e^{-\lambda_{\ell_0}^n t} \varphi_\ell^n(x) \right| \leq \sqrt{2n} \sum_{\ell=1}^{\ell_0} \left| c_\ell^n(u_0) - c_\ell(u_0) \right| + \sqrt{n} \sum_{\ell=\ell_0+1}^{n-1} 2e^{-\lambda_\ell^n t}.$$

By Lemma 2.4.6,

$$\sqrt{2n} \sum_{\ell=1}^{\ell_0} \left| c_\ell^n(u_0) - c_\ell(u_0) \right| \leq \frac{C(\|u'\|_\infty, \ell_0)}{\sqrt{n}}.$$

By Lemma 2.4.1,

$$\sqrt{n} \sum_{\ell=\ell_0+1}^{n-1} 2e^{-\lambda_\ell^n t} \leq \sqrt{n} \sum_{\ell=\ell_0+1}^{\infty} 2e^{-\frac{\lambda_{\ell_0+1}^n}{\ell_0+1} \ell t} = \frac{2\sqrt{n} e^{-\lambda_{\ell_0+1}^n t}}{1 - e^{-\lambda_{\ell_0+1}^n t}}.$$

Observe that

$$\lim_{n \to \infty} \frac{\lambda_{\ell_0+1}^n}{\lambda_{\ell_0}^n} = \left(1 + \tfrac{1}{\ell_0}\right)^2.$$

Therefore, there exists $n_0 = n_0(\ell_0)$ such that $\frac{\lambda_{\ell_0+1}^n}{\lambda_{\ell_0}^n} \geq 1 + \frac{1}{\ell_0}$ for every $n \geq n_0$. Observe that the function $s \mapsto \frac{e^{-s}}{1-e^{-s}}$ is decreasing in s. Therefore, for every $b \in [-B, B]$,

$$\sqrt{n} \sum_{\ell=\ell_0+1}^{n} 2e^{-\lambda_\ell^n t^n(b)} \leq \frac{2\sqrt{n} e^{-(1+\frac{1}{\ell_0})(\frac{1}{2}\log n - B)}}{1 - e^{-(1+\frac{1}{\ell_0})(\frac{1}{2}\log n - B)}} \leq \frac{2e^{2B}}{n^{\frac{1}{2\ell_0}}(1 - e^{-2B})} \leq \frac{C(B)}{n^{\frac{1}{2\ell_0}}}.$$

The numerical value of the constant $C(B)$ is not really important, the decay in n is what we need. Observing that $\sqrt{n}e^{-\lambda^n_{\ell_0} t^n(b)} = e^{-b}$, the lemma is proved. $\square$

2.5 The relative entropy method

The goal of this section is to prove Theorem 2.2.2. The proof relies in the so-called *Yau's relative entropy method*, introduced in [19].

Let us recall the definition of the *carré du champ* operator, placed just above Theorem 2.2.4. Let us use the Bernoulli product measure $\bar{\nu}^n_\rho$ as a reference measure in Ω_n. Recall the reference measures $\nu^n_t = \nu_{u^n_t(\cdot)}$ defined after (2.2.2). Let $\psi^n_t : \Omega_n \to [0,\infty)$ be the Radon-Nikodym derivative of ν^n_t with respect to $\bar{\nu}^n_\rho$, that is, $\psi^n_t(\eta) = \frac{\nu^n_t(\eta)}{\bar{\nu}^n_\rho(\eta)}$ for every $\eta \in \Omega_n$. Let $\mathfrak{L}^*_{n,t}$ be the adjoint of $\mathfrak{L}_n$ with respect to ν^n_t. The action of $\mathfrak{L}^*_{n,t}$ over a function $g : \Omega_n \to \mathbb{R}$ is given by

$$
\begin{aligned}
\mathfrak{L}^*_{n,t} g(\eta) &:= n^2 \sum_{x=1}^{n-2} \Big(g(\eta^{x,x+1}) \frac{\nu^n_t(\eta^{x,x+1})}{\nu^n_t(\eta)} - g(\eta) \Big) \\
&+ n^2 \sum_{x \in \{1,n-1\}} \Big(\big(\rho\eta(x) + (1-\rho)(1-\eta(x))\big) g(\eta^x) \frac{\nu^n_t(\eta^x)}{\nu^n_t(\eta)} - \big(\rho(1-\eta(x)) + (1-\rho)\eta(x)\big) g(\eta) \Big)
\end{aligned} \tag{2.5.1}
$$

for every $\eta \in \Omega_n$.

Now we invoke Yau's relative entropy inequality:

Proposition 2.5.1 (Yau's inequality)**.** *For each $t \geq 0$, let μ^n_t be the law of η_t with respect to $\mathbb{P}_{\nu^n_0}$ and let f^n_t be the Radon-Nikodym derivative of μ^n_t with respect to ν^n_t. Recall that $H_n(t) := H_{\nu^n_t}(f^n_t)$. We have that*

$$
\tfrac{d}{dt} H_n(t) \leq - \int \Gamma_n \sqrt{f^n_t} d\nu^n_t + \int f^n_t \big(\mathfrak{L}^*_{n,t} \mathbb{1} - \partial_t \log \psi^n_t \big) d\nu^n_t,
$$

where $\mathbb{1}$ is the constant function equal to 1.

Proof. This proof can be found in [12, Lemma A.1] in a more general scenario. Let $\mathfrak{L}^*_{n,t}$ be the adjoint of $\mathfrak{L}_n$ with respect to the reference measure ν^n_t. The forward Fokker-Planck equation asserts that

$$
\frac{d}{dt}\big(f^n_t\, \psi^n_t\big) = \mathfrak{L}^*_{n,t}\big(f^n_t\, \psi^n_t\big)
$$

for any $t \geq 0$, from where

$$
\frac{d}{dt} f^n_t = \frac{1}{\psi^n_t} \Big(\mathfrak{L}^*_{n,t}\big(f^n_t \psi^n_t\big) - f^n_t \frac{d}{dt} \psi^n_t \Big).
$$

Therefore, rewriting $H_n(t)$ as $\int f_t^n \log f_t^n \psi_t^n d\bar{\nu}_\rho^n$, we see that

$$\begin{aligned}\frac{d}{dt}H_n(t) &= \int (1+\log f_t^n)\big(\mathfrak{L}_{n,t}^*\big(f_t^n\psi_t^n\big) - f_t^n\frac{d}{dt}\psi_t^n\big)d\bar{\nu}_\rho^n \\ &\quad + \int f_t^n \log f_t^n \frac{d}{dt}\psi_t^n d\bar{\nu}_\rho^n \\ &= \int f_t^n \mathfrak{L}_n \log f_t^n d\nu_t^n - \int f_t^n \frac{d}{dt}\log\psi_t^n d\nu_t^n.\end{aligned}$$

Now, since $a(\log b - \log a) \leq 2\sqrt{a}(\sqrt{b}-\sqrt{a})$, we obtain

$$\begin{aligned}f_t^n(\eta)\mathfrak{L}_n \log f_t^n(\eta) &= \sum_{\xi\in\Omega_n} r(\eta,\xi) f_t(\eta)\big(\log f_t^n(\xi) - \log f_t^n(\eta)\big) \\ &\leq \sum_{\xi\in\Omega_n} 2r(\eta,\xi)\sqrt{f_t^n(\eta)}\big(\sqrt{f_t^n(\xi)}-\sqrt{f_t^n(\eta)}\big)\end{aligned}$$

for any $\eta\in\Omega_n$, where $r(\eta,\xi)$ stands for the jump rate from configuration η to ξ. Moreover, since $2\sqrt{a}(\sqrt{b}-\sqrt{a}) = -(\sqrt{b}-\sqrt{a})^2 + b - a$, we conclude that

$$2\,r(\eta,\xi)\sqrt{f_t^n(\eta)}\big(\sqrt{f_t^n(\xi)}-\sqrt{f_t^n(\eta)}\big) = -r(\eta,\xi)\big(\sqrt{f_t^n(\xi)}-\sqrt{f_t^n(\eta)}\big)^2 + r(\eta,\xi)\big(f_t^n(\xi)-f_t^n(\eta)\big).$$

Therefore, $2\sqrt{f_t^n}\,\mathfrak{L}_n\sqrt{f_t^n} = -\Gamma_n\sqrt{f_t^n} + \mathfrak{L}_n f_t^n$. Hence, we obtain that

$$\frac{d}{dt}H_n(t) \leq -\int \Gamma_n\sqrt{f_t^n}d\nu_t^n + \int\Big(\mathfrak{L}_n f_t^n - f_t^n\frac{d}{dt}\log\psi_t^n\Big)d\nu_t^n,$$

which implies the desired inequality due to the fact that $\int \mathfrak{L}_n f_t^n d\nu_t^n = \int \mathfrak{L}_{n,t}^*\mathbb{1} f_t^n d\nu_t^n$. □

Now that we know Yau's inequality, observe that

$$\psi_t^n(\eta) = \prod_{x\in\Lambda_n}\bigg(\eta(x)\frac{u_t^n(x)}{\rho} + (1-\eta(x))\frac{1-u_t^n(x)}{1-\rho}\bigg). \tag{2.5.2}$$

Using this expression and (2.5.1), it is possible to compute $\mathfrak{L}_{n,t}^*\mathbb{1} - \partial_t\log\psi_t^n$, explicitly. Indeed, for each $x\in\Lambda_n$, let us define

$$\omega_x := \frac{\eta(x)-u_t^n(x)}{u_t^n(x)(1-u_t^n(x))}.$$

Observe that ω_x also depends on t. In order to compact notation, we will not include this dependence in ω_x. We have

$$\begin{aligned}\mathfrak{L}_{n,t}^*\mathbb{1}(\eta) &= n^2\sum_{x\in\Lambda_n}\eta(x+1)(1-\eta(x))\Big(\frac{\nu_t^n(\eta^{x,x+1})}{\nu_t^n(\eta)}-1\Big) + n^2\sum_{x\in\partial\Lambda_n}\eta(x)\Big(\rho\frac{\nu_t^n(\eta^x)}{\nu_t^n(\eta)}-(1-\rho)\Big) \\ &+ n^2\sum_{x\in\Lambda_n}\eta(x)(1-\eta(x+1))\Big(\frac{\nu_t^n(\eta^{x,x+1})}{\nu_t^n(\eta)}-1\Big) + n^2\sum_{x\in\partial\Lambda_n}(1-\eta(x))\Big((1-\rho)\frac{\nu_t^n(\eta^x)}{\nu_t^n(\eta)}-\rho\Big)\end{aligned}$$

$$
\begin{aligned}
&= n^2 \sum_{x\in\Lambda_n} \eta(x+1)(1-\eta(x))\Big(\frac{u_t^n(x)(1-u_t^n(x+1))}{u_t^n(x+1)(1-u_t^n(x))} - 1\Big) \\
&+ n^2 \sum_{x\in\Lambda_n} \eta(x)(1-\eta(x+1))\Big(\frac{u_t^n(x+1)(1-u_t^n(x))}{u_t^n(x)(1-u_t^n(x+1))} - 1\Big) \\
&+ n^2 \sum_{x\in\partial\Lambda_n} \frac{\eta(x)}{u_t^n(x)} \left(\rho(1-u_t^n(x)) - (1-\rho)u_t^n(x)\right) \\
&+ n^2 \sum_{x\in\partial\Lambda_n} \frac{1-\eta(x)}{1-u_t^n(x)} \left((1-\rho)u_t^n(x) - \rho(1-u_t^n(x))\right).
\end{aligned}
$$

Observe that the above identity can be rewritten as

$$
\begin{aligned}
\mathfrak{L}_{n,t}^* \mathbb{1}(\eta) &= n^2 \sum_{x\in\Lambda_n} \frac{\eta(x+1)(1-\eta(x))}{u_t^n(x+1)(1-u_t^n(x))}\Big(u_t^n(x)(1-u_t^n(x+1)) - u_t^n(x+1)(1-u_t^n(x))\Big) \\
&+ n^2 \sum_{x\in\Lambda_n} \frac{\eta(x)(1-\eta(x+1))}{u_t^n(x)(1-u_t^n(x+1))}\Big(u_t^n(x+1)(1-u_t^n(x)) - u_t^n(x)(1-u_t^n(x+1))\Big) \\
&+ n^2 \sum_{x\in\partial\Lambda_n} \left(\frac{\eta(x)}{u_t^n(x)} - \frac{1-\eta(x)}{1-u_t^n(x)}\right)\Big(\rho(1-u_t^n(x)) - (1-\rho)u_t^n(x)\Big) \\
&= n^2 \sum_{x\in\Lambda_n} \left(\frac{\eta(x+1)(1-\eta(x))}{u_t^n(x+1)(1-u_t^n(x))} - \frac{\eta(x)(1-\eta(x+1))}{u_t^n(x)(1-u_t^n(x+1))}\right) \\
&\qquad\qquad \times \Big(u_t^n(x)(1-u_t^n(x+1)) - u_t^n(x+1)(1-u_t^n(x))\Big) \\
&+ n^2 \sum_{x\in\partial\Lambda_n} \left(\frac{\eta(x)}{u_t^n(x)} - \frac{1-\eta(x)}{1-u_t^n(x)}\right)\Big(\rho(1-u_t^n(x)) - (1-\rho)u_t^n(x)\Big) \\
&= n^2 \sum_{x\in\Lambda_n} \left(\frac{\eta(x+1)(1-\eta(x))}{u_t^n(x+1)(1-u_t^n(x))} - \frac{\eta(x)(1-\eta(x+1))}{u_t^n(x)(1-u_t^n(x+1))}\right)\Big(u_t^n(x) - u_t^n(x+1)\Big) \\
&+ n^2 \sum_{x\in\partial\Lambda_n} \left(\frac{\eta(x)}{u_t^n(x)} - \frac{1-\eta(x)}{1-u_t^n(x)}\right)\Big(\rho - u_t^n(x)\Big).
\end{aligned}
$$

Since $u_t^n(x) = \rho$ for any $x \in \partial\Lambda_n$ we have

$$
\mathfrak{L}_{n,t}^* \mathbb{1}(\eta) = n^2 \sum_{x\in\Lambda_n} \left(\frac{\eta(x+1)(1-\eta(x))}{u_t^n(x+1)(1-u_t^n(x))} - \frac{\eta(x)(1-\eta(x+1))}{u_t^n(x)(1-u_t^n(x+1))}\right)\Big(u_t^n(x) - u_t^n(x+1)\Big).
$$

The next step is to rewrite $\frac{\eta(x+1)(1-\eta(x))}{u_t^n(x+1)(1-u_t^n(x))} - \frac{\eta(x)(1-\eta(x+1))}{u_t^n(x)(1-u_t^n(x+1))}$ as a linear combination of 1, ω_x, ω_{x+1} and $\omega_x\omega_{x+1}$. Indeed, assume that

$$
\frac{\eta(x+1)(1-\eta(x))}{u_t^n(x+1)(1-u_t^n(x))} - \frac{\eta(x)(1-\eta(x+1))}{u_t^n(x)(1-u_t^n(x+1))} = a + b\omega_x + c\omega_{x+1} + d\omega_x\omega_{x+1}
$$

for some $a, b, c, d \in \mathbb{R}$. Taking the expectation of the above identity with respect to ν_t^n, we see that $a = 0$. Evaluating this identity at $\eta(x) = 1$ and $\eta(x+1) = u_t^n(x+1)$ (which is equivalent to taking expectations with respect to Bern$(1)\times$Bern$(u_t^n(x))$), we see that $b = -1$. Similarly, evaluating the identity at $\eta(x+1) = 1$ and $\eta(x) = u_t^n(x)$, we see that

$c=1$. Last, evaluating the identity at $\eta(x)=\eta(x+1)=1$, we obtain that

$$\frac{d}{u_t^n(x)u_t^n(x+1)}-\frac{1}{u_t^n(x)}+\frac{1}{u_t^n(x+1)}=0,$$

from which we conclude that $d=u_t^n(x+1)-u_t^n(x)$. Hence,

$$\mathfrak{L}_{n,t}^*\mathbb{1}(\eta)=n^2\sum_{x\in\Lambda_n}\Big(\omega_{x+1}-\omega_x+(u_t^n(x+1)-u_t^n(x))\omega_x\omega_{x+1}\Big)\Big(u_t^n(x)-u_t^n(x+1)\Big).$$

Summing the above expression by parts and using the fact that $u_t^n(x)=\rho$ for any $x\in\partial\Lambda_n$, we conclude that

$$\mathfrak{L}_{n,t}^*\mathbb{1}(\eta)=\sum_{x\in\Lambda_n}\Delta_n u_t^n(x)\omega_t(x)-\sum_{x=1}^{n-2}n^2\big(u_t^n(x+1)-u_t^n(x)\big)^2\omega_x\omega_{x+1}, \tag{2.5.3}$$

Now observe that

$$\begin{aligned}\frac{d}{dt}\log\psi_t^n(\eta)&=\frac{d}{dt}\sum_{x\in\Lambda_n}\left(\eta(x)\log(2u_t^n(x))+(1-\eta(x))\log(2(1-u_t^n(x)))\right)\\&=\sum_{x\in\Lambda_n}\left(\frac{\eta(x)}{u_t^n(x)}-\frac{1-\eta(x)}{1-u_t^n(x)}\right)\frac{d}{dt}u_t^n(x)\\&=\sum_{x\in\Lambda_n}\omega_x\frac{d}{dt}u_t^n(x).\end{aligned}$$

Since $\frac{d}{dt}u_t^n(x)=(\Delta_n u_t^n)(x)$ for any $x\in\Lambda_n$ we conclude that

$$\mathfrak{L}_{n,t}^*\mathbb{1}-\partial_t\log\psi_t^n=-\sum_{x=1}^{n-2}n^2\big(u_t^n(x+1)-u_t^n(x)\big)^2\omega_x\omega_{x+1}.$$

Observe that integrating a function that depends on η_t with respect to $f_t^n d\nu_t^n$ is equivalent to taking expectations with respect to the law $\mathbb{P}_{\nu_0^n}$. Therefore,

$$\tfrac{d}{dt}H_n(t)\leq-\int\Gamma_n\sqrt{f_t^n}d\nu_t^n-\sum_{x=1}^{n-2}n^2\big(u_t^n(x+1)-u_t^n(x)\big)^2\mathbb{E}_{\nu_0^n}\big[\omega_x\omega_{x+1}\big]. \tag{2.5.4}$$

We see that it would be good to have an estimate for $\mathbb{E}_{\nu_0^n}[\omega_x\omega_{x+1}]$. We recall that the expression inside the previous expectation depends on t. The following proposition follows from [14, Lemma 4.1 and Proposition 4.4]:

Proposition 2.5.2. *For every $n\in\{2,3,\dots\}$, every profile $u\in\mathcal{U}_{\varepsilon_0,\kappa}$, every $x\in\Lambda_{n-1}$ and every $t\geq 0$,*

$$\big|\mathbb{E}_{\nu_0^n}[\omega_x\omega_{x+1}]\big|\leq\frac{4\kappa(2+\kappa)}{\varepsilon_0^2 n}$$

and

$$n\big|u_t^n(x+1)-u_t^n(x)\big|\leq\kappa. \tag{2.5.5}$$

We are finally ready to prove Theorem 2.2.2.

Proof of Theorem 2.2.2. By Proposition 2.5.2 and (2.5.4), we have

$$\tfrac{d}{dt}H_n(t) \le -\int \Gamma_n \sqrt{f_t^n}\, d\nu_t^n + \frac{4\kappa^3(2+\kappa)}{\varepsilon_0^2}. \tag{2.5.6}$$

By Theorem 2.2.4,

$$\tfrac{d}{dt}H_n(t) \le -K_0 H_n(t) + \frac{4\kappa^3(2+\kappa)}{\varepsilon_0^2}.$$

Thus, by Grönwall's Lemma we conclude that

$$H_n(t) \le \frac{4\kappa^3(2+\kappa)}{K_0\varepsilon_0^2} \tag{2.5.7}$$

for every $t \ge 0$, which means that $H_n(t)$ is uniformly bounded in t by a constant that only depends on ρ, ε_0 and κ.

In order to show that $H_n(t)$ decays to 0 in t, we need to take advantage of the presence of the discrete gradient $n(u_t^n(x+1) - u_t^n(x))$ in the expression for $\mathfrak{L}_{n,t}^*\mathbb{1} - \partial_t \log \psi_t^n$. By Lemma 2.4.3,

$$n\big|u_t^n(x+1) - u_t^n(x)\big| \le 8\pi e^{-\lambda_1^n t}$$

for every $n \in \{2, 3, \dots\}$, every $x \in \Lambda_n$ and every $t \ge \frac{1}{\lambda_1^n}\log 2$. Therefore, for $t \ge t_0^n := \frac{1}{\lambda_1^n}\log 2$,

$$\tfrac{d}{dt}H_n(t) \le -K_0 H_n(t) + \frac{2^8\pi^2\kappa(2+\kappa)e^{-2\lambda_1^n t}}{\varepsilon_0^2}.$$

Assume that $2\lambda_1^n > K_0$. Integrating last inequality between t_0^n and $t_0^n + t$ and using (2.5.7) to estimate $H_n(t_0)$, we conclude that

$$H_n(t_0^n + t) \le \Big(\frac{4\kappa^3(2+\kappa)}{K_0\varepsilon_0^2} + \frac{2^8\pi^2\kappa(2+\kappa)}{(2\lambda_1^n - K_0)\varepsilon_0^2}\Big)e^{-K_0 t}.$$

If $2\lambda_1^n < K_0$, the estimate holds with exponential factor $e^{-2\lambda_1^n t}$. If $2\lambda_1^n = K_0$, the estimate holds with exponential factor $e^{-(2\lambda_1^n - \delta)t}$ for any $\delta > 0$. By Lemma 2.4.2, $\lambda_1^n \ge \frac{3\pi^2}{4}$. Taking $t_0 = \frac{4}{3\pi^2}\log 2$, in each case Theorem 2.2.2 is proved. □

2.6 Total variation distance between profile measures

We finally prove Theorem 2.2.3.

Proof of Theorem 2.2.3. By definition,

$$\|\nu_t^n - \bar{\nu}_\rho^n\|_{TV} = \frac{1}{2}\int |\psi_t^n - 1|\, d\bar{\nu}_\rho^n.$$

Recall that ψ_t^n has an explicit formula, see (2.5.2). Observe that ψ_t^n can be rewritten in the form

$$\psi_t^n := \exp\Big(\sum_{x \in \Lambda_n} \big(a_t^n(x)(\eta_x - \rho) - b_t^n(x) \big) \Big)$$

for

$$a_t^n(x) = \log \tfrac{u_t^n(x)}{\rho} - \log \tfrac{1-u_t^n(x)}{1-\rho}$$

and

$$b_t^n(x) = -\rho \log \tfrac{u_t^n(x)}{\rho} - (1-\rho) \log \tfrac{1-u_t^n(x)}{1-\rho}.$$

The sum

$$\sum_{x \in \Lambda_n} a_t^n(x)(\eta_x - \rho)$$

can be understood as a triangular array of independent, centered random variables, and in particular it should converge, after a suitable renormalization, to a Gaussian random variable.

Let us define

$$s_t^n := \Big(\rho(1-\rho) \sum_{x \in \Lambda_n} a_t^n(x)^2 \Big)^{1/2},$$

$$X_t^n := \frac{1}{s_t^n} \sum_{x \in \Lambda_n} a_t^n(x)(\eta_x - \rho),$$

$$b_t^n := \sum_{x \in \Lambda_n} b_t^n(x).$$

With these notations, we have that

$$\psi_t^n = \exp\{ s_t^n X_t^n - b_t^n \},$$

and the analysis of $\|\nu_t^n - \bar{\nu}_\rho^n\|_{TV}$ reduces to the analysis of s_t^n, X_t^n and b_t^n. Recall that we are interested in the behavior of these quantities for $t = \frac{1}{2\pi^2 \ell_0^2} \log n + \frac{b}{\pi^2 \ell_0^2}$. In order to simplify notation, from now on we fix $B > 0$ and we will take

$$t = t^n(b) := \frac{1}{2\lambda_{\ell_0}^n} \log n + \frac{1}{\lambda_{\ell_0}^n} b$$

with $b \in [-B, B]$. Here and below we denote by $R_t^{n,i}(x)$ an error term that goes to 0 as $n \to \infty$, uniformly in $x \in \Lambda_n$ and $b \in [-B, B]$. The index i serves to indicate the places at which the error term changes. By Lemma 2.4.8,

$$u_t^n(x) = \rho + \tfrac{1}{\sqrt{n}} \big(c_{\ell_0}(u_0) e^{-b} \varphi_{\ell_0}^n(x) + R_t^{n,1}(x) \big).$$

By Taylor's formula, $\log(1+x) = x - x^2/2 + \mathcal{O}(x^3)$. Therefore,

$$a_t^n(x) = \frac{1}{\sqrt{n}} \Big(\frac{c_{\ell_0}(u_0) e^{-b} \varphi_{\ell_0}^n(x)}{\rho(1-\rho)} + R_t^{n,2}(x) \Big). \tag{2.6.1}$$

Now we can compute s_t^n:

$$(s_t^n)^2 = \rho(1-\rho)\sum_{x\in\Lambda_n} a_t^n(x)^2 = \frac{\rho(1-\rho)}{n}\sum_{x\in\Lambda_n}\Big(\frac{c_{\ell_0}(u_0)^2 e^{-2b}\varphi_{\ell_0}^n(x)^2}{\rho^2(1-\rho)^2} + R_t^{n,3}(x)\Big).$$

Since

$$\frac{1}{n}\sum_{x\in\Lambda_n}\varphi_{\ell_0}^n(x)^2$$

is a Riemann sum of the integral $2\int_0^1 \sin^2(\pi\ell_0 x)dx$, which is equal to 1, we see that

$$(s_t^n)^2 = \frac{c_{\ell_0}(u_0)^2 e^{-2b}}{\rho(1-\rho)} + R_t^{n,4},$$

from where

$$\lim_{n\to\infty}\sup_{b\in[-B,B]}\Big|s_t^n - \frac{|c_{\ell_0}(u_0)|e^{-b}}{\sqrt{\rho(1-\rho)}}\Big| = 0.$$

In order to compute b_t^n, we need to go one step further in Taylor's expansion of $\log(1+x)$. More precisely,

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \frac{x^3}{3} + \mathcal{O}(x^4).$$

Proceeding as before we see that

$$b_t^n = \frac{1}{n}\sum_{x\in\Lambda_n}\Big(\frac{c_{\ell_0}(u_0)^2 e^{-2b}\varphi_\ell^n(x)^2}{2\rho(1-\rho)} + R_t^{n,5}(x)\Big) = \frac{c_{\ell_0}(u_0)^2 e^{-2b}}{2\rho(1-\rho)} + R_t^{n,6},$$

and in particular we see that b_t^n and $\frac{1}{2}(s_t^n)^2$ have the same limit as $n\to\infty$.

Recall that we want to obtain the limit as $n\to\infty$ of

$$\frac{1}{2}\int\big|\exp\{s_t^n X_t^n - b_t^n\} - 1\big| d\nu_\rho. \tag{2.6.2}$$

Up to here, we have proved the convergence of s_t^n and b_t^n. By Lyapounov's criterion with fourth moment condition (Theorem A.2.1 with $\delta = 2$), X_t^n converges in law to a standard Gaussian law if

$$\lim_{n\to\infty}\frac{1}{(s_t^n)^4}\sum_{x\in\Lambda_n} a_t^n(x)^4\int(\eta_x-\rho)^4 d\nu_\rho = 0.$$

Observe that $\int(\eta_x-\rho)^4 d\nu_\rho = \rho(1-\rho)(1-3\rho+3\rho^2)$. The actual value of this integral is not relevant; it is only relevant that it is constant in n, x and t. Since s_t^n has a non-zero limit, we only need to prove that

$$\lim_{n\to\infty}\sum_{x\in\Lambda_n} a_t^n(x)^4 = 0.$$

From (2.6.1), we see that there exists a finite constant $C = C(u_0,\ell_0,B)$ such that

$|a_t^n(x)| \le \frac{C}{\sqrt{n}}$ for every $n \in \{2, 3, \dots\}$, every $x \in \Lambda_n$ and every $b \in [-B, B]$. Therefore,

$$\sum_{x \in \Lambda_n} a_t^n(x)^4 \le \frac{C^4}{n}$$

and Lyapounov's condition is satisfied. Therefore, by the Central Limit Theorem (Theorem A.2.1) and the above estimates, $s_t^n X_t^n - b_t^n$ converges in law to $\gamma e^{-b} X - \frac{1}{2}\gamma^2 e^{-2b}$, where $\gamma = \frac{|c_{\ell_0}(u_0)|}{\sqrt{\rho(1-\rho)}}$ and X has a standard Gaussian law. Since the exponential function is not bounded, one needs an additional argument in order to prove that (2.6.2) converges. A sufficient condition so that the integral (2.6.2) converges to

$$\tfrac{1}{2}\mathbb{E}\big[\big|e^{\gamma e^{-b} X - \frac{1}{2}\gamma^2 e^{-2b}} - 1\big|\big]$$

is that the sequence $\{e^{s_t^n X_t^n}; n \in \{2, 3, \dots\}\}$ is uniformly integrable. Since L^p-boundedness implies uniform integrability (see [2, Theorem 25.12]) for $p > 1$, it is enough to show that $\int e^{p s_t^n X_t^n} d\nu_\rho$ is uniformly bounded for some $p > 1$. From Hoeffding's Lemma (Lemma A.3.1), for any $p > 1$

$$\int e^{p s_t^n X_t^n} d\nu_\rho \le \exp\Big\{\sum_{x \in \Lambda_n} \tfrac{1}{8} p^2 a_t^n(x)^2\Big\} \le \exp\big\{\tfrac{1}{8} C^2 p^2\big\}.$$

Therefore, for any $b \in \mathbb{R}$,

$$\lim_{n \to \infty} \frac{1}{2} \int |\psi_t^n - 1| d\nu_\rho = \frac{1}{2}\mathbb{E}\big[\big|e^{\gamma e^{-b} X - \frac{1}{2}\gamma^2 e^{-2b}} - 1\big|\big] = \mathcal{G}(\gamma e^{-b}),$$

which proves the theorem. □

Chapter 3

Central limit theorem for non-equilibrium stationary states

In this chapter we perform a qualitative analysis of the stationary states of an example of what is known in the literature as a reaction-diffusion model [4]. Although there are some similar, or even matching, definitions, the notation in this chapter is independent of the one introduced in Chapter 2.

3.1 Reaction-diffusion model

Let $\mathbb{T}_n^d = (V_n, E_n)$ be the d-dimensional discrete torus. The vertex set of this graph can be represented by $V_n = \mathbb{Z}/n\mathbb{Z}$ and if we denote by e_i the i-th vector of the canonical basis of $\mathbb{R}^d$, then the edges in E_n are the pairs xy such that $y = x \pm e_i$ for some $i \in \{1, \ldots, d\}$. Abusing notation, from now on we will often write $\mathbb{T}_n^d$ for its vertex set V_n.

Let $\Omega_n = \{0,1\}^{\mathbb{T}_n^d}$ be the set of functions $\eta : \mathbb{T}_n^d \to \{0,1\}$. We will call an element $\eta \in \Omega_n$ a configuration of particles, meaning that vertices with value 1 are occupied by a particle and that vertices with value 0 are empty. There are two types of interactions in the *reaction-diffusion* model: the symmetric simple exclusion dynamics and the Glauber dynamics. The exclusion dynamics just flip edges of the graph with rate 1, exchanging the occupation numbers of adjacent vertices. This makes particles perform nearest-neighbor random walks with the *exclusion rule*, which forbids a vertex to be occupied by more than one particle. The Glauber dynamics creates a particle in an empty vertex $x \in \mathbb{T}_n^d$ with rate $c_x^+(\eta) = 1 + \lambda \sum_{j=1}^d \eta(x - e_j)\eta(x + e_j)$, where $\lambda \geq 0$ is fixed, and annihilates a particle in an occupied vertex x with rate $c_x^-(\eta) = 1$. The infinitesimal generator of this reaction-diffusion model is given by

$$\mathfrak{L}_n f(\eta) = n^2 \mathfrak{L}_n^{exc} f(\eta) + \mathfrak{L}_n^{Glauber} f(\eta), \tag{3.1.1}$$

where

$$\mathfrak{L}_n^{exc} f(\eta) = \sum_{i=1}^{d} \sum_{x \in \mathbb{T}_n^d} \Big(f(\eta^{x,x+e_i}) - f(\eta) \Big)$$

and

$$\mathfrak{L}_n^{Glauber} f(\eta) = \sum_{x \in \mathbb{T}_n^d} \Big((1-\eta(x))c_x^+(\eta) + \eta(x)c_x^-(\eta) \Big) \Big(f(\eta^x) - f(\eta) \Big)$$

on any $f : \Omega_n \to \mathbb{R}$. Above $\eta^{x,y}$ stands for the configuration obtained from η after exchanging the occupation numbers of vertices x and y, and η^x stands for the configuration obtained from η after changing the occupation number of vertex x. Namely,

$$\eta^{x,y}(z) = \begin{cases} \eta(x), & \text{if } z = y; \\ \eta(y), & \text{if } z = x; \\ \eta(z), & \text{if } z \neq \{x, y\} \end{cases}$$

and

$$\eta^{x}(z) = \begin{cases} 1-\eta(x), & \text{if } z = x; \\ \eta(z), & \text{if } z \neq x. \end{cases}$$

3.1.1 Law of the process

Let μ_n be a probability measure on Ω_n. We will denote by $\{\eta_t; t \geq 0\}$ the continuous-time Markov process with generator $\mathfrak{L}_n$ and initial measure μ_n. We will denote by S_t the semigroup associated with $\mathfrak{L}_n$. Let $\mathcal{D}([0,T], \Omega_n)$ be the path space of càdlàg trajectories with values in Ω_n, known as the *Skorohod space*. Given a probability measure $\mu_n \in \Omega_n$, we denote by $\mathbb{P}_{\mu_n}$ the probability measure on $\mathcal{D}([0,T], \Omega_n)$ induced by the initial measure μ_n and the Markov process $\{\eta_t\,;\, t \geq 0\}$. We denote by $\mathbb{E}_{\mu_n}$ denote the expectation with respect to $\mathbb{P}_{\mu_n}$. We also use the notation $\eta_\cdot$ to represent elements of the Skorohod space $\mathcal{D}([0,T], \Omega_n)$, that is, the time trajectories of the reaction-diffusion model. This notation $\eta_\cdot$ should not be confused with the notation η for elements of Ω_n. Given any other probability measure μ and a random variable X we will use the notation $\mu(X)$ for the expectation of X with respect to μ.

3.1.2 Non-equilibrium scenario

Now we will discuss a little about the difficulties of the model. Given a function $u : \mathbb{T}_n^d \to [0,1]$, the Bernoulli product measure $\nu_{u(\cdot)}$ associated with the profile u is defined on Ω_n by

$$\nu_{u(\cdot)}(\eta) = \prod_{x \in \mathbb{T}_n^d} \Big\{ \eta(x)u(x) + (1-\eta(x))(1-u(x)) \Big\}.$$

When u is constant equal to p, we will write ν_p instead of $\nu_{u(\cdot)}$. Let $p \in (0,1)$ and let us define

$$F(p) := \int \mathfrak{L}_n \eta(0) d\nu_p = \int \Big\{ (1-\eta(0))c_0^+(\eta) - \eta(0)c_0^-(\eta) \Big\} d\nu_p. \tag{3.1.2}$$

In the above definition, the vertex 0 can be replaced by any other vertex.

The exclusion dynamics is already reversible with respect to the measures ν_p, $p \in [0,1]$. If λ were zero then we would obtain a completely reversible dynamics with respect to the measure $\nu_{1/2}$. In this case the process is said to be in equilibrium and the Bernoulli product measure with parameter $1/2$ is stationary, that is, $\int \mathfrak{L}_n f(\eta) d\nu_{1/2} = 0$ for every $f : \Omega_n \to \mathbb{R}$. In particular, $F(1/2) = 0$. In the non-equilibrium scenario, that is, $\lambda > 0$, we still can find $\rho \in (0,1)$ that satisfies $F(\rho) = 0$. Indeed, a simple computation shows that ρ is given by

$$\rho = (1-\rho)(1+\lambda d \rho^2). \tag{3.1.3}$$

Let $\Theta(\rho) = (1-\rho)(1+\lambda d\rho^2)$. Since $\Theta(1) = 0$, $\Theta(0) = 1$ and Θ is continuous, the Intermediate Value Theorem asserts that there exists $\rho \in (0,1)$ such that $\Theta(\rho) = \rho$. On the other hand, ν_ρ is not the stationary measure of the process: it can be checked, for instance, that

$$\int \mathfrak{L}_n \left(\eta(0)\eta(1)\right) d\nu_\rho \neq 0.$$

3.2 Density fluctuation field

Let the inner product in $L^2(\mathbb{T}^d)$ be given by $\langle f, g\rangle = \int_{\mathbb{T}^d} f(u)\overline{g(u)}du$. For each $k \in \mathbb{Z}^d$ let $\psi_k : \mathbb{T}^d \to \mathbb{C}$ be defined by

$$\psi_k(x) = e^{2\pi k i x} \tag{3.2.1}$$

for any $x \in \mathbb{T}^d$. The family $\{\psi_k; k \in \mathbb{Z}^d\}$ forms an orthonormal basis of $L^2(\mathbb{T}^d)$. For each bounded function $f : \mathbb{T}^d \to \mathbb{R}$, let $\hat{f} : \mathbb{Z}^d \to \mathbb{C}$ be the Fourier coefficient of f of order k, that is,

$$\hat{f}(k) := \langle f, \psi_k \rangle .$$

For each $f \in C^\infty(\mathbb{T}^d)$ and each $m \in \mathbb{R}$ let us define

$$\|f\|_m := \left(\sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 (1+|k|^2)^m \right)^{1/2},$$

where $|k| := (k_1^2 + \cdots + k_d^2)^{1/2}$. Observe that $\|f\|_m < \infty$ for each $m \in \mathbb{R}$ because $f \in C^\infty(\mathbb{T}^d)$. The Sobolev space $\mathcal{H}_m$ is defined as the closure of $C^\infty(\mathbb{T}^d)$ with respect to the norm $\|f\|_m$.

The *density fluctuation field* is defined on functions $G \in \mathcal{H}_k(\mathbb{T}^d)$, by

$$X_t^n(G) = \frac{1}{n^{d/2}} \sum_{x \in \mathbb{T}_n^d} (\eta_t(x) - \rho)\, G\left(\frac{x}{n}\right). \tag{3.2.2}$$

By duality, (3.2.2) defines a process $\{X_t^n(G); t \geq 0\}$ with values in $\mathcal{H}_{-k}(\mathbb{T}^d)$ for any $k > d/2$ (see [12, comments after Proposition C.5]).

3.2.1 Review of some results in the literature

The main result in [11], where the authors consider this same model, is the following:

Theorem 3.2.1. *Let $G \in \mathcal{H}_k(\mathbb{T}^d)$, $k > d/2$. Let us fix a time horizon $[0,T]$, $T > 0$. Assume that η_0 has law ν_ρ. For any $d \leq 3$, the sequence $\{X_t^n(G); n \in \mathbb{N}\}$ converges, with respect to the J_1-Skorohod topology on the Sobolev space $\mathcal{D}([0,T], \mathcal{H}_{-k}(\mathbb{T}^d))$, to the unique solution of the Ornstein-Uhlenbeck equation*

$$dX_t(G) = (\Delta - \Phi(\rho))\, X_t(G)dt + d\mathcal{W}_t(G)$$

where $d\mathcal{W}_t(G)$ denotes space-time white noise with quadratic variation

$$2t \left(\chi(\rho) \|\nabla G\|^2_{L^2(\mathbb{T}^d)} + \rho \|G\|^2_{L^2(\mathbb{T}^d)} \right),$$

$\chi(u) = u(1-u)$ and

$$\Phi(\rho) := -F'(\rho) = 2 + \lambda \rho^2 d - 2\lambda\chi(\rho) d > 0. \tag{3.2.3}$$

In particular, for $G \in \mathcal{H}_k(\mathbb{T}^d)$, the sequence $\{X_t^n(G); n \in \mathbb{N}\}$ is tight in the J_1-Skorohod topology of $\mathcal{D}([0,T], \mathbb{R})$ and if $X_\cdot(G)$ is a limit point, then the processes

$$\mathcal{M}_t(G) := X_t(G) - X_0(G) - \int_0^t X_s \left(\Delta G + \left(2\lambda\chi(\rho) d - 2 - \lambda \rho^2 d \right) G \right) ds$$

and

$$\mathcal{N}_t(G) := (\mathcal{M}_t(G))^2 - 2t \left(\chi(\rho) \|\nabla G\|^2_{L^2(\mathbb{T}^d)} + \rho \|G\|^2_{L^2(\mathbb{T}^d)} \right)$$

are mean-zero martingales with respect to the filtration $\mathcal{F}_t := \sigma\{X_s(g) : s \leq t$ and $g \in \mathcal{H}_k(\mathbb{T}^d)\}$.

The main ingredient to prove Theorem 3.2.1 is the following result:

Theorem 3.2.2. *Let $\rho \in (0,1)$ be given by (3.1.3). Let f_t denote the Radon-Nikodym derivative of the measure $\nu_\rho S_t$ with respect to the measure ν_ρ. Let $H(t) = \int f_t \log f_t d\nu_\rho$ be the relative entropy of f_t with respect to the measure ν_ρ. Thus, there exists a positive constant such that*

$$H(t) \leq C\, t\, n^{d-2} g_d(n), \tag{3.2.4}$$

where

$$g_d(n) = \begin{cases} n, & \text{if } d = 1; \\ \log n, & \text{if } d = 2; \\ 1, & \text{if } d \geq 3. \end{cases}$$

3.2.2 Improvement on the entropy bound

In Lemma 3.3.5 we state an improvement in the upper bound on the relative entropy $H(t)$ in Theorem 3.2.2. More precisely, we will remove the multiplicative factor t on the right-hand side of (3.2.4). This allows us to pass the inequality to the limit when $t \to \infty$ with fixed n. Our aim is to prove the following result:

Theorem 3.2.3. *Let $\rho \in (0,1)$ be given by (3.1.3) and let μ_{ss}^n be the stationary measure of the process $\{\eta_t; t \geq 0\}$. Let f denote the Radon-Nikodym derivative of the measure μ_{ss}^n with respect to the measure ν_ρ and let $H(\mu_{ss}^n|\nu_\rho) = \int f \log f d\nu_\rho$ be the relative entropy of f with respect to the measure ν_ρ. There exist positive constants λ^*, C such that for any $\lambda < \lambda^*$,*

$$H(\mu_{ss}^n|\nu_\rho) \leq C\,\lambda\, n^{d-2} g_d(n).$$

3.2.3 Hydrostatic limit

The hydrodynamic limit of the process $\{\eta_t; t \geq 0\}$ was also proven in [11]. It is a consequence of Theorem 3.2.2 and it states the following:

Theorem 3.2.4 (Hydrodynamic limit). *Let ρ_t be the strong solution of the reaction-diffusion equation*

$$\begin{cases} \frac{d}{dt}\rho_t(q) = \Delta \rho_t(q) + F(\rho_t(q)) & , q \in \mathbb{T}^d, \\ \rho_0(q) = f(q) & , q \in \mathbb{T}^d. \end{cases} \tag{3.2.5}$$

For any $\delta > 0$, any $t \geq 0$ and any $H \in C(\mathbb{T}^d)$,

$$\lim_{n\to\infty} \mathbb{P}_{\nu_{f(\cdot/n)}} \left(\eta_\cdot \, ; \Big| \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \eta_t(x)\, H\left(\frac{x}{n}\right) - \int_{\mathbb{T}^d} \rho_t(u)\, H(u)\, du \Big| > \delta \right) = 0.$$

The hydrodynamic limit above can be seen as a law of large numbers with respect to the law of the process starting from the measure $\nu_{f(\cdot/n)}$. Below we use Theorem 3.2.3 to prove the same kind of result, but now starting from the stationary measure, the *hydrostatic limit*:

Corollary 3.2.5 (Hydrostatic limit). *Let $\rho \in (0,1)$ be given by (3.1.3). There exists $\lambda^* > 0$ such that for any $\lambda < \lambda^*$, any $\delta > 0$, any $t \geq 0$ and any $H \in C(\mathbb{T}^d)$,*

$$\lim_{n\to\infty} \mu_{ss}^n \left(\eta\, ; \Big| \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \eta_t(x)\, H\left(\frac{x}{n}\right) - \rho \int_{\mathbb{T}^d} H(u)\, du \Big| > \delta \right) = 0.$$

Proof. Fix $\delta > 0$, $t \in [0,T]$ and $H \in C(\mathbb{T}^d)$ and let us define the random variable

$$A_{t,H} := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \eta_t(x)\, H\left(\frac{x}{n}\right) - \rho \int_{\mathbb{T}^d} H(u)\, du \tag{3.2.6}$$

and the event $A_{t,\delta,H} := \{\eta\, ; |A_{t,H}| > \delta\}$.

By Corollary B.3.2, we have

$$\mu_{ss}^n(A_{t,\delta,H}) \leq \frac{\log 2 + H\left(\mu_{ss}^n \,|\, \nu_\rho\right)}{\log\left(1 + 1/\nu_\rho\left(A_{t,\delta,H}\right)\right)}.$$

Our goal is to show that there exists a positive constant $C = C\left(\delta\right)$ such that for any sufficiently large $n \in \mathbb{N}$ we have $\nu_\rho\left(A_{t,\delta,H}\right) < e^{-C\,n^d}$. Thus,

$$\mu_{ss}^n(A_{t,\delta,H}) \leq \frac{\log 2 + H\left(\mu_{ss}^n \,|\, \nu_\rho\right)}{C\,n^d}.$$

Assuming Theorem 3.2.3 we have $H\left(\mu_{ss}^n \,|\, \nu_\rho\right) \leq C\,\lambda\,n^{d-2}g_d(n)$ from which we can conclude the proof. Indeed, let us define

$$B_{t,H} := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \left(\eta_t(x) - \rho\right) H\left(\frac{x}{n}\right).$$

and

$$C_{t,H} := \frac{\rho}{n^d} \sum_{x \in \mathbb{T}_n^d} H\left(\frac{x}{n}\right) - \rho \int_{\mathbb{T}^d} H\left(u\right) du.$$

Thus,

$$\begin{aligned}
\nu_\rho\,\left(A_{t,\delta,H}\right) &= \nu_\rho\,\left(A_{t,H} > \delta\right) + \nu_\rho\,\left(A_{t,-H} > \delta\right) \\
&= \nu_\rho\,\left(B_{t,H} + C_{t,H} > \delta\right) + \nu_\rho\,\left(B_{t,-H} + C_{t,-H} > \delta\right).
\end{aligned}$$

Since $\frac{1}{n^d}\sum_{x \in \mathbb{T}_n^d} H\left(\frac{x}{n}\right)$ converges to $\int_{\mathbb{T}^d} H\left(u\right) du$, for sufficiently large n we have

$$-\frac{\delta}{2} < C_{t,H} < \frac{\delta}{2}.$$

Therefore, $\nu_\rho\,\left(A_{t,\delta,H}\right) \leq \nu_\rho\,\left(B_{t,H} > \delta/2\right) + \nu_\rho\,\left(B_{t,-H} > \delta/2\right)$. Moreover, by Chebyshev's exponential inequality and since ν_ρ is a product measure, for any $a > 0$ we have

$$\begin{aligned}
\nu_\rho\,\left(B_{t,H} > \delta/2\right) \leq e^{-a\,\delta/2}\nu_\rho[e^{a\,B_{t,H}}] &= e^{-a\,\delta/2}\;\nu_\rho\,\left[\prod_{x \in \mathbb{T}_n^d} \exp\left\{\frac{a\,H\left(\frac{x}{n}\right)}{n^d}\left(\eta_t(x) - \rho\right)\right\}\right] \\
&= e^{-a\,\delta/2} \prod_{x \in \mathbb{T}_n^d} \nu_\rho\,\left[\exp\left\{\frac{a\,H\left(\frac{x}{n}\right)}{n^d}\left(\eta_t(x) - \rho\right)\right\}\right] \\
&= e^{-a\,\delta/2}\,\exp\left\{\log \prod_{x \in \mathbb{T}_n^d} \nu_\rho\,\left[\exp\left\{\frac{a\,H\left(\frac{x}{n}\right)}{n^d}\left(\eta_t(x) - \rho\right)\right\}\right]\right\} \\
&= e^{-a\,\delta/2}\,\exp\left\{\sum_{x \in \mathbb{T}_n^d} \log\,\nu_\rho\,\left[\exp\left\{\frac{a\,H\left(\frac{x}{n}\right)}{n^d}\left(\eta_t(x) - \rho\right)\right\}\right]\right\}.
\end{aligned}$$

Now, by Hoeffding's Lemma (Lemma A.3.1), we conclude that

$$\nu_\rho\left(A_{t,\delta,H}\right) \leq 2\, e^{-a\,\delta/2}\, \exp\left\{\frac{a^2}{n^{2d}}\sum_{x\in\mathbb{T}_n^d}\left|H\left(\frac{x}{n}\right)\right|^2\right\} = \exp\left\{\frac{a^2}{n^d}\|H\|_\infty^2 + \log 2 - \frac{a\,\delta}{2}\right\}.$$

Choosing $a = \frac{\delta\, n^d}{4\,\|H\|_\infty^2}$ we obtain that

$$\nu_\rho\left(A_{t,\delta,H}\right) \leq \exp\left\{-\frac{\delta^2}{16\,\|H\|_\infty^2}\, n^d + \log 2\right\}$$

Hence, there exists a constant $C = C\left(\delta\right)$ such that for any sufficiently large n we have

$$\nu_\rho\left(A_{t,\delta,H}\right) < e^{-C\, n^d}.$$

□

3.2.4 Gaussian limit for the density fluctuation field

The goal of Section 3 is to prove that under the non-equilibrium stationary states, which are unknown, the fluctuation field converges to a Gaussian field:

Theorem 3.2.6. *Let $d \leq 3$. Let $\lambda^* > 0$ be sufficiently small. Let $\lambda \in [0, \lambda^*]$. Let $\rho \in (0,1)$ be given by (3.1.3). Let $H \in \mathcal{H}_k(\mathbb{T}^d)$, $k > d/2$. Let $X^n(H) := X_0^n(H)$ be the initial density fluctuation field defined in (3.2.2) starting from μ_{ss}^n. For each $t \geq 0$ let $T_t H$ be the solution of the Fokker-Planck equation*

$$\begin{cases} \frac{d}{dt}T_t H = \Delta T_t H - \Phi(\rho) T_t H, \\ T_0 H = H. \end{cases} \tag{3.2.7}$$

Recall that $\chi(\rho) = \rho(1-\rho)$. The sequence $\{X^n(H); n \in \mathbb{N}\}$ converges to a mean-zero Gaussian field $X(H)$ in $\mathcal{H}_{-k}(\mathbb{T}^d)$, $k > d/2$, with variance given by

$$\begin{aligned} \mathbb{E}\left[(X(H))^2\right] &= 2\chi(\rho)\int_0^\infty \|\nabla T_r H\|_{L^2(\mathbb{T}^d)}^2\, dr + 2\rho\int_0^\infty \|T_r H\|_{L^2(\mathbb{T}^d)}^2\, dr \\ &= \chi(\rho)\left(\|H\|_{L^2(\mathbb{T}^d)}^2 + \left(\frac{2\rho}{\chi(\rho)} - 2\Phi(\rho)\right)\int_0^\infty \|T_r H\|_{L^2(\mathbb{T}^d)}^2 dr\right). \end{aligned} \tag{3.2.8}$$

Remark 3.2.7. *The assumption $d \leq 3$ comes from the fact that the upper bound obtained for relative entropy in Theorem 3.2.3 becomes weaker as d increases. Thus, some of the estimates obtained in Section 3.5 are not strong enough to conclude the result in higher dimensions. This is better explained further ahead.*

Remark 3.2.8. *In order to see that the integral in (3.2.8) is finite, observe that*

$$\begin{aligned}\frac{d}{dt}\|T_tH\|^2_{L^2(\mathbb{T}^d)} &= 2\int_{\mathbb{T}^d}(T_tH(x))\Big(\frac{d}{dt}T_tH(x)\Big)dx = 2\int_{\mathbb{T}^d}(T_tH(x))((\Delta-\Phi(\rho))T_tH(x))dx\\ &= -2\int_{\mathbb{T}^d}(\nabla T_tH(x))^2dx - 2\Phi(\rho)\int_{\mathbb{T}^d}(T_tH(x))^2dx\\ &\le -2\Phi(\rho)\|T_tH\|^2_{L^2(\mathbb{T}^d)}.\end{aligned}$$

Thus, for any $t \ge 0$

$$\|T_tH\|_{L^2(\mathbb{T}^d)} \le \|H\|_{L^2(\mathbb{T}^d)}e^{-\Phi(\rho)t}. \tag{3.2.9}$$

From the linearity of the equation (3.2.7), $\nabla T_tH = T_t\nabla H$*. Thus, by (3.2.8), we conclude that the right-hand side of (3.2.9) is bounded from above by*

$$2\left(\chi(\rho)\,\|\nabla H\|^2_{L^2(\mathbb{T}^d)} + \rho\,\|H\|^2_{L^2(\mathbb{T}^d)}\right)\int_0^\infty e^{-2\Phi(\rho)r}dr = \frac{\chi(\rho)\,\|\nabla H\|^2_{L^2(\mathbb{T}^d)} + \rho\,\|H\|^2_{L^2(\mathbb{T}^d)}}{\Phi(\rho)} < \infty.$$

While it is simple to obtain an upper bound on the $L^2(\mathbb{T}^d)$-norm of T_tH, we need to do some more effort when considering its $L^\infty(\mathbb{T}^d)$-norm:

Proposition 3.2.9. *For any* $t > \frac{\log 2}{4\pi^2}$ *we have*

$$\|T_tH\|_\infty \le 5^d\|H\|_\infty e^{-\Phi(\rho)t}.$$

In particular, there exists a positive constant c_0 *such that* $\|T_tH\|_\infty < c_0$ *for any* $t \ge 0$.

Proof. Recall the Fourier coefficients $\hat{H}(k)$ of the function H with respect to the basis $\{\psi_k; k \in \mathbb{Z}^d\}$ defined in the beginning of Section 3.2. For each $k \in \mathbb{Z}^d$ let $\lambda_k = -4\pi^2|k|^2$ and observe that $T_t\psi_k(x) = \psi_k(x)e^{-(\lambda_k+\Phi(\rho))t}$ for every $x \in \mathbb{T}^d$. Since $H(x) = \sum_{k\in\mathbb{Z}^d}\hat{H}(k)\psi_k(x)$ we have

$$\begin{aligned}|T_tH(x)| &= \left|\sum_{k\in\mathbb{Z}^d}\hat{H}(k)T_t\psi_k(x)\right| = \left|\sum_{k\in\mathbb{Z}^d}\hat{H}(k)\psi_k(x)e^{-(\lambda_k+\Phi(\rho))t}\right|\\ &\le \sum_{k\in\mathbb{Z}^d}\left|\hat{H}(k)\right|e^{-(\lambda_k+\Phi(\rho))t}.\end{aligned}$$

Since $\hat{H}(k) \le \|H\|_\infty$, we have

$$|T_tH(x)| \le \|H\|_\infty\, e^{-\Phi(\rho)t}\sum_{k\in\mathbb{Z}^d}e^{-4\pi^2k^2t} = \|H\|_\infty\, e^{-\Phi(\rho)t}\left(\sum_{\ell\in\mathbb{Z}}e^{-4\pi^2\ell^2t}\right)^d.$$

Moreover, since $\ell^2 \ge |\ell|$ for every $\ell \in \mathbb{Z}$ we obtain

$$|T_tH(x)| \le \|H\|_\infty\, e^{-\Phi(\rho)t}\left(1 + 2\sum_{\ell=1}^\infty\left(e^{-4\pi^2t}\right)^\ell\right)^d = \|H\|_\infty\, e^{-\Phi(\rho)t}\left(1+\frac{2e^{-4\pi^2t}}{1-e^{-4\pi^2t}}\right)^d.$$

Using the inequality $(1-e^{-a})^{-1} < 2$ for any $a > \log 2$ we conclude that for any $x \in \mathbb{T}^d$

$$|T_t H(x)| \leq 5^d \|H\|_\infty \, e^{-\Phi(\rho)t}$$

for every $t > \frac{\log 2}{4\pi^2}$. $\square$

3.3 Relative entropy method

In this section we prove Theorem 3.2.3. As in Chapter 2, the main ingredient on the entropy estimate is a log-Sobolev inequality.

3.3.1 Logarithmic-Sobolev inequality

Let $\Lambda_n := \{0, \dots, n-1\}$ be the path of length n. Let $\rho : \Lambda_n \to (0,1)$. Let $S_n = \{0,1\}^{\Lambda_n}$ and let ν_ρ be the Bernoulli product measure in S_n of parameter ρ.

Fix $\gamma > 0$. For $f : S_n \to \mathbb{R}$ and $x, y \in \Lambda_n$, let us define

$$\mathcal{D}_x(f) := \int \big(f(\eta^x) - f(\eta)\big)^2 d\nu_\rho,$$

$$\mathcal{D}_{x,y}(f) := \int \big(f(\eta^{x,y}) - f(\eta)\big)^2 d\nu_\rho,$$

$$\mathcal{D}^k(f) := \sum_{x=0}^{k-2} \mathcal{D}_{x,x+1}(f) + \frac{\gamma}{n^2} \sum_{x=0}^{k-1} \mathcal{D}_x(f),$$

and $\mathcal{D}(f) := \mathcal{D}^n(f)$. First, we prove the following:

Lemma 3.3.1 (Comparison of quadratic forms)**.** *There exists a positive constant $C = C(\rho)$ such that for any function $f : S_n \to \mathbb{R}$*

$$\mathcal{D}_{n-1}(f) \leq Cn\mathcal{D}(f).$$

Proof. For each $f : S_n \to \mathbb{R}$ and $x, y \in \Lambda_n$, define $\nabla_{x,y} f, \nabla_x f : S_n \to \mathbb{R}$ as

$$\nabla_{x,y} f(\eta) := f(\eta^{x,y}) - f(\eta),$$

$$\nabla_x f(\eta) = f(\eta^x) - f(\eta)$$

for any $\eta \in S_n$. For x, y we have that

$$\nabla_x f(\eta) = \nabla_{x,y} f(\eta) + \nabla_y f(\eta^{x,y}) + \nabla_{x,y} f\big((\eta^{x,y})^y\big).$$

Using the inequality

$$(a+b+c)^2 \leq 2(1+\beta)(a^2+b^2) + (1+\tfrac{1}{\beta})c^2,$$

valid for any $a, b, c \in \mathbb{R}$, $\beta > 0$, we see that

$$\mathcal{D}_x(f) \le 2(1+\beta) \int \Big[\big(\nabla_{x,y} f\big)^2 + \big(\nabla_{x,y} f\big((\eta^{x,y})^y\big)\big)^2\Big] \nu_\rho(d\eta) \\ + \big(1 + \tfrac{1}{\beta}\big) \int \big(\nabla_y f(\eta^{x,y})\big)^2 \nu_\rho(d\eta).$$

Performing some changes of variables, we see that

$$\int \big(\nabla_{x,y} f\big((\eta^{x,y})^y\big)\big)^2 \nu_\rho(d\eta) = \int \big(\nabla_{x,y} f(\eta)\big)^2 \frac{\nu_\rho\big((\eta^y)^{x,y}\big)}{\nu_\rho(\eta)} \nu_\rho(d\eta),$$

$$\int \big(\nabla_y f(\eta^{x,y})\big)^2 d\nu_\rho \le \int \big(\nabla_y f\big)^2 \frac{\nu_\rho(\eta^{x,y})}{\nu_\rho(\eta)} d\nu_\rho.$$

Since ν_ρ in invariant under transpositions, the Jacobian factors satisfy

$$\frac{\nu_\rho\big((\eta^y)^{x,y}\big)}{\nu_\rho(\eta)} \le \frac{1}{\rho} - 1, \quad \frac{\nu_\rho(\eta^{x,y})}{\nu_\rho(\eta)} = 1. \tag{3.3.1}$$

We conclude that

$$\mathcal{D}_x(f) \le \tfrac{2(1+\beta)}{\rho} \mathcal{D}_{x,y}(f) + \big(1 + \tfrac{1}{\beta}\big) \mathcal{D}_y(f).$$

Now let us choose $x = n-1$ and $\beta = n$. Let us sum the above inequality over all $y \neq n-1$ and divide both sides by $n-1$. We obtain

$$\mathcal{D}_{n-1}(f) \le \tfrac{2(1+n)}{\rho(n-1)} \sum_{y=0}^{n-2} \mathcal{D}_{n-1,y}(f) + \frac{\big(1 + \frac{1}{n}\big)}{n-1} \sum_{y=0}^{n-1} \mathcal{D}_y(f).$$

Therefore, the proof is complete once we show that

$$\sum_{y=0}^{n-2} \mathcal{D}_{n-1,y}(f) \le Cn \sum_{x=0}^{n-2} \mathcal{D}_{x,x+1}(f).$$

Indeed, for each $x, y \in \Lambda_n$ let us define $(x,y) \circ \eta := \eta^{x,y}$. Observe that for any $x > y$, $\eta^{x,y}$ can be rewritten as

$$(x-1,x)\circ(x-2,x-1)\circ\cdots\circ(y+1,y+2)\circ(y,y+1)\circ(y+1,y+2)\circ\cdots\circ(x-2,x-1)\circ(x-1,x)\circ\eta.$$

Thus, there exists a finite sequence $\xi_1, \dots, \xi_m$, with $m \le 2(x-y)$, such that $\xi_1 = \eta$, $\xi_m = \eta^{x,y}$ and for any $2 \le k \le m$ we have $\xi_k = (\ell, \ell+1) \circ \xi_{k-1}$ for some ℓ. Therefore, we can write

$$f(\eta^{x,y}) - f(\eta) = \sum_{k=1}^{m-1} \left(f(\xi_{k+1}) - f(\xi_k)\right).$$

By Cauchy-Schwarz inequality, for any $x > y$ we obtain that

$$\mathcal{D}_{x,y}(f) = \int \left(f(\eta^{x,y}) - f(\eta)\right)^2 \nu_{\rho(\cdot)}(d\eta) \leq m \sum_{k=1}^{m-1} \int \left(f(\xi_{k+1}) - f(\xi_k)\right)^2 \nu_\rho(d\eta) \tag{3.3.2}$$

$$= m \sum_{k=1}^{m-1} \int \left(f(\xi_{k+1}) - f(\xi_k)\right)^2 \left(\frac{\nu_\rho(\eta)}{\nu_\rho(\xi_k)}\right) \nu_\rho(d\xi_k). \tag{3.3.3}$$

Moreover, since ν_ρ in invariant under transpositions, $\nu_\rho(\eta) = \nu_\rho(\xi_k)$ for every $k \in \{1, \ldots, m-1\}$. Hence,

$$\mathcal{D}_{x,y}(f) \leq m \sum_{k=1}^{m-1} \int \left(f(\xi_{k+1}) - f(\xi_k)\right)^2 \nu_\rho(d\xi_k) \tag{3.3.4}$$

$$\leq 2n \sum_{x=1}^{n-2} \mathcal{D}_{x,x+1}(f). \tag{3.3.5}$$

□

Observe that although the statements of Lemma 2.3.2 and Lemma 3.3.1 are similar, their proofs are different because they use the dynamics of the system to move particles. Once the comparison of quadratic forms are done, we can prove a log-Sobolev inequality for a reaction-diffusion model on S_n, following the same recipe given in the proof of Theorem 2.3.1:

Theorem 3.3.2. *There exists a finite constant $K = K(\gamma, \rho)$ such that*

$$\int f \log f d\nu_\rho \leq \frac{1}{K} n^2 \mathcal{D}\left(\sqrt{f}\right) \tag{3.3.6}$$

for any density f with respect to ν_ρ.

Now prove the validity of the result for the process evolving on the d-dimensional tori:

Theorem 3.3.3 (Log-Sobolev inequality)**.** *Let ν be a probability measure on Ω_n and define $\mathfrak{D}(f, \nu) = \int \Gamma f d\nu$ where Γ is the carré du champ operator associated with $\mathfrak{L}_n$ given by*

$$\Gamma f(\eta) = n^2 \sum_{i=1}^{d} \sum_{x \in \mathbb{T}_n^d} \left(f(\eta^{x,x+e_i}) - f(\eta)\right)^2 + \sum_{x \in \mathbb{T}_n^d} c_x(\eta) \left(f(\eta^x) - f(\eta)\right)^2 \tag{3.3.7}$$

for any function $f : \Omega_n \to \mathbb{R}$. Let $\rho \in (0, 1)$ be given by (3.1.3). There exists a positive constant K_{LS}, independent of n (but depending on ρ), such that

$$\int f \log f d\nu_\rho \leq \frac{1}{K_{LS}} \mathfrak{D}\left(\sqrt{f}, \nu_\rho\right)$$

for any density f with respect to ν_ρ.

Proof. For each $n \in \mathbb{N}$ let $G_1 = (V_1, E_1)$ be such that $V_1 = \{0, 1 \ldots, n-1\}$ and $E_1 = \{\{x, x+1\,(\text{mod n})\}\,;\, x \in V_1\}$. Notice that G_1 and $\mathbb{T}_n^1$ are isomorphic. By comparison of Dirichlet forms, Theorem 3.3.2 gives a logarithmic-Sobolev inequality for the generator of the exclusion process on G_1 with Glauber dynamics of density γ/n at all vertices, and the measure ν_ρ.

Now, for each $d \geq 2$ let us consider n copies of $G_{d-1} = (V_d, E_d)$, which we label as $G_{d-1}^i = (V_d^i, E_d^i)$, $i \in \{0, \ldots, n-1\}$. The vertex set V_d^i and the edge set E_d^i are defined by

$$V_d := \cup_{i=1}^{n-1} V_d^i \;\text{ and }\; E_d := \cup_{i=1}^{n-1} E_d^i.$$

We will let the process evolve in each of the graphs G_{d-1}^i, independently. By [5, Lemma 3.2] and since the resulting process on G_d can be seen as a product chain of the one on G_1, the inverse of log-Sobolev constant associated with the quadratic form $\mathfrak{D}(\cdot, \nu_\rho)$ is or order 1. In order to obtain the tori in this graph construction, for each vertex $v \in V_d$ let v^i denote its copy at V_d^i. Define

$$E_d^* := \{\{v^i, v^{i+1\,(\text{mod n})}\}\,;\, v \in V_{d-1}\}.$$

The graph $\widetilde{G}_d = (V_d, E_d \cup E_d^*)$ is now a torus. The result follows from the fact that adding more edges can only decrease the log-Sobolev constant of the process. ☐

3.3.2 Entropy estimate

Let $u : \mathbb{T}_n^d \to [\varepsilon_0, 1-\varepsilon_0]$ for some $\varepsilon_0 \in (0, 1/2]$. Assume that

$$n|u(x+e_i) - u(x)| \leq \kappa$$

for any $x \in \mathbb{T}_n^d$ and any $i \in \{1, \ldots, d\}$.

Let $\mathbb{O}^- = \{x \in \mathbb{Z}^d; z_i \leq 0 \text{ for any } i \in \{1, \ldots, d\}\}$ be the non-positive orthant and let $A \subset \mathbb{O}^-$ be finite. Note that A is projected into $\mathbb{T}_n^d$ when n is sufficiently large.

Now, for each $x \in \mathbb{T}_n^d$ let us define

$$\omega(x) = \frac{\eta(x) - u(x)}{\chi(u(x))},$$

which is well-defined because $\varepsilon_0 > 0$. Moreover, define

$$\omega(x+A) = \prod_{y \in A} \omega(x+y)$$

and, for each $i \in \{1, \ldots, d\}$ and each $G : \mathbb{T}_n^d \to \mathbb{R}$ define

$$V_i(\eta, G, A) = \sum_{x \in \mathbb{T}_n^d} \omega(x+A)\omega(x+e_i)G(x) \tag{3.3.8}$$

For any density f with respect to $\nu_{u(\cdot)}$ define $H\left(f|\nu_{u(\cdot)}\right) = \int f \log f d\nu_{u(\cdot)}$. With the above notation, the following replacement lemma was proven in [12, Lemma 3.1]:

Lemma 3.3.4 (Static Replacement)**.** *There exists a finite constant $C = (\varepsilon_0, A)$ such that for any $G : \mathbb{T}_n^d \to \mathbb{R}$, any density f with respect to $\nu_{u(\cdot)}$ and any $\delta > 0$,*

$$\int V_i(\eta, G, A) f d\nu_{u(\cdot)} \leq \delta n^2 \int \sum_{i=1}^{d} \sum_{x \in \mathbb{T}_n^d} \left(\sqrt{f(\eta^{x,x+e_i})} - \sqrt{f(\eta)}\right)^2 d\nu_{u(\cdot)}$$
$$+ \frac{C(1+\kappa)}{\delta} \left(\|G\|_\infty + \|G\|_\infty^2\right) \left(H\left(f|\nu_{u(\cdot)}\right) + n^{d-2} g_d(n)\right).$$

Using the log-Sobolev inequality and Lemma 3.3.4, we are able to prove the following result:

Lemma 3.3.5. *Let $\rho \in (0,1)$ be given by (3.1.3). Let f_t be the Radon-Nikodym derivative of $\nu_\rho S_t$ with respect to the measure ν_ρ. Let $H(t) := H(f_t|\nu_\rho)$. Then, there exist positive constants λ^*, C such that for any $\lambda < \lambda^*$ and any $t \geq 0$ we have*

$$H(t) \leq C \lambda n^{d-2} g_d(n).$$

Proof. Let ψ be the Radon-Nikodym derivative of ν_ρ with respect to the measure $\nu_{1/2}$. Let f_t denote the Radon-Nikodym derivative of $\nu_\rho S_t$ with respect to ν_ρ and let $H(t) = H(f_t, \nu_\rho)$. By Yau's entropy inequality (Proposition 2.5.1),

$$\frac{d}{dt} H(t) \leq -\mathfrak{D}\left(\sqrt{f_t}, \nu_\rho\right) + \int \left(\mathfrak{L}_t^* \mathbb{1} - \frac{d}{dt}\psi\right) f_t \, d\nu_\rho,$$

where $\mathfrak{L}_t^*$ stands for the adjoint of $\mathfrak{L}_n$ with respect to ν_ρ. After a very long, but elementary computation (similar to the one done in (2.5.3)), we can see that for any density f with respect to ν_ρ we have

$$\int \left(\mathfrak{L}_t^* \mathbb{1} - \frac{d}{dt}\psi\right) f \, d\nu_\rho = \int \sum_{x \in \mathbb{T}_n^d} \omega(x) \left(F(\rho)\right) f \, d\nu_\rho \tag{3.3.9}$$
$$+ 2\lambda (\chi(\rho))^2 \int \sum_{x \in \mathbb{T}_n^d} \omega(x) \sum_{j=1}^{d} \omega(x+e_j) \, f \, d\nu_\rho$$
$$+ \lambda \frac{(\chi(\rho))^3}{\rho} \int \sum_{x \in \mathbb{T}_n^d} \omega(x) \sum_{j=1}^{d} \omega(x-e_j) \omega(x+e_j) \, f \, d\nu_\rho.$$

Therefore, since $F(\rho) = 0$ and $f = f_t$ we obtain that

$$\frac{d}{dt}H(t) \leq -\mathfrak{D}\left(\sqrt{f_t}, \nu_\rho\right)$$
$$+ 2\lambda(\chi(\rho))^2 \int \sum_{i=1}^{d} \sum_{x\in\mathbb{T}_n^d} \omega(x)\omega(x+e_i)\, f_t\, d\nu_\rho$$
$$+ \lambda\frac{(\chi(\rho))^3}{\rho} \int \sum_{j=1}^{d} \sum_{x\in\mathbb{T}_n^d} \omega(x)\omega(x-e_j)\omega(x+e_j)\, f_t\, d\nu_\rho.$$

Now, by Lemma 3.3.4, there exists a positive constant C such that for any $\delta > 0$

$$2\lambda(\chi(\rho))^2 \int \sum_{i=1}^{d} \sum_{x\in\mathbb{T}_n^d} \omega(x)\omega(x+e_i)\, f_t\, d\nu_\rho$$
$$+ \lambda\frac{(\chi(\rho))^3}{\rho} \int \sum_{j=1}^{d} \sum_{x\in\mathbb{T}_n^d} \omega(x)\omega(x-e_j)\omega(x+e_j)\, f_t\, d\nu_\rho$$

is bounded from above by

$$\lambda\delta n^2 \int \sum_{i=1}^{d} \sum_{x\in\mathbb{T}_n^d} \left(\sqrt{f_t(\eta^{x,x+e_i})} - \sqrt{f_t(\eta)}\right)^2 d\nu_\rho + \frac{C\lambda}{\delta}\left(H(t) + n^{d-2}\, g_d(n)\right).$$

Thus, since $n^2 \int \sum_{i=1}^{d} \sum_{x\in\mathbb{T}_n^d} \left(\sqrt{f_t(\eta^{x,x+e_i})} - \sqrt{f_t(\eta)}\right)^2 d\nu_\rho \leq \mathfrak{D}\left(\sqrt{f_t}, \nu_\rho\right)$ we obtain

$$\frac{d}{dt}H(t) \leq -(1-\lambda\delta)\mathfrak{D}\left(\sqrt{f_t}, \nu_\rho\right) + \frac{C\lambda}{\delta}\left(H(t) + n^{d-2}\, g_d(n)\right).$$

Let us use the main ingredient of this proof. By Theorem 3.3.3, for any $\delta \leq \lambda^{-1}$,

$$\frac{d}{dt}H(t) \leq -\left((1-\lambda\delta)K_{LS} - \frac{C\lambda}{\delta}\right) H(t) + \frac{C\lambda}{\delta} n^{d-2}\, g_d(n).$$

Let us choose our parameter δ. We will take one such that the function $f(\delta) := (1 - \lambda\delta)K_{LS} - C\lambda/\delta$ is positive. Since $f(\delta) > 0$ if and only if the polynomial $p(\delta) := \delta^2 - \lambda^{-1}\delta + C/K_{LS}$ is negative and since the discriminant of $p(\delta)$ equals $\lambda^{-2} - 4C/K_{LS}$, our choice on δ is possible only if $\lambda < \sqrt{K_{LS}/(4C)} =: \lambda^*$ (this forces our assumption that λ is small enough). Moreover, since the roots of $p(\delta)$ are $\lambda^{-1} \pm \sqrt{(2\lambda)^{-2} - C/K_{LS}}$ and we had required that $\delta \leq \lambda^{-1}$, we must choose some $\delta^* \in \left(\lambda^{-1} - \sqrt{(2\lambda)^{-2} - C/K_{LS}}, \lambda^{-1}\right)$. Therefore,

$$\frac{d}{dt}H(t) \leq -f(\delta^*)H(t) + \frac{C\lambda}{\delta^*} n^{d-2}\, g_d(n).$$

Finally, by Gronwall's inequality, we conclude the Lemma. □

Observe that Theorem 3.2.3 follows from Lemma 3.3.5.

Proof of Theorem 3.2.3. By [13, Proposition 8.1], the entropy is lower semicontinuous. Therefore, by Lemma 3.3.5 we see that

$$H(\mu_{ss}^n|\nu_\rho) \le \limsup_{t\to\infty} H(t) \le C\lambda n^{d-2}\, g_d(n).$$

$\square$

3.4 Estimates of some functionals

In this section we use the entropy bound of Theorem 3.2.3 to estimate some functionals.

Proposition 3.4.1. *There exist positive constants λ^* and $C^* = C^*(\lambda^*)$ such that for any $\lambda \in [0, \lambda^*]$, any function $G \in C^{1,\infty}(\mathbb{R}^+ \times \mathbb{T}^d)$, any $t \ge 0$ and any sufficiently large n we have*

$$\mathbb{E}_{\mu_{ss}^n}\left[\Big|\frac{1}{n^{d/2}}\sum_{x\in\mathbb{T}_n^d}(\eta_0(x)-\rho)G_t\left(\frac{x}{n}\right)\Big|\right] \le C^*\, n^{d/2-1}\sqrt{g_d(n)}\|G_t\|_{L^2(\mathbb{T}^d)}.$$

Proof. By the entropy inequality (Proposition B.3.1), for any $\gamma > 0$ we have

$$\mathbb{E}_{\mu_{ss}^n}\left[\Big|\frac{1}{n^{d/2}}\sum_{x\in\mathbb{T}_n^d}(\eta_0(x)-\rho)G_t\left(\frac{x}{n}\right)\Big|\right] \le \frac{H(\mu_{ss}^n|\nu_\rho)}{\gamma}$$
$$+\frac{1}{\gamma}\log\int\exp\left\{\gamma\Big|\frac{1}{n^{d/2}}\sum_{x\in\mathbb{T}_n^d}(\eta_0(x)-\rho)G_t\left(\frac{x}{n}\right)\Big|\right\}d\mathbb{P}_{\nu_{\rho(\cdot)}}.$$

Moreover, by Theorem 3.2.3, we can bound the right-hand side of the above inequality by

$$\frac{C\lambda n^{d-2}g_d(n)}{\gamma}+\frac{1}{\gamma}\log\int\exp\left\{\gamma\Big|\frac{1}{n^{d/2}}\sum_{x\in\mathbb{T}_n^d}(\eta_0(x)-\rho)G_t\left(\frac{x}{n}\right)\Big|\right\}d\mathbb{P}_{\nu_\rho}. \tag{3.4.1}$$

Since $e^{|x|} \le e^x + e^{-x}$ and

$$\limsup_{n\to\infty}\frac{1}{n}\log(b_n+c_n) = \max\left\{\limsup_{n\to\infty}\frac{1}{n}\log b_n\,,\ \limsup_{n\to\infty}\frac{1}{n}\log c_n\right\},$$

we can remove the absolute value on the right-hand side of (3.4.1), obtaining

$$\frac{C\lambda n^{d-2}g_d(n)}{\gamma}+\frac{1}{\gamma}\log\int\exp\left\{\frac{\gamma}{n^{d/2}}\sum_{x\in\mathbb{T}_n^d}(\eta_0(x)-\rho)G_t\left(\frac{x}{n}\right)\right\}d\mathbb{P}_{\nu_\rho}. \tag{3.4.2}$$

Now, by Hoeffding's Lemma (Lemma A.3.1), the previous expression is bounded from above by

$$\frac{C\lambda n^{d-2} g_d(n)}{\gamma} + \frac{\gamma}{8n^d} \sum_{x\in\mathbb{T}_n^d} G_t^2\left(\frac{x}{n}\right).$$

Since $n^{-d}\sum_{x\in\mathbb{T}_n^d} G_t^2(x/n)$ converges to $\|G_t\|^2_{L^2(\mathbb{T}^d)}$, for sufficiently large n we have

$$\frac{1}{n^d}\sum_{x\in\mathbb{T}_n^d} G_t^2\left(\frac{x}{n}\right) \le 2\|G_t\|^2_{L^2(\mathbb{T}^d)}.$$

Therefore,

$$\mathbb{E}_{\mu_{ss}^n}\left[\Big|\frac{1}{n^{d/2}}\sum_{x\in\mathbb{T}_n^d}(\eta_0(x)-\rho)G_t\left(\frac{x}{n}\right)\Big|\right] \le \frac{C\lambda n^{d-2} g_d(n)}{\gamma} + \frac{\gamma\,\|G_t\|^2_{L^2(\mathbb{T}^d)}}{4}.$$

The proof finishes taking $\gamma = n^{d/2-1}\sqrt{g_d(n)}/\|G_t\|_{L^2(\mathbb{T}^d)}$. □

Proposition 3.4.2. *Let $G\in C^{1,\infty}(\mathbb{R}^+\times\mathbb{T}^d)$. Assume that there exists a positive constant c_0 such that*

$$\int_0^t \|G_s\|^2_{L^2(\mathbb{T}^d)} ds \le c_0$$

for every $t\ge 0$. There exist positive constants λ^* and $C^* = C^*(c_0)$ such that for any $\lambda\in[0,\lambda^*]$, any $t\in[0,T]$, any $a\in\mathbb{R}$ and any sufficiently large n, we have

$$\mathbb{E}_{\mu_{ss}^n}\left[\Big|\int_0^t \frac{1}{n^a}\sum_{x\in\mathbb{T}_n^d}(\eta_s(x)-\rho)G_s\left(\frac{x}{n}\right)ds\Big|\right] \le C^*\,(t+1)\,n^{d-1-a}\sqrt{g_d(n)}.$$

Proof. By Jensen's inequality and Fubini's Theorem

$$\mathbb{E}_{\mu_{ss}^n}\left[\Big|\int_0^t \frac{1}{n^a}\sum_{x\in\mathbb{T}_n^d}(\eta_s(x)-\rho)G_s\left(\frac{x}{n}\right)ds\Big|\right] \le \int_0^t \mathbb{E}_{\mu_{ss}^n}\left[\Big|\frac{1}{n^a}\sum_{x\in\mathbb{T}_n^d}(\eta_s(x)-\rho)G_s\left(\frac{x}{n}\right)\Big|\right]ds.$$

Let $\gamma>0$. By the entropy inequality (Proposition B.3.1), the right-hand side above is bounded from above by

$$\int_0^t \left[\frac{H\left(\mu_{ss}^n|\nu_\rho\right)}{\gamma} + \frac{1}{\gamma}\log\int\exp\left\{\frac{\gamma}{n^a}\left|\sum_{x\in\mathbb{T}_n^d}(\eta_s(x)-\rho)G_s\left(\frac{x}{n}\right)\right|\right\}d\nu_\rho\right]ds \qquad (3.4.3)$$

Moreover, since $e^{|x|}\le e^x+e^{-x}$ and

$$\limsup_{n\to\infty}\frac{1}{n}\log(b_n+c_n) = \max\left\{\limsup_{n\to\infty}\frac{1}{n}\log b_n\,,\ \limsup_{n\to\infty}\frac{1}{n}\log c_n\right\},$$

the expression (3.4.3) is not larger than

$$\frac{H\left(\mu_{ss}^n|\nu_\rho\right)\,t}{\gamma}+\frac{1}{\gamma}\int_0^t\log\int\exp\left\{\frac{\gamma}{n^a}\sum_{x\in\mathbb{T}_n^d}(\eta_s(x)-\rho)G_s\left(\frac{x}{n}\right)\right\}d\nu_\rho ds$$

By Theorem 3.2.3 and Hoeffding's Lemma (Lemma A.3.1), the above expression is bounded from above by

$$\frac{C\lambda t\,n^{d-2}g_d(n)}{\gamma}+\frac{\gamma}{8n^{2a}}\int_0^t\sum_{x\in\mathbb{T}_n^d}G_s^2\left(\frac{x}{n}\right)ds.$$

Taking $\gamma=\sqrt{g_d(n)}/n^{1-a}$ and since $n^{-d}\sum_{x\in\mathbb{T}_n^d}G_s^2(x/n)$ converges to $\|G_s\|^2_{L^2(\mathbb{T}^d)}$, we conclude that there exists $C^*=C^*(c_0)>0$ such that for any sufficiently large n, we have

$$\mathbb{E}_{\mu_{ss}^n}\left[\Big|\int_0^t\frac{1}{n^a}\sum_{x\in\mathbb{T}_n^d}(\eta_s(x)-\rho)G_s\left(\frac{x}{n}\right)ds\Big|\right]\le C^*\,(t+1)\,n^{d-1-a}\sqrt{g_d(n)}.$$

□

3.4.1 Boltzmann-Gibbs principle

Now we prove the so-called *Boltzmann-Gibbs principle*:

Theorem 3.4.3 (Boltzmann-Gibbs principle)**.** *Let $G\in C^{1,\infty}(\mathbb{R}^+\times\mathbb{T}^d)$. Assume that there exists a positive constant c_0, independent of t, such that*

$$\|G_t\|_\infty<c_0\ \ \textit{for any}\ t\ge 0.$$

Recall the definition of $V_i(\eta,G,A)$ given in (3.3.8). There exist positive constants $\lambda^*,C=C(c_0)$ such that for any $t>0$, any $i\in\{1,\dots,d\}$, any $A\subset\mathbb{O}^-$ and any $\lambda<\lambda^*$, we have

$$\mathbb{E}_{\mu_{ss}^n}\left[\Big|\int_0^t\frac{1}{n^{d/2}}V_i(\eta_s,G_s,A)ds\Big|\right]\le C(t+1)\,n^{d/2-2}\,g_d(n).$$

Proof. We will prove that

$$\mathbb{E}_{\mu_{ss}^n}\left[\Big|\int_0^t\frac{\lambda}{n^{d/2}}V_i(\eta_s,G_s,A)ds\Big|\right]\le C\lambda(t+1)\,n^{d/2-2}\,g_d(n)$$

from where the proof ends. As we will see, the factor λ will be needed to show that the argument only works when this parameter is small enough. Indeed, by the entropy inequality (Proposition B.3.1), for any $\gamma > 0$, and Jensen's inequality we have

$$\mathbb{E}_{\mu_{ss}^n}\left[\Big|\int_0^t \frac{\lambda}{n^{d/2}} V_i(\eta_s, G_s, A) ds\Big|\right] \leq \frac{H\left(\mu_{ss}^n | \nu_\rho\right)}{\gamma} + \frac{1}{\gamma}\log\int \exp\left\{\gamma\Big|\int_0^t \frac{\lambda}{n^{d/2}} V_i(\eta_s, G_s, A) ds\Big|\right\} d\mathbb{P}_{\nu_\rho}.$$

Therefore, by Theorem 3.2.3, for any λ small enough, we have

$$\mathbb{E}_{\mu_{ss}^n}\left[\Big|\int_0^t \frac{\lambda}{n^{d/2}} V_i(\eta_s, G_s, A) ds\Big|\right] \leq \frac{C\lambda\, n^{d-2} g_d(n)}{\gamma} + \frac{1}{\gamma}\log\int \exp\left\{\gamma\Big|\int_0^t \frac{\lambda}{n^{d/2}} V_i(\eta_s, G_s, A) ds\Big|\right\} d\mathbb{P}_{\nu_\rho}. \tag{3.4.4}$$

From now we ignore the first parcel on the right-hand side of the above inequality and we deal with the second one. Since $e^{|x|} \leq e^x + e^{-x}$ and

$$\limsup_{n\to\infty} \frac{1}{n}\log(b_n + c_n) = \max\left\{\limsup_{n\to\infty}\frac{1}{n}\log b_n\,,\ \limsup_{n\to\infty}\frac{1}{n}\log c_n\right\},$$

the second parcel is bounded from above by

$$\frac{1}{\gamma}\log\int\exp\left\{\int_0^t \frac{\gamma\lambda}{n^{d/2}} V_i(\eta_s, G_s, A) ds\right\} d\mathbb{P}_{\nu_\rho}.$$

Moreover, by the Feynman-Kac's formula (Lemma B.4.1), we can bound the previous expression from above by

$$\int_0^t \sup_f \left\{-\frac{1}{\gamma}\mathfrak{D}\left(\sqrt{f}, \nu_\rho\right) + \int \frac{\lambda}{n^{d/2}} V_i(\eta, G_s, A) f d\nu_\rho + \frac{1}{2\gamma}\int\left(\mathfrak{L}_t^* \mathbb{1} - \frac{d}{d_t}\psi\right) f\, d\nu_\rho\right\} ds,$$

where the supremum runs over all densities f with respect to ν_ρ. Now recall the identity (3.3.9). Thus, we can rewrite the expression inside the above supremum as

$$-\frac{1}{\gamma}\mathfrak{D}\left(\sqrt{f}, \nu_\rho\right)$$

$$+\int \frac{\lambda}{n^{d/2}} V_i(\eta, G_s, A) f d\nu_\rho + \frac{2\lambda(\chi(\rho))^2}{2\gamma}\int \sum_{i=1}^d \sum_{x\in\mathbb{T}_n^d} \omega(x)\omega(x+e_i)\, f\, d\nu_\rho \tag{3.4.5}$$

$$+\frac{\lambda(\chi(\rho))^3}{2\gamma\rho}\int\sum_{j=1}^d\sum_{x\in\mathbb{T}_n^d}\omega(x)\omega(x-e_j)\omega(x+e_j)\, f\, d\nu_\rho. \tag{3.4.6}$$

We will choose $\gamma = L\, n^{d/2}$, for some $L > 0$, so that all the integrals in the sum of (3.4.5) and (3.4.6) have the same order. Recall that there exists $c_0 > 0$ such that $\|G_s\|_\infty \leq c_0$ for any $s \geq 0$. Applying Lemma 3.3.4 to these integrals and using the fact that

$$n^2 \int \sum_{i=1}^{d} \sum_{x \in \mathbb{T}_n^d} \left(\sqrt{f(\eta^{x,x+e_i})} - \sqrt{f(\eta)} \right)^2 d\nu_\rho \leq \mathfrak{D}\left(\sqrt{f}, \nu_\rho\right),$$

we conclude that there exists a constant C such that (3.4.5) has the upper bound

$$-\frac{1}{Ln^{d/2}} \mathfrak{D}\left(\sqrt{f}, \nu_\rho\right) + \frac{\lambda\delta}{Ln^{d/2}} \mathfrak{D}\left(\sqrt{f}, \nu_\rho\right) + \frac{C\lambda}{\delta\, Ln^{d/2}} \left(H\left(f|\nu_\rho\right) + n^{d-2}\, g_d(n) \right)$$

for any $\delta > 0$. Invoking Theorem 3.3.3, which is the key ingredient in this proof (as in the entropy estimate), we can bound the previous expression from above by

$$-\left(\frac{1}{Ln^{d/2}} - \frac{\lambda\delta}{Ln^{d/2}} - \frac{C\lambda}{\delta\, L\, K_{LS} n^{d/2}} \right) \mathfrak{D}\left(\sqrt{f}, \nu_\rho\right) + \frac{C\lambda}{\delta\, L} n^{d/2-2}\, g_d(n). \tag{3.4.7}$$

Finally, let us choose constants L and δ such that the function $r(L,\delta) := \frac{1}{Ln^{d/2}} - \frac{\lambda\delta}{Ln^{d/2}} - \frac{C\lambda}{\delta\, L\, K_{LS} n^{d/2}}$ becomes strictly positive. Thus, we can discard the term in (3.4.7) involving $\mathfrak{D}\left(\sqrt{f}, \nu_\rho\right)$. Finding such constants is possible because we can take λ sufficiently small. Therefore, the sum in (3.4.7) is bounded from above by $C\lambda n^{d/2-2}\, g_d(n)$. Plugging the above estimates in (3.4.4) we get

$$\mathbb{E}_{\mu_{ss}^n}\left[\Big| \int_0^t \frac{\lambda}{n^{d/2}} V_i(\eta_s, G_s, A) ds \Big| \right] \leq C\lambda\, (t+1)\; n^{d/2-2} g_d(n).$$

□

3.5 Central limit theorem

In this section we finally prove Theorem 3.2.6. Our approach relies on the semigroup method.

3.5.1 Some Dynkin's martingales

For each $x \in \mathbb{T}_n^d$, each $i \in \{1, \ldots, d\}$ and each $G : \mathbb{T}^d \to \mathbb{R}$ let us define the discrete derivative on direction e_i by

$$\nabla_n^i G\left(\frac{x}{n}\right) = n \left(G\left(\frac{x+e_i}{n}\right) - G\left(\frac{x}{n}\right) \right).$$

Similarly, for each $x \in \mathbb{T}_n^d$ and each $G : \mathbb{T}^d \to \mathbb{R}$, the discrete Laplacian is the operator given by

$$\Delta_n G\left(\frac{x}{n}\right) = n^2 \sum_{i=1}^{d} \left(G\left(\frac{x+e_i}{n}\right) + G\left(\frac{x-e_i}{n}\right) - 2G\left(\frac{x}{n}\right) \right).$$

Let $\overline{\eta}_x := \eta(x) - \rho$. An elementary computation shows that for any $\eta \in \Omega_n$

$$\mathfrak{L}_n \overline{\eta}_x = \Delta_n \overline{\eta}_x - (2 + \lambda\rho^2 d)\overline{\eta}_x + \lambda\chi(\rho) \sum_{i=1}^{d} \left(\overline{\eta}_{x-e_i} + \overline{\eta}_{x+e_i}\right)$$
$$- \lambda(1-\rho) \sum_{i=1}^{d} \overline{\eta}_{x-e_i}\overline{\eta}_{x+e_i} - \lambda\rho \sum_{i=1}^{d} \overline{\eta}_x \left(\overline{\eta}_{x-e_i} + \overline{\eta}_{x+e_i}\right) - \lambda \sum_{i=1}^{d} \overline{\eta}_{x-e_i}\overline{\eta}_x\overline{\eta}_{x+e_i}.$$

Thus, for any $G : \mathbb{T}^d \to \mathbb{R}$ we have

$$\begin{aligned}
\mathfrak{L}_n X^n(G) &= X^n(\Delta_n G) + X^n\left((2\lambda\chi(\rho)d - 2 - \lambda\rho^2 d)G\right) \\
&+ \frac{\lambda\chi(\rho)}{n^{1+d/2}} \sum_{x\in\mathbb{T}_n^d} \overline{\eta}_x \sum_{i=1}^{d} \left(\nabla_n^i G\left(\frac{x}{n}\right) + \nabla_n^i G\left(\frac{x-e_i}{n}\right)\right) \\
&- \frac{\lambda(1-\rho)}{n^{d/2}} \sum_{x\in\mathbb{T}_n^d} G\left(\frac{x}{n}\right) \sum_{i=1}^{d} \overline{\eta}_{x-e_i}\overline{\eta}_{x+e_i} \\
&- \frac{\lambda\rho}{n^{d/2}} \sum_{x\in\mathbb{T}_n^d} \sum_{i=1}^{d} \left(G\left(\frac{x}{n}\right) + G\left(\frac{x+e_i}{n}\right)\right) \overline{\eta}_x\overline{\eta}_{x+e_i} \\
&- \frac{\lambda}{n^{d/2}} \sum_{x\in\mathbb{T}_n^d} \sum_{i=1}^{d} G\left(\frac{x}{n}\right) \overline{\eta}_{x-e_i}\overline{\eta}_x\overline{\eta}_{x+e_i}.
\end{aligned}$$

Recall (3.2.3). Let us fix a time horizon $[0,\tau]$, $\tau > 0$. By Dynkin's formula (Lemma B.2.1), for any function $G \in C^{1,\infty}([0,\tau]\times\mathbb{T}^d)$, any $\tau \geq 0$ and any $t \in [0,\tau]$, the process

$$M_{t,\tau}^n(G) := X_t^n(G_t) - X_0^n(G_0) - \int_0^t \left(\left(\Delta - \Phi(\rho) + \frac{d}{ds}\right) X_s^n(G_s)\right) ds \tag{3.5.1}$$
$$- \int_0^t \left((\Delta_n - \Delta)\, X_s^n(G_s)\right) ds \tag{3.5.2}$$
$$- \int_0^t \frac{\lambda\chi(\rho)}{n^{1+d/2}} \sum_{x\in\mathbb{T}_n^d} (\eta_s(x) - \rho) \sum_{i=1}^{d} \left(\nabla_n^i G_s\left(\frac{x}{n}\right) + \nabla_n^i G_s\left(\frac{x-e_i}{n}\right)\right) ds \tag{3.5.3}$$
$$+ \int_0^t \frac{\lambda}{n^{d/2}} \sum_{x\in\mathbb{T}_n^d} (1-\rho) \sum_{i=1}^{d} (\eta_s(x-e_i) - \rho)\,(\eta_s(x+e_i) - \rho)\, G_s\left(\frac{x}{n}\right) ds \tag{3.5.4}$$
$$+ \int_0^t \frac{\lambda}{n^{d/2}} \sum_{x\in\mathbb{T}_n^d} \rho \sum_{i=1}^{d} (\eta_s(x) - \rho)\,(\eta_s(x+e_i) - \rho) \left(G_s\left(\frac{x}{n}\right) + G_s\left(\frac{x+e_i}{n}\right)\right) ds \tag{3.5.5}$$
$$+ \int_0^t \frac{\lambda}{n^{d/2}} \sum_{x\in\mathbb{T}_n^d} (\eta_s(x-e_i) - \rho)\,(\eta_s(x) - \rho)\,(\eta_s(x+e_i) - \rho)\, G_s\left(\frac{x}{n}\right) ds \tag{3.5.6}$$

is a mean-zero martingale with respect to the natural filtration and its quadratic variation is given by

$$\langle M^n(G)\rangle_{t,\tau} = \int_0^t \frac{1}{n^d}\sum_{x\in\mathbb{T}_n^d}\sum_{i=1}^d \left(\nabla_n^i G_s\left(\frac{x}{n}\right)\right)^2 (\eta_s(x+e_i)-\eta_s(x))^2\, ds$$
$$+\int_0^t \frac{1}{n^d}\sum_{x\in\mathbb{T}_n^d}\left\{(1-\eta_s(x))c_x^+(\eta_s)+\eta_s(x)c_x^-(\eta_s)\right\}\left(G_s\left(\frac{x}{n}\right)\right)^2 ds.$$

From now on we consider the process $\{\eta_t; t\in[0,\tau]\}$ starting from the stationary measure μ_{ss}^n. For each $H\in C^\infty(\mathbb{T}^d)$ let $\{T_tH; t\in[0,\tau]\}$ be the solution of (3.2.7). Replacing G_t by $T_{\tau-t}H$ in (3.5.1), we conclude that

$$X_t^n(T_{\tau-t}H) := \mathcal{M}_{t,\tau}^n(H) + X_0^n(T_\tau H) \tag{3.5.7}$$
$$+\int_0^t \left((\Delta_n-\Delta)\, X_s^n(T_{\tau-s}H)\right) ds \tag{3.5.8}$$
$$+\int_0^t \frac{\lambda\chi(\rho)}{n^{1+d/2}}\sum_{x\in\mathbb{T}_n^d}(\eta_s(x)-\rho)\sum_{i=1}^d\left(\nabla_n^i T_{\tau-s}H\left(\frac{x}{n}\right)+\nabla_n^i T_{\tau-s}H\left(\frac{x-e_i}{n}\right)\right) ds \tag{3.5.9}$$
$$-\int_0^t \frac{\lambda}{n^{d/2}}\sum_{x\in\mathbb{T}_n^d}(1-\rho)\sum_{i=1}^d(\eta_s(x-e_i)-\rho)(\eta_s(x+e_i)-\rho)\,T_{\tau-s}H\left(\frac{x}{n}\right) ds \tag{3.5.10}$$
$$-\int_0^t \frac{\lambda}{n^{d/2}}\sum_{x\in\mathbb{T}_n^d}\rho\sum_{i=1}^d(\eta_s(x)-\rho)(\eta_s(x+e_i)-\rho)\left(T_{\tau-s}H\left(\frac{x}{n}\right)+T_{\tau-s}H\left(\frac{x+e_i}{n}\right)\right) ds \tag{3.5.11}$$

$$-\int_0^t \frac{\lambda}{n^{d/2}}\sum_{x\in\mathbb{T}_n^d}(\eta_s(x-e_i)-\rho)(\eta_s(x)-\rho)(\eta_s(x+e_i)-\rho)\,T_{\tau-s}H\left(\frac{x}{n}\right) ds, \tag{3.5.12}$$

where $\mathcal{M}_{t,\tau}^n(H)$ is a mean-zero martingale with quadratic variation given by

$$\langle\mathcal{M}^n(H)\rangle_{t,\tau} = \int_0^t \frac{1}{n^d}\sum_{x\in\mathbb{T}_n^d}\sum_{i=1}^d\left(\nabla_n^i T_{\tau-s}H\left(\frac{x}{n}\right)\right)^2(\eta_s(x+e_i)-\eta_s(x))^2\, ds \tag{3.5.13}$$
$$+\int_0^t \frac{1}{n^d}\sum_{x\in\mathbb{T}_n^d}\left\{(1-\eta_s(x))c_x^+(\eta_s)+\eta_s(x)c_x^-(\eta_s)\right\}\left(T_{\tau-s}H\left(\frac{x}{n}\right)\right)^2 ds. \tag{3.5.14}$$

In what follows, we use Proposition 3.4.2 and Theorem 3.4.3 to estimate some functionals. In order the verify that these functionals satisfy the hypotheses of these results, recall (3.2.9) and Proposition 3.2.9, and observe that for any smooth function H we have

$$\int_0^t \|T_{\tau-s}H\|_{L^2(\mathbb{T}^d)}ds \leq \|H\|_{L^2(\mathbb{T}^d)} \int_0^t e^{-\Phi(\rho)(\tau-s)}ds = \frac{\|H\|_{L^2(\mathbb{T}^d)}}{\Phi(\rho)}\left(e^{-\Phi(\rho)(\tau-t)} - e^{-\Phi(\rho)\tau}\right)$$
$$\leq \frac{\|H\|_{L^2(\mathbb{T}^d)}}{\Phi(\rho)}.$$

Moreover, we use the notation $f_n = \mathcal{O}(g_n)$ if there exists a positive constant C such that $f_n \leq C g_n$. Sometimes we write the sequence g_n depending on time, inside the notation, because we want to consider sequences of time which converge to ∞ as $n \to \infty$.

By Proposition 3.4.2, Theorem 3.4.3 and since $F(\rho) = 0$, for any $t \in [0, \tau]$ and any $d \leq 3$ we have

$$\langle \mathcal{M}^n(H) \rangle_{t,\tau} = \int_0^t \frac{2\chi(\rho)}{n^d} \sum_{x \in \mathbb{T}_n^d} \sum_{i=1}^d \left(\nabla_n^i T_{\tau-s} H\left(\frac{x}{n}\right)\right)^2 ds \tag{3.5.15}$$

$$+ \int_0^t \frac{2\rho}{n^d} \sum_{x \in \mathbb{T}_n^d} \left(T_{\tau-s} H\left(\frac{x}{n}\right)\right)^2 ds + \mathcal{O}\left(\frac{t}{\sqrt{n}}\right). \tag{3.5.16}$$

We remark that the above error is estimated in the $L^1(\mathbb{P}_{\mu_{ss}^n})$-norm. Thus, using Taylor's expansion on $T_{\tau-s}H$ and the fact that $\log n \leq n$, we obtain

$$\langle \mathcal{M}^n(H) \rangle_{t,\tau} = 2\chi(\rho) \int_0^t \|\nabla T_{\tau-s}H\|^2_{L^2(\mathbb{T}^d)}\, ds + 2\rho \int_0^t \|T_{\tau-s}H\|^2_{L^2(\mathbb{T}^d)}\, ds + \mathcal{O}\left(\frac{t}{\sqrt{n}}\right),$$

which, after the change of variables $\tau - s \mapsto r$, becomes

$$\langle \mathcal{M}^n(H) \rangle_{t,\tau} = 2\chi(\rho) \int_{\tau-t}^{\tau} \|\nabla T_r H\|^2_{L^2(\mathbb{T}^d)}\, dr + 2\rho \int_{\tau-t}^{\tau} \|T_r H\|^2_{L^2(\mathbb{T}^d)}\, dr + \mathcal{O}\left(\frac{t}{\sqrt{n}}\right). \tag{3.5.17}$$

3.5.2 Convergence to the Gaussian field

Proof of Theorem 3.2.6. First, observe that, by Proposition 3.4.2,

(1) the $L^1(\mathbb{P}_{\mu_{ss}^n})$-norm of the sum of (3.5.8) and (3.5.9) is of order $\mathcal{O}\left(t n^{d/2-2}\sqrt{g_d(n)}\right)$.

Furthemore, by Theorem 3.4.3,

(2) the $L^1(\mathbb{P}_{\mu_{ss}^n})$-norm of the sum of (3.5.10), (3.5.11) and (3.5.12) is of order $\mathcal{O}\left(t n^{d/2-2}\, g_d(n)\right)$.

Last, but not least, by inequality (3.2.9) and Proposition 3.4.1,

(3) the $L^1(\mathbb{P}_{\mu_{ss}^n})$-norm of the term $X^n_{0,\tau}(T_\tau H)$ in (3.5.7) is of order $\mathcal{O}\left(n^{d/2-1}\sqrt{g_d(n)}\, e^{-2\Phi(\rho)\tau}\right)$.

From the identity (3.5.7) and from the three observations above made, for any $\tau > 0$, and any $H \in C^\infty(\mathbb{T}^d)$

$$\mathbb{E}_{\mu^n_{ss}}\left[\left|X^n_\tau(H) - \mathcal{M}^n_{\tau,\tau}(H)\right|\right] = \begin{cases} \mathcal{O}\left(\frac{\tau}{\sqrt{n}} + e^{-2\Phi(\rho)\tau}\right) & \text{, if } d = 1, \\ \mathcal{O}\left(\frac{\tau \log n}{n} + \sqrt{\log n}\, e^{-2\Phi(\rho)\tau}\right) & \text{, if } d = 2, \\ \mathcal{O}\left(\tau n^{d/2-2} + n^{d/2-1}\, e^{-2\Phi(\rho)\tau}\right) & \text{, if } d \geq 3. \end{cases} \tag{3.5.18}$$

Let us set $\tau_n = \log n$. Since the process $\{\eta_t; t \geq 0\}$ starts from the stationary measure μ^n_{ss}, $X^n_t(H)$ and $X^n_0(H)$ have the same law, for any $H \in \mathcal{H}_k(\mathbb{T}^d)$ and any $t \geq 0$. In particular, the fluctuation field has the same law as the limit of $X^n_{\tau_n}(H)$ as $n \to \infty$, provided this limit exists. Moreover, by (3.5.18) and since $\lim_{\lambda \to 0} \Phi(\rho) = 2$, we can find λ^* small enough so that

$$\lim_{n\to\infty} \mathbb{E}_{\mu^n_{ss}}\left[\left|X^n_{\tau_n}(H) - \mathcal{M}^n_{\tau_n,\tau_n}(H)\right|\right] = 0 \tag{3.5.19}$$

for any $d \leq 3$.

Remark 3.5.1. *Notice that we can find λ^* such that the second parcel in the error of (3.5.18) converges to zero for any $d < 10$. The first parcel is what makes us restrict to $d < 4$.*

By (3.5.19), we are left to show that $\mathcal{M}^n_{\tau_n,\tau_n}(H)$ converges to the Gaussian field stated in Theorem 3.2.6. To do so, let us fix some $t_0 > 0$ and let us construct the sequence of martingales $\{\mathcal{N}^n_{t,\tau}(H); t \in [0, t_0]\}$ which we define by

$$\mathcal{N}^n_{t,\tau}(H) := \mathcal{M}^n_{\tau-t_0+t,\tau}(H) - \mathcal{M}^n_{\tau-t_0,\tau}(H).$$

From the definition of $\mathcal{N}^n_{t,\tau}(H)$ and by (3.5.17) we have

$$\begin{aligned}
\mathbb{E}_{\mu^n_{ss}}\left[\left(\mathcal{M}^n_{\tau,\tau}(H) - \mathcal{N}^n_{t_0,\tau}(H)\right)^2\right] &= \mathbb{E}_{\mu^n_{ss}}\left[\left(\mathcal{M}^n_{\tau-t_0,\tau}(H)\right)^2\right] = \mathbb{E}_{\mu^n_{ss}}\left[\langle \mathcal{M}^n(H)\rangle_{\tau-t_0,\tau}\right] \\
&= 2\chi(\rho)\int_{t_0}^{\tau} \|\nabla T_r H\|^2_{L^2(\mathbb{T}^d)}\, dr + 2\rho\int_{t_0}^{\tau} \|T_r H\|^2_{L^2(\mathbb{T}^d)}\, dr + \mathcal{O}\left(\frac{\tau}{n}\right) \\
&\leq 2\rho\int_{t_0}^{\tau} \|\nabla T_r H\|^2_{L^2(\mathbb{T}^d)} + \|T_r H\|^2_{L^2(\mathbb{T}^d)}\, dr + \mathcal{O}\left(\frac{\tau}{n}\right) \\
&\leq 2\rho\int_{t_0}^{\infty} \|\nabla T_r H\|^2_{L^2(\mathbb{T}^d)} + \|T_r H\|^2_{L^2(\mathbb{T}^d)}\, dr + \mathcal{O}\left(\frac{\tau}{n}\right).
\end{aligned}$$

Thus, from (3.2.9) and the linearity of the equation (3.2.7), we have

$$\begin{aligned}
\mathbb{E}_{\mu^n_{ss}}\left[\left(\mathcal{M}^n_{\tau,\tau}(H) - \mathcal{N}^n_{t_0,\tau}(H)\right)^2\right] &\leq 2\rho\left(\|\nabla H\|^2_{L^2(\mathbb{T}^d)} + \|H\|^2_{L^2(\mathbb{T}^d)}\right)\int_{t_0}^{\infty} e^{-2\Phi(\rho) r} dr + \mathcal{O}\left(\frac{\tau}{n}\right) \\
&= \frac{\rho}{\Phi(\rho)}\left(\|\nabla H\|^2_{L^2(\mathbb{T}^d)} + \|H\|^2_{L^2(\mathbb{T}^d)}\right) e^{-2\Phi(\rho) t_0} + \mathcal{O}\left(\frac{\tau}{n}\right).
\end{aligned}$$

Therefore,

$$\lim_{t_0\to\infty}\lim_{n\to\infty}\mathbb{E}_{\mu^n_{ss}}\left[\left(\mathcal{M}^n_{\tau_n,\tau_n}(H)-\mathcal{N}^n_{t_0,\tau_n}(H)\right)^2\right]=0. \tag{3.5.20}$$

Now we claim that $\{\mathcal{N}^n_{t,\tau_n}(H); t\in[0,t_0]\}$ converges to a mean-zero Gaussian martingale $\{\mathcal{N}_t(H); t\in[0,t_0]\}$ with quadratic variation

$$\langle\mathcal{N}(H)\rangle_t = 2\chi(\rho)\int_{t_0-t}^{t_0}\|\nabla T_r H\|^2_{L^2(\mathbb{T}^d)}\,dr + 2\rho\int_{t_0-t}^{t_0}\|T_r H\|^2_{L^2(\mathbb{T}^d)}\,dr. \tag{3.5.21}$$

Hence, by (3.5.20) we conclude that $\mathcal{M}^n_{\tau_n,\tau_n}(H)$ converges, as $n\to\infty$, to the limit of $\mathcal{N}_{t_0}(H)$ as $t_0\to\infty$, which is a Gaussian random variable with variance

$$2\chi(\rho)\int_0^\infty\|\nabla T_r H\|^2_{L^2(\mathbb{T}^d)}\,dr + 2\rho\int_0^\infty\|T_r H\|^2_{L^2(\mathbb{T}^d)}\,dr.$$

Finally, we prove the aforementioned claim. We will use Theorem B.5.1 for the sequence of martingales $\{\mathcal{N}^n_{t,\tau_n}(H); t\in[0,t_0]\}$ so we need to verify each of its three hypotheses. Indeed, from (3.5.17) and the definition of $\mathcal{N}^n_{t,\tau}(H)$, the process $\{\mathcal{N}^n_{t,\tau_n}(H); t\in[0,t_0]\}$, has quadratic variation given by

$$\begin{aligned}\langle\mathcal{N}^n(H)\rangle_t &= \langle\mathcal{M}^n(H)\rangle_{\tau_n-t_0+t,\tau_n} - \langle\mathcal{M}^n(H)\rangle_{\tau_n-t_0,\tau_n}\\ &= 2\chi(\rho)\int_{t_0-t}^{t_0}\|\nabla T_r H\|^2_{L^2(\mathbb{T}^d)}\,ds + 2\rho\int_{t_0-t}^{t_0}\|T_r H\|^2_{L^2(\mathbb{T}^d)}\,dr + \mathcal{O}\left(\frac{\tau_n}{n}\right),\end{aligned}$$

which clearly converges to $\langle\mathcal{N}(H)\rangle_t$ for any $t\in[0,t_0]$. This proves the third hypothesis of Theorem B.5.1. Furthermore, the explicit form of the quadratic variation of the martingale directly implies that it has continuous trajectories in time, proving the first hypothesis of Theorem B.5.1. To see that the second, and last, hypothesis of the theorem holds, observe that

$$\begin{aligned}\left|\mathcal{N}^n_{s,\tau_n}(H)-\mathcal{N}^n_{s-,\tau_n}(H)\right| &= \left|\mathcal{M}^n_{\tau_n-t_0+s,\tau}(H)-\mathcal{M}^n_{\tau_n-t_0+(s-),\tau}(H)\right|\\ &= \left|X^n_s(T_{\tau_n-s}H)-X^n_{s-}(T_{\tau_n-(s-)}H)\right|,\end{aligned}$$

being the last equality true due to the fact that the integrals in (3.5.8), (3.5.9), (3.5.10), (3.5.11) and (3.5.12) are finite. Thus, since $T_{\tau_n-t}H:\mathbb{T}^d\to[0,1]$ is continuous,

$$\begin{aligned}&\mathbb{E}_{\mu^n_{ss}}\left[\sup_{s\in[0,t_0]}\left|\mathcal{N}^n_{s,\tau_n}(H)-\mathcal{N}^n_{s-,\tau_n}(H)\right|\right]\\ &\quad=\frac{1}{n^{d/2}}\mathbb{E}_{\mu^n_{ss}}\left[\sup_{s\in[0,t_0]}\left|\sum_{x\in\mathbb{T}^d_n}\left\{(\eta_s(x)-\rho)\,T_{\tau_n-s}H\left(\frac{x}{n}\right)-(\eta_{s-}(x)-\rho)\,T_{\tau_n-(s-)}H\left(\frac{x}{n}\right)\right\}\right|\right]\\ &\quad\le\frac{1}{n^{d/2}}\mathbb{E}_{\mu^n_{ss}}\left[\sup_{s\in[0,t_0]}\left|\sum_{x\in\mathbb{T}^d_n}\left\{\eta_s(x)\,T_{\tau_n-s}H\left(\frac{x}{n}\right)-\eta_{s-}(x)\,T_{\tau_n-(s-)}H\left(\frac{x}{n}\right)\right\}\right|\right]\end{aligned}$$

$$
+ \frac{1}{n^{d/2}} \mathbb{E}_{\mu^n_{ss}} \left[\sup_{s\in[0,t_0]} \left| \rho \sum_{x\in\mathbb{T}^d_n} \left\{ T_{\tau_n-(s-)} H\left(\frac{x}{n}\right) - T_{\tau_n-s} H\left(\frac{x}{n}\right) \right\} \right| \right]
$$

$$
= \frac{1}{n^{d/2}} \mathbb{E}_{\mu^n_{ss}} \left[\sup_{s\in[0,t_0]} \left| \sum_{x\in\mathbb{T}^d_n} (\eta_s(x) - \eta_{s-}(x)) \; T_{\tau_n-s} H\left(\frac{x}{n}\right) \right| \right]. \qquad (3.5.22)
$$

Now observe that in an infinitesimal time only one update occurs and it changes the occupations of a configuration in either one vertex (in case of a reaction update) or two vertices (in case of an exclusion update). In the first case, there exists $y \in \mathbb{T}^d_n$ such that $\eta_s(y) \neq \eta_{s-}(y)$, $\eta_s(x) = \eta_{s-}(x)$ for all $x \neq y$ and

$$
\left| \sum_{x\in\mathbb{T}^d_n} (\eta_s(x) - \eta_{s-}(x)) \; T_{\tau_n-s} H\left(\frac{x}{n}\right) \right| = \left| (\eta_s(y) - \eta_{s-}(y)) \; T_{\tau_n-s} H\left(\frac{y}{n}\right) \right| = \left| T_{\tau_n-s} H\left(\frac{y}{n}\right) \right|.
$$

In the second case, there exist $y, z \in \mathbb{T}^d_n$ such that $\eta_s(y) = \eta_{s-}(z) = 1$, $\eta_s(z) = \eta_{s-}(y) = 0$, $\eta_s(x) = \eta_{s-}(x)$ for all $x \in \mathbb{T}^d_n \setminus \{y, z\}$ and

$$
\begin{aligned}
\left| \sum_{x\in\mathbb{T}^d_n} (\eta_s(x) - \eta_{s-}(x)) \; T_{\tau_n-s} H\left(\frac{x}{n}\right) \right| &= \left| (1-0) \, T_{\tau_n-s} H\left(\frac{y}{n}\right) + (0-1) \, T_{\tau_n-s} H\left(\frac{z}{n}\right) \right| \\
&= \left| \, T_{\tau_n-s} H\left(\frac{y}{n}\right) - \, T_{\tau_n-s} H\left(\frac{z}{n}\right) \right|.
\end{aligned}
$$

In any case $\left| \sum_{x\in\mathbb{T}^d_n} (\eta_s(x) - \eta_{s-}(x)) \; T_{\tau_n-s} H\left(\frac{x}{n}\right) \right|$ is uniformly bounded in n. Therefore, by (3.5.22)

$$
\lim_{n\to\infty} \mathbb{E}_{\mu^n_{ss}} \left[\sup_{s\in[0,t_0]} \left| \mathcal{N}^n_{s,\tau_n}(H) - \mathcal{N}^n_{s-,\tau_n}(H) \right| \right] = 0.
$$

Hence, we can apply Theorem B.5.1 and conclude the claim. □

Chapter 4

Conclusion and future directions

In Chapter 2 we developed a method to study the time that a particle systems takes to have its law close to equilibrium. We illustrate the strategy with the diagram depicted in Figure 4.1:

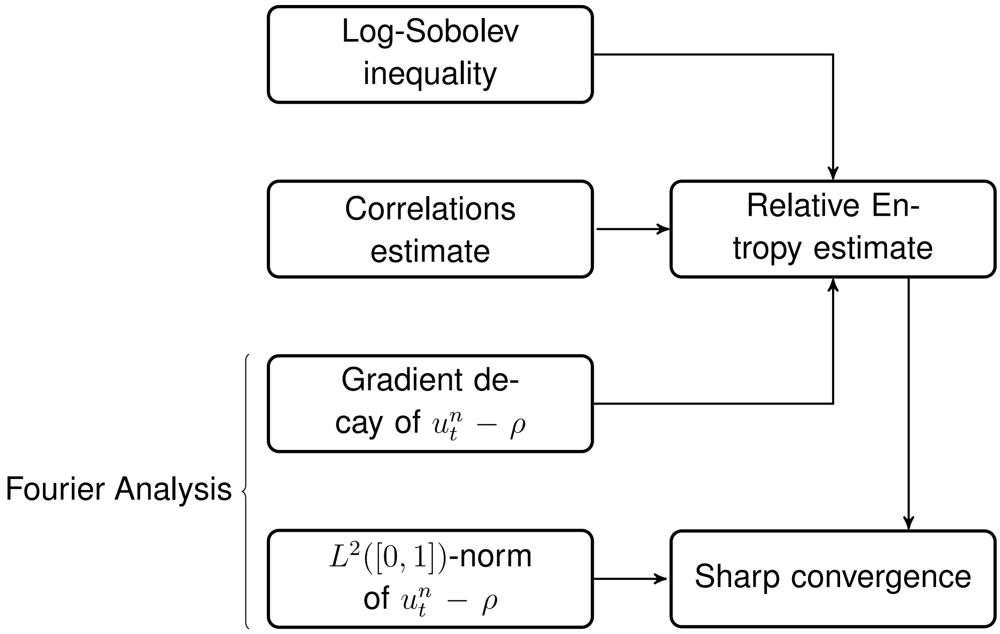

Figure 4.1: Diagram of tasks to prove sharp convergence to equilibrium of particle systems with Glauber dynamics.

Heuristic arguments suggest that in a bidimensional version of this model (as the one depicted in Figure 4.2) the correlations in Proposition 2.5.2 should be of order $\mathcal{O}\left(\frac{\log n}{n^2}\right)$. These estimates could be done as in [14] or one could avoid them with the static replacement stated in Lemma 3.3.4. If one proves that, then, following the recipe given in the above diagram, one can obtain a sharp convergence result as the one stated in Theorem 2.1.1.

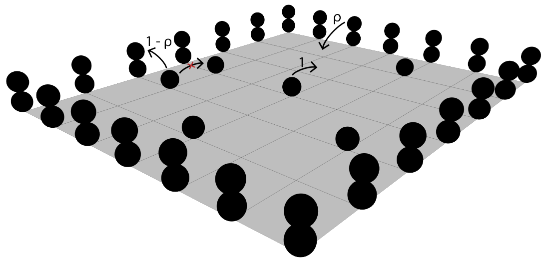

Figure 4.2: Exclusion process on the unit square in contact with reservoirs of density ρ in all four boundary sides.

The above method can be applied to other models. Due to the non-linearity of the hydrodynamic equation, it would be a personal challenge to do it for the porous-media model in contact with reservoirs [3] and we leave it for a future work. Another problem is to study the non-equilibrium case.

In Chapter 3 we proved a central limit theorem for the fluctuation field of a reaction-diffusion model that is out of equilibrium depending on a parameter $\lambda \geq 0$. More precisely, the larger λ is, further the process is from the equilibrium states. The strategy, which we illustrate with the diagram depicted in Figure 4.3, works when λ is small enough. An interesting problem would be to understand what happens when λ is large.

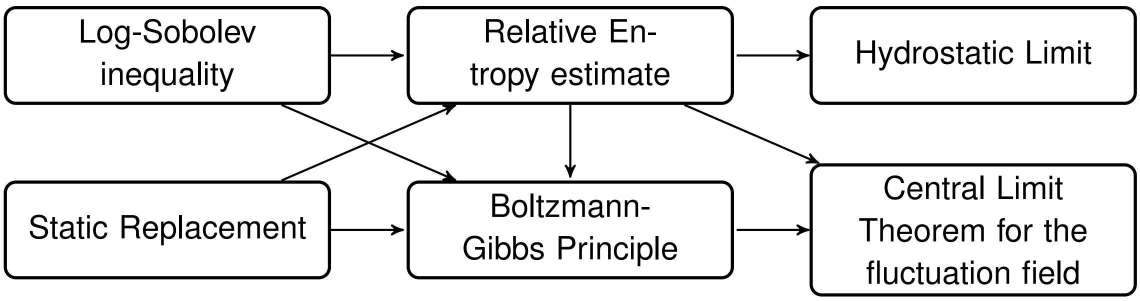

Figure 4.3: Diagram of tasks to prove hydrostatic limit and stationary fluctuations for particle systems with Glauber dynamics.

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Appendix A

Some results for sub-Gaussian random variables

A.1 Comparison of Gaussian densities

Proposition A.1.1. *Let X be a standard Gaussian random variable, that is, $X \sim \mathcal{N}(0,1)$. Recall the Gauss error function in (2.1.5). For any $m \geq 0$ we have*

$$\mathcal{G}(m) = \frac{1}{2}\,\mathbb{E}\Big[\Big|e^{mX-\frac{m^2}{2}} - 1\Big|\Big] = \textit{erf}\left(\tfrac{m}{2\sqrt{2}}\right).$$

Proof. Let us recall that the probability density function f_m of a Gaussian random variable with mean m and variance 1 is defined as

$$f_m(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-m)^2/2}.$$

Observe that

$$\begin{aligned}
\mathbb{E}\left[\left|e^{mX-\frac{m^2}{2}} - 1\right|\right] &= \int_{-\infty}^{\infty} \left|e^{mx-\frac{m^2}{2}} - 1\right| f_0(x)dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left|e^{mx-\frac{m^2}{2}} - 1\right| e^{-x^2/2}dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left|e^{-(x-m)^2/2} - e^{-x^2/2}\right| dx \\
&= \int_{-\infty}^{\infty} |f_m(x) - f_0(x)|\, dx = 2\,\mathcal{G}(m).
\end{aligned}$$

Therefore, we aim to compute the area between the red curve and the blue curve in Figure A.1.

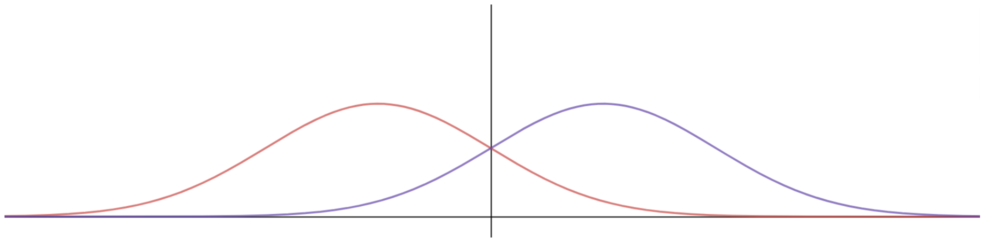

Figure A.1: Red curve representing the probability density function f_0 and blue curve representing f_m. The vertical axis represents the line $x = m/2$ where the two functions intersect.

Furthermore, since the above figure is symmetric with respect to the axis $x = m/2$, the desired area is the double of the area on the left side. Hence, we conclude that

$$\mathbb{E}\left[\left|e^{mX-\frac{m^2}{2}} - 1\right|\right] = 2\int_{-\infty}^{m/2} (f_0(x) - f_m(x))\, dx = 2\,\text{erf}\left(\frac{m}{2\sqrt{2}}\right).$$

□

A.2 Lyapounov Central Limit Theorem

Theorem A.2.1 (Lyapounov Central Limit Theorem)**.** *Let* $\{\xi_x; x \in \{1, \ldots, N\}\}$ *be a sequence of independent random variables with mean* $\mathbb{E}[\xi_x] = m_x$ *and variance Var*$[\xi_x] = v_x$*. Define* $s_N^2 = \sum_{x=1}^N v_x$*. If for some* $\delta > 0$*, the Lyapounov's condition*

$$\lim_{N\to\infty} \frac{1}{s_N^{2+\delta}} \sum_{x=1}^{N} \mathbb{E}[|\xi_x - m_x|^{2+\delta}] = 0$$

holds, then

$$\frac{1}{s_N} \sum_{x=1}^{N} (\xi_x - m_x)$$

converges, in law, to a Standard Gaussian random variable.

Proof. See [2, Theorem 27.3]. □

A.3 Hoeffding's Lemma

Lemma A.3.1 (Hoeffding's Lemma)**.** *Let* η *be a random variable taking values on* $[0, 1]$*, with mean* $m = \mathbb{E}[\eta]$*. Then, for any* $\theta \in \mathbb{R}$*, we have* $\log \mathbb{E}\left[e^{\theta(\eta - m)}\right] \le \frac{\theta^2}{8}$*.*

Proof. We refer to [17, Lemma 2.6] for a very detailed proof of a more general version of this result. □

Appendix B

Classical results for hydrodynamics and fluctuations

In this appendix we collect some classical results that are often used to prove scaling limits of interacting particle systems. We refer the reader to see their proofs where we believe that it is simpler to follow.

B.1 *Carré du champ* operator

Proposition B.1.1. *Let $\mathcal{L}$ be the infinitesimal generator of an irreducible continuous-time Markov chain with state space Ω. Let $r(\eta,\xi)$ denote the jump rate from state η to state ξ. Recall the carré du champ operator, which is defined on functions $f:\Omega\to\mathbb{R}$ by $\Gamma f(\eta)=\mathcal{L}f^2(\eta)-2f(\eta)\mathcal{L}f(\eta)$. We have*

$$\Gamma f(\eta)=\sum_{\xi\in\Omega} r(\eta,\xi)\left(f(\xi)-f(\eta)\right)^2.$$

Proof.

$$\begin{aligned}\Gamma f(\eta)&=\sum_{\xi\in\Omega} r(\eta,\xi)\left(f^2(\xi)-f^2(\eta)\right)-2f(\eta)\sum_{\xi\in\Omega} r(\eta,\xi)\left(f(\xi)-f(\eta)\right)\\&=\sum_{\xi\in\Omega} r(\eta,\xi)\left[f^2(\xi)-f^2(\eta)-2f(\eta)\left(f(\xi)-f(\eta)\right)\right]\\&=\sum_{\xi\in\Omega} r(\eta,\xi)\left(f(\xi)-f(\eta)\right)^2.\end{aligned}$$

□

B.2 Dynkin's martingales

Lemma B.2.1. *Let $F:[0,+\infty)\times\Omega_n\to\mathbb{R}$ be a bounded function. Assume that F is smooth in the first coordinate uniformly over the second: for each $\eta\in\Omega_n$, $F(\cdot,\eta)$ is* *twice continuously differentiable and there exists a finite constant C such that*

$$\sup_{(s,\eta)} |(\partial_s^j F)(s,\eta)| \le C$$

for each $j = 1, 2$. Above, $(\partial_s^j F)$ stands for the j-th derivative of $F(\cdot,\eta)$. Denote by $\{\mathcal{F}_t; t \ge 0\}$ the filtration induced by the Markov process $\{\eta_t; t \ge 0\}$: $\mathcal{F}_t = \sigma(\eta_s; s \le t)$. Thus, the processes

$$\mathcal{M}_t(F) := F(t,\eta_t) - F(0,\eta_0) - \int_0^t (\partial_s + \mathfrak{L}_n)F(s,\eta_s)ds,$$

and

$$\mathcal{N}_t(F) := (\mathcal{M}_t(F))^2 - \int_0^t \Gamma F(s,\eta_s)ds$$

are zero-mean $\mathcal{F}_t$-martingales.

Proof. We refer to [13, Appendix 1, Lemma 5.1]. □

B.3 Entropy inequality

Proposition B.3.1 (Entropy inequality). *Let $B > 0$. Let μ and ν be two probability measures in a finite space Ω_n. Let g be the Radon-Nikodym derivative of the measure μ with respect to ν and let $H(\mu|\nu) = \int g \log g d\nu$ be the relative entropy of g with respect to ν. Let $f : \Omega_n \to \mathbb{R}$ be any function. We have*

$$\int f(\eta)\, \mu(d\eta) \le \frac{1}{B} \left(H(\mu\,|\,\nu) + \log \int e^{\{B\, f(\eta)\}}\, \nu(d\eta) \right).$$

Proof. We refer to [12, Proposition F.2]. □

An immediate consequence of the entropy inequality is the following:

Corollary B.3.2. *Let $A \subset \Omega_n$. Let μ and ν be two probability measures in Ω_n. Let g be the Radon-Nikodym derivative of the measure μ with respect to ν and let $H(\mu|\nu)$ be the relative entropy of g with respect to ν. We have*

$$\mu(A) \le \frac{\log 2 + H(\mu\,|\,\nu)}{\log(1 + 1/\nu(A))}.$$

Proof. Take $f = \mathbb{1}_A$ and $B = \log(1 + 1/\nu(A))$ in the previous proposition. □

B.4 Feynman-Kac's inequality

The following result is a consequence of the Feynman-Kac's formula:

Lemma B.4.1. *For any $V : [0,T] \times \Omega_n \to \mathbb{R}$ and any $t \in [0,T]$,*

$$\log \mathbb{E}_{\mu_n}\left[\exp\left\{\int_0^t V_s(\eta_s)ds\right\}\right] \\ \leq \int_0^t \sup_f \left\{-\mathfrak{D}\left(\sqrt{f},\nu_\rho\right) + \int V f d\nu_\rho + \frac{1}{2}\int\left(\mathfrak{L}_s^* \mathbb{1} - \frac{d}{ds}\psi\right) f d\nu_\rho\right\} ds,$$

where the supremum runs over all densities f with respect to ν_ρ.

Proof. We refer to [12, Lemma A.2]. □

B.5 Convergence of martingales

Theorem B.5.1. *Let $\{M_t^n; t \in [0,T]\}_{n\in\mathbb{N}}$ be a sequence of continuous-time martingales living in the Skorohod space $D([0,T],\mathbb{R})$. For each $n \in \mathbb{N}$ and each $t \in [0,T]$ denote by $\langle M^n \rangle_t$ the quadratic variation of M_t^n. Assume that*

1. *for any $n \in \mathbb{N}$, the quadratic variation process $\{\langle M^n \rangle_t; t \in [0,T]\}$ has continuous trajectories almost surely;*
2. $\lim_{n\to\infty} \mathbb{E}_{\mu_n}\left[\sup_{s\in[0,T]} \left|M_s^n - M_{s-}^n\right|\right] = 0$*;*
3. *for each $t \in [0,T]$ the sequence of random variables $\{\langle M^n \rangle_t\}_{n\in\mathbb{N}}$ converges in probability to a deterministic function $\langle M \rangle_t$.*

Thus, the sequence $\{M_t^n; t \in [0,T]\}_{n\in\mathbb{N}}$ converges, in law, in $D([0,T];\mathbb{R})$, as $n \to \infty$, to a zero mean Gaussian process $\{M_t; t \in [0,T]\}$ which is a martingale with continuous trajectories and whose quadratic variation is given by $\langle M \rangle_t$, for any $t \in [0,T]$.

Proof. We refer to [10, Theorem VIII, 3.12]. □