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Steady Navier–Stokes Equations with Regularized Directional Do-Nothing Boundary Condition: Optimal Boundary Control for a Velocity Tracking Problem

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Abstract

We consider the steady Navier–Stokes equations with mixed boundary conditions, where a regularized directional do-nothing (RDDN) condition is defined on the Neumann boundary portion. An auxiliary Stokes reference flow, which also works as a lifting of the inhomogeneous Dirichlet boundary values, is used to define the RDDN condition. Our aim is to study the minimization of a velocity tracking cost functional with controls localized on a part of the boundary. We prove the existence of a solution for this optimal control problem and derive the corresponding first order necessary optimality conditions in terms of dual variables. All results are obtained under appropriate assumptions on the size of the data and the controls, which, however, are less restrictive when compared with the case of the classical do-nothing outflow condition. This is further confirmed by the numerical examples presented, which include scenarios where only noisy data is available.

Keywords Navier–Stokes equations · Mixed boundary conditions · Regularized directional do-nothing boundary condition · Optimal boundary control · Velocity tracking

Mathematics Subject Classification 76D05 · 35J66 · 76D55 · 49J20 · 49K20

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1 Introduction

In Fluid Mechanics, including a wide range of applications in Engineering and Biomedicine, flow control by acting on the boundary conditions of the system poses interesting theoretical and computational challenges [1–8].

Both in the context of the forward problem and the control problem, when carrying out numerical simulations of fluid flows in exterior domains, channels and pipes, artificial boundaries have to be introduced in order to define or reduce the size of computational domains and lower the computational cost of the simulations [8–20]. This motivates several theoretical and numerical studies of Navier-Stokes flow control with open boundary conditions [1, 2, 6, 21–23].

A mathematical study on modelling and optimal control of blood flow redirected by a bypass surgery in a tract of an artery was carried out in [6]. Specifically, in [6], the authors consider the following optimal control problem: tracking a desired velocity field for a Navier–Stokes fluid with inputs on a portion of the boundary of the fluid domain and a classical do-nothing boundary condition (CDN) imposed on the outflow part of the domain. The CDN condition involves the stress vector on a portion of the boundary.

The so-called directional do-nothing condition (DDN) is an improved Neumann type boundary condition which has been successfully used to address classical Navier–Stokes flow problems encountered in simulations with open boundary conditions [11, 13, 14]. It takes into account the presence of the convection term in the Navier–Stokes equations and is used to prevent backflow in outflow boundary portions. In [10, 11], the authors show that the DDN outflow condition also provides accuracy and stability for higher Reynolds numbers. In [2], alternative convective-like energy-stable open boundary conditions are tested in some numerical simulations for optimal flow control.

Inspired by the models [2, 11, 16] and by the related control problems [4, 6], our aim is to investigate the replacement of the classical Neumann type condition in the steady Navier-Stokes equations by a regularized directional do-nothing boundary condition (RDDN), which is defined in terms of a reference flow associated with the nonhomogeneous Dirichlet boundary values [13, 14, 18, 24]. The mathematical model consists in solving a suitable Stokes problem and, afterwards, use this reference flow in the outflow boundary conditions of the Navier-Stokes problem. In this paper, we intend to investigate this alternative RDDN condition in the boundary control problem for tracking of fluid flow velocity fields [4, 6]. The DDN condition corrects the CDN condition by an inward velocity term which is defined using the (nonsmooth) negative part of a function. Since we are interested in the derivation of first order necessary conditions of optimality, wherein Gâteaux differentiation is an essential tool, we opted for a RDDN outflow condition, which is similar to those considered in [2], but takes into account the use of the reference flow. It will be clear that the reference flow also depends on the control and this will lead to a nontrivial control-to-state mapping and additional difficulties in the characterization of the first order optimality conditions, namely the identification of the adjoint system.

2 Formulation of the Problem, Structure of the Paper and Main Results

Throughout this work, $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, represents a bounded domain with a Lipschitz continuous boundary $\Gamma := \partial\Omega$. We assume that Γ is divided into two disjoint parts, the Dirichlet part Γ_D and the "Neumann part" Γ_N, $\Gamma = \overline{\Gamma_D \cup \Gamma_N}$, such that $|\Gamma_D|, |\Gamma_N| > 0$ (see [8, p. 99]). Our aim is to study a velocity tracking problem via controls with support on Γ_D. Here, we will replace the CDN outflow condition,

$$\nu(\boldsymbol{n} \cdot \nabla)\boldsymbol{v} - p\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma_N, \tag{2.1}$$

with an alternative, more robust, Neumann type boundary condition. In (2.1) and in what follows, $\boldsymbol{n}$ denotes the outward unit normal to Γ, $\nu > 0$ is the kinematical viscosity coefficient of the fluid, $\boldsymbol{v}$ is the velocity and p is the corresponding pressure.

The gradient of a vector-valued function of several variables is defined as the transpose of the Jacobian matrix, $(\nabla \boldsymbol{v})_{ij} = \frac{\partial \boldsymbol{v}_j}{\partial x_i}$, $i, j = 1, ..., n$, so that in (2.1) we can write $(\boldsymbol{n} \cdot \nabla)\boldsymbol{v} = \boldsymbol{n} \cdot \nabla \boldsymbol{v}$. Usually, the notation $\mathbb{T}(\boldsymbol{v}, p)$ is reserved for the stress tensor, $\mathbb{T}(\boldsymbol{v}, p) := 2\nu\mathbb{D}(\boldsymbol{v}) - p\mathbb{I}$, where $\mathbb{D}(\boldsymbol{v})$ is the stretching tensor, $\mathbb{D}(\boldsymbol{v}) = \frac{1}{2}\left(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^\top\right)$, and $\mathbb{I}$ is the identity tensor. We will use the notation $\widetilde{\mathbb{T}}(\boldsymbol{v}, p) := \nu\nabla\boldsymbol{v} - p\mathbb{I}$ for the so-called pseudo-stress tensor and the adjoint pseudo-stress tensor will be denoted by $\widetilde{\mathbb{T}}^*(\boldsymbol{v}, p) := \nu\nabla\boldsymbol{v} + p\mathbb{I}$.

The divergence of a tensor-valued function $\boldsymbol{T}$ is defined by $(\nabla \cdot \boldsymbol{T})_j = \frac{\partial \boldsymbol{T}_{ij}}{\partial x_i}$, $j = 1, ..., n$. When $\nabla \cdot \boldsymbol{v} = 0$, we have $\nabla \cdot \mathbb{T}(\boldsymbol{v}, p) = \nu\Delta\boldsymbol{v} - \nabla p = \nabla \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}, p)$.

The Navier–Stokes system with CDN boundary condition on Γ_N can be written in the form

$$\begin{cases} -\nabla \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}, p) + \boldsymbol{v} \cdot \nabla\boldsymbol{v} = \boldsymbol{f} & \text{in } \Omega \\ \nabla \cdot \boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v} = \boldsymbol{g} & \text{on } \Gamma_D \\ \boldsymbol{n} \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}, p) = \boldsymbol{0} & \text{in } \Gamma_N. \end{cases} \tag{2.2}$$

In [13, 14] (see also [11] for the case of homogeneous Dirichlet boundary values) a modification of the CDN condition was proposed which can be written as

$$\begin{cases} -\nabla \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}, p) + \boldsymbol{v} \cdot \nabla\boldsymbol{v} = \boldsymbol{f} & \text{in } \Omega \\ \nabla \cdot \boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v} = \boldsymbol{g} & \text{on } \Gamma_D \\ \boldsymbol{n} \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}, p) + \frac{1}{2}\left[\boldsymbol{v} \cdot \boldsymbol{n}\right]^- (\boldsymbol{v} - \boldsymbol{v}_r) = \boldsymbol{n} \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}_r, p_r) & \text{on } \Gamma_N \end{cases} \tag{2.3}$$

where

$$[y]^- := \max\{0, -y\} = (|y| - y)/2 \quad (y \in \mathbb{R})$$

and $(\boldsymbol{v}_r, p_r)$ is a reference flow associated with the nonhomogeneous Dirichlet boundary condition. This reference flow can be obtained, for example, by solving, in a first step, a steady Stokes problem. Following [13], we will consider $(\boldsymbol{v}_r, p_r)$ solution of the system

$$\begin{cases} \nabla \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}_r, p_r) = \mathbf{0} & \text{in } \Omega \\ \nabla \cdot \boldsymbol{v}_r = 0 & \text{in } \Omega \\ \boldsymbol{v}_r = \boldsymbol{g} & \text{on } \Gamma_D \\ \boldsymbol{n} \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}_r, p_r) = \mathbf{0} & \text{on } \Gamma_N. \end{cases} \tag{2.4}$$

Following [11], the Neumann boundary condition defined in (2.3) is called directional do-nothing (DDN) boundary condition. Using the reference flow (2.4), we will formulate the problem with RDDN boundary condition as

$$\begin{cases} -\nabla \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}, p) + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \boldsymbol{f} & \text{in } \Omega \\ \nabla \cdot \boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v} = \boldsymbol{g} & \text{on } \Gamma_D \\ \boldsymbol{n} \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}, p) + \dfrac{1}{2}\Psi_\delta(\boldsymbol{v} \cdot \boldsymbol{n})(\boldsymbol{v} - \boldsymbol{v}_r) = \mathbf{0} & \text{on } \Gamma_N \end{cases} \tag{2.5}$$

where Ψ_δ is a regular function that depends on a parameter $0 < \delta \ll 1$. From the computational point of view, the advantage of working with (2.5) is that the RDDN condition allows the application of Newton's method. Possible choices for Ψ_δ are

$$\Psi_\delta(y) := \frac{1}{2}\left[y \tanh\left(\frac{y}{\delta}\right) - y + \delta\right], \; y \in \mathbb{R}, \tag{2.6}$$

and

$$\Psi_\delta(y) := \begin{cases} \sqrt{y^2 + \delta^2} & \text{if } y \leq 0 \\ \delta & \text{if } y > 0. \end{cases} \tag{2.7}$$

The main properties assumed for Ψ_δ are:

(i) when $\delta \to 0$, $\psi_\delta(\cdot) \to [\cdot]^-$ a.e in $\mathbb{R}^3$,
(ii) there exists $\tilde{c} > 0$ such that for all $0 < \delta \ll 1$,

$$[y]^- < \Psi_\delta(y) \leq \tilde{c}\delta + [y]^-, \quad \forall y \in \mathbb{R}, \tag{2.8}$$

(iii) for all $0 < \delta \ll 1$, $\psi'_\delta \in C^{0,1}(\mathbb{R})$ and therefore we can define the constants

$$M_{\Psi'_\delta} = \|\Psi'_\delta\|_{\infty,\mathbb{R}}, \qquad L_{\Psi'_\delta} = \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|\Psi'_\delta(x) - \Psi'_\delta(y)|}{|x - y|}. \tag{2.9}$$

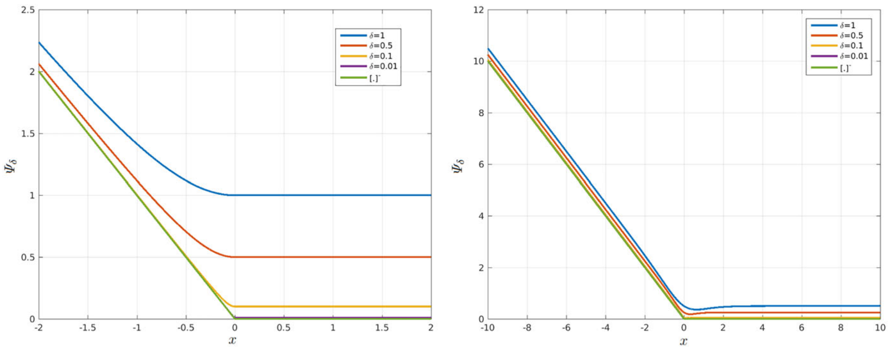


Fig. 1 Different profiles of the regularization $\Psi_\delta(x)$. Function (2.7) on the left, (2.6) on the right

Condition (iii) will ensure the Gâteaux differentiability of the control-to-state operator, while (ii) implies

$$\boldsymbol{v}\cdot\boldsymbol{n} = [\boldsymbol{v}\cdot\boldsymbol{n}]^+ - [\boldsymbol{v}\cdot\boldsymbol{n}]^- > [\boldsymbol{v}\cdot\boldsymbol{n}]^+ - \Psi_\delta(\boldsymbol{v}\cdot\boldsymbol{n})$$

which is of crucial importance for establishing certain estimates in the existence and uniqueness results. The well-posedness of the coupled problem (2.4), (2.5) for $(\boldsymbol{v}_r, \boldsymbol{v}, p)$ was studied in [24] under the above assumptions for Ψ_δ.

The problem we are interested in is the minimization of the cost functional

$$\mathcal{J}(\boldsymbol{v}, \boldsymbol{g}) := \frac{1}{2}\|\boldsymbol{v} - \boldsymbol{v}_\Omega\|^2_{2,\Omega} + \frac{\tau}{2}\|\boldsymbol{g}\|^2_{1/2,\Gamma_D},$$

subject to (2.5), by acting on Γ_D through boundary values $\boldsymbol{g}$. The parameter $\tau \geq 0$ and the target velocity $\boldsymbol{v}_\Omega \in L^2(\Omega)^n$ are given.

Observe that the state equations (2.4), (2.5) are a kind of coupled system and therefore, when studying the control problem, we will have to take into account that (2.4) also depends on the control.

Structure of the paper and main results. Notations and auxiliary results, in particular, the relevant functions spaces and operators used in the paper are presented in Sect. 3. In Sect. 4, we give the mixed weak formulation of the state equations (simultaneously for the velocity and pressure variables). In Theorem 4.4 we recall a result on existence and uniqueness of weak solutions. The formulation of the control problem is given in Sect. 5. Existence of an optimal solution is proved in Theorem 5.1, under appropriate restrictions on the size of the data and the controls. Lipschitz estimates for the control-to-state mapping are obtained in Sect. 6, namely, Theorem 6.2, and then, in Sect. 7, the Gâteaux differentiability of the control-to-state mapping is investigated. The main result of Sect. 7 is Theorem 7.1. The adjoint system is presented in Sect. 8, where Theorem 8.1 gives existence and uniqueness of solution for such system. First order optimality conditions and the main result of the paper, Theorem 9.1, are deduced in Sect. 9. The theoretical results are illustrated with a numerical experiment in Sect. 10. Finally, Sect. 11 highlights the main results of the paper.

3 Notations and Auxiliary Results

In this section, we present the framework for stating and proving the results of the next sections.

Throughout the paper, we shall use the classical notations for Lebesgue and Sobolev spaces, namely, $L^s(\Omega)$ and $H^1(\Omega)$, with norms $\|.\|_{s,\Omega}$ and $\|.\|_{1,2,\Omega}$, respectively. We recall the Hilbert space

$$H^{1/2}(\Gamma) = \Big\{u \in L^2(\Gamma) : \|u\|_{1/2,\Gamma} := \left(\|u\|_{2,\Gamma}^2 + \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^2}{|x-y|^n} \mathrm{d}\sigma_x \mathrm{d}\sigma_y\right)^{\frac{1}{2}} < \infty\Big\}$$

and the Lions-Magenes space

$$H_{00}^{1/2}(\Gamma_D) := \Big\{u \in H^{1/2}(\Gamma_D) : \tilde{u} \in H^{1/2}(\Gamma)\Big\},$$

where $\tilde{u}$ denotes extension to Γ by zero. The space $H_{00}^{1/2}(\Gamma_D)$ will be endowed with the $H^{1/2}$-norm. The following notations will be used for inner products: $(u, v)_\Omega := (u, v)_{L^2(\Omega)}$ and $(u, v)_{\Gamma_D} := (u, v)_{H^{1/2}(\Gamma_D)}$. The duality product between $H^{-1/2}(\Gamma_D)$ and $H_{00}^{1/2}(\Gamma_D)$ will be denoted by $\langle u, v\rangle_{\Gamma_D}$.

The non-homogeneous Dirichlet boundary values of (2.4), (2.5) will be prescribed in the space $\boldsymbol{T}_D := H_{00}^{1/2}(\Gamma_D)^n$. The velocity $\boldsymbol{v}$ and pressure p will be searched in the functional spaces $\boldsymbol{X} := H^1(\Omega)^n$ and $Q := L^2(\Omega)$, respectively.

Using the results of [25, Sect. III.3] for the equation $\nabla \cdot \boldsymbol{v} = f$ one can show:

Lemma 3.1 *Given $f \in Q$ and $\boldsymbol{g} \in \boldsymbol{T}_D$, there exists $\boldsymbol{v} \in \boldsymbol{X}$ satisfying*

$$\begin{cases} \nabla \cdot \boldsymbol{v} = f & \text{in } \Omega \\ \boldsymbol{v} = \boldsymbol{g} & \text{on } \Gamma_D \end{cases} \tag{3.1}$$

and the estimate ($\kappa_b = \kappa_b(\Omega) > 0$)

$$\|\boldsymbol{v}\|_{\boldsymbol{X}} \leq \kappa_b \left(\|f\|_Q + \|\boldsymbol{g}\|_{\boldsymbol{T}_D}\right). \tag{3.2}$$

Let

$$b : \boldsymbol{X} \times Q \to \mathbb{R}, \quad b(\boldsymbol{v}, q) = -\int_\Omega (\nabla \cdot \boldsymbol{v}) q \, \mathrm{d}x \tag{3.3}$$

and consider the subspaces of $\boldsymbol{X}$

$$\boldsymbol{U} := \Big\{\boldsymbol{u} \in H^1(\Omega)^n : \boldsymbol{u}|_{\Gamma_D} = \boldsymbol{0}\Big\},$$

$$\boldsymbol{V} := \Big\{\boldsymbol{u} \in \boldsymbol{U} : b(\boldsymbol{u}, \phi) = 0, \forall \phi \in Q\Big\}.$$

It is clear that $|\boldsymbol{u}|_{\boldsymbol{U}} := \|\nabla \boldsymbol{u}\|_{2,\Omega}$ defines a seminorm in $\boldsymbol{X}$ and is a norm in $\boldsymbol{U}$. By Poincaré inequality, there exists a constant $C_P > 0$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{X}} \leq \sqrt{1 + C_P^2}|\boldsymbol{u}|_{\boldsymbol{U}} =: \lambda_P |\boldsymbol{u}|_{\boldsymbol{U}}, \quad \forall \boldsymbol{u} \in \boldsymbol{U}. \tag{3.4}$$

The decomposition $\boldsymbol{U} = \boldsymbol{V} \oplus \boldsymbol{V}^{\perp}$, where $\boldsymbol{V}^{\perp}$ denotes the orthogonal of $\boldsymbol{V}$ in $\boldsymbol{U}$ with respect to the inner product $(\cdot, \cdot)_{\boldsymbol{U}} := \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \mathrm{d}x$, is valid. The annihilator of $\boldsymbol{V}$ in $\boldsymbol{U}'$ can be identified isometrically with $(\boldsymbol{V}^{\perp})'$ (see, for example the proof of [26, Corollary 2.4, p. 24]):

$$\boldsymbol{V}^{\circ} := \left\{ \boldsymbol{F} \in \boldsymbol{U}' : \langle \boldsymbol{F}, \boldsymbol{u} \rangle_{\boldsymbol{U}', \boldsymbol{U}} = 0, \ \forall \boldsymbol{u} \in \boldsymbol{V} \right\} \cong (\boldsymbol{V}^{\perp})'.$$

Analogously,

$$(\boldsymbol{V}^{\perp})^{\circ} := \left\{ \boldsymbol{F} \in \boldsymbol{U}' : \langle \boldsymbol{F}, \boldsymbol{v} \rangle_{\boldsymbol{U}', \boldsymbol{U}} = 0, \ \forall \boldsymbol{v} \in \boldsymbol{V}^{\perp} \right\} \cong \boldsymbol{V}'.$$

Then $\boldsymbol{U}' \cong (\boldsymbol{V}^{\perp})^{\circ} \oplus \boldsymbol{V}^{\circ} \cong \boldsymbol{V}' \oplus (\boldsymbol{V}^{\perp})'$.

Consider the operator

$$B : \boldsymbol{U} \to Q, \quad (B\boldsymbol{v}, q)_Q = b(\boldsymbol{v}, q) = -\int_\Omega (\nabla \cdot \boldsymbol{v}) q \mathrm{d}x.$$

From Lemma 3.1 with $\boldsymbol{g} = \boldsymbol{0}$, it follows:

Lemma 3.2 (i) *the operator B is an isomorphism from $\boldsymbol{V}^{\perp}$ to Q and*

$$|\boldsymbol{v}|_{\boldsymbol{U}} \leq \kappa_b \|B\boldsymbol{v}\|_Q, \quad \forall \boldsymbol{v} \in \boldsymbol{V}^{\perp};$$

(ii) *the operator B^* is an isomorphism from Q to $\boldsymbol{V}^{\circ}$ and*

$$\|q\|_Q \leq \kappa_b \|B^* q\|_{\boldsymbol{U}'}, \quad \forall q \in Q.$$

4 Well-Posedness of the State Equations

Let

$$a : \boldsymbol{X} \times \boldsymbol{U} \to \mathbb{R}, \quad a(\boldsymbol{u}, \boldsymbol{v}) = \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \mathrm{d}x.$$

Given $\boldsymbol{g} \in \boldsymbol{T}_D$, the mixed weak formulation of the Stokes reference flow (2.4) is: find $(\boldsymbol{v}_r, p_r) \in \boldsymbol{X} \times Q$ such that

$$\begin{cases} a(\boldsymbol{v}_r, \boldsymbol{\varphi}) + b(p_r, \boldsymbol{\varphi}) - b(\boldsymbol{v}_r, \phi) = 0, \ \forall (\boldsymbol{\varphi}, \phi) \in \boldsymbol{U} \times Q \\ \gamma_D \boldsymbol{v}_r = \boldsymbol{g}. \end{cases} \tag{4.1}$$

In the last equation and in what follows, $\gamma_D : H^1(\Omega) \to H^{1/2}(\Gamma_D)$ is the trace operator.

Lemma 4.1 *The weak formulation* (4.1) *has a unique solution and the following estimates hold*

$$\|\boldsymbol{v}_r\|_{\boldsymbol{X}} \le 2\kappa \|\boldsymbol{g}\|_{\boldsymbol{T}_D}, \tag{4.2}$$

$$\|p_r\|_Q \le 2\nu\kappa^2 \|\boldsymbol{g}\|_{\boldsymbol{T}_D}. \tag{4.3}$$

where $\kappa := \kappa_b \lambda_P$, κ_b *being the constant from Lemmas* 3.1 *and* 3.2 *and* λ_P *the constant defined in* (3.4).

Proof Let $\boldsymbol{g} \in \boldsymbol{T}_D$. From Lemma 3.1, there exists $\boldsymbol{G} \in \boldsymbol{X}$ such that

$$\|\boldsymbol{G}\|_X \le \kappa_b \|\boldsymbol{g}\|_{T_D}$$

where κ_b is a positive constant. The Stokes velocity is given by $\boldsymbol{v}_r = \boldsymbol{G} + \boldsymbol{z}$, where $\boldsymbol{z} \in \boldsymbol{U}$ satisfies

$$a(\boldsymbol{z}, \boldsymbol{\varphi}) = -a(\boldsymbol{G}, \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}$$

and the estimate

$$\|\nabla \boldsymbol{z}\|_2 \le \|\nabla \boldsymbol{G}\|_2 \le \|\boldsymbol{G}\|_X \le \kappa_b \|\boldsymbol{g}\|_{\boldsymbol{T}_D}. \tag{4.4}$$

From (4.4) and the fact that $\lambda_P \ge 1$, we obtain

$$\|\boldsymbol{v}_r\|_X \le \|\boldsymbol{G}\|_X + \lambda_p \|\nabla \boldsymbol{z}\|_2 \le 2\lambda_P \kappa_b \|\boldsymbol{g}\|_{\boldsymbol{T}_D}$$

To bound the pressure norm $\|p_r\|_Q$, we use inf-sup continuity, which yields

$$\|p_r\|_Q \le \kappa_b \nu \|\nabla \boldsymbol{v}_r\|_2 \le \nu 2\kappa_b^2 \|\boldsymbol{g}\|_{\boldsymbol{T}_D}.$$

□

Assuming the reference flow $(\boldsymbol{v}_r, p_r)$ is available from (4.1), we now consider the Navier–Stokes system. The notation $\mathbf{H}^{-1}(\Omega)$ will be used for the dual space $H^{-1}(\Omega)^n$, where $H^{-1}(\Omega) = H_0^1(\Omega)'$, and the corresponding norm will be denoted by $\|\cdot\|_{-1,\Omega}$.

Lemma 4.2 *Let* $H^{-1/2}(\Gamma_N) := H_{00}^{1/2}(\Gamma_N)'$ *and* $U = \{u \in H^1(\Omega) : u|_{\Gamma_D} = 0\}$. *Then*

$$U' \cong H^{-1}(\Omega) \oplus H^{-1/2}(\Gamma_N).$$

Proof Since $H_0^1(\Omega)$ is a closed subspace of U, the space U admits the orthogonal decomposition $U = H_0^1(\Omega) \oplus \left(H_0^1(\Omega)\right)^{\perp}$ (with respect to the inner product defined in

U). The trace operator $\gamma_N : U \to H_{00}^{1/2}(\Gamma_N)$ and the lifting operator $\ell : H_{00}^{1/2}(\Gamma_N) \to U$ satisfy

$$\mathrm{Ker}(\gamma_N) = H_0^1(\Omega), \quad \mathrm{Im}(\ell) = \left(H_0^1(\Omega)\right)^\perp, \quad (\gamma_N \circ \ell)(b) = b, \; b \in H_{00}^{1/2}(\Gamma_N).$$

The restriction $\gamma_N|_{\left(H_0^1(\Omega)\right)^\perp} =: \mathcal{I}$ is linear bijective between $\left(H_0^1(\Omega)\right)^\perp$ and $H_{00}^{1/2}(\Gamma_N)$. By the properties of the trace and lifting operators, $\mathcal{I} : H_0^1(\Omega)^\perp \to H_{00}^{1/2}(\Gamma_N)$ and $\mathcal{I}^{-1} : H_{00}^{1/2}(\Gamma_N) \to H_0^1(\Omega)^\perp$ are continuous:

$$\begin{aligned} \|\mathcal{I}u\|_{1/2,2,\Gamma_N} &= \|\gamma_N u\|_{1/2,2,\Gamma_N} \le C(\Omega)\|u\|_{1,2,\Omega}, \quad u \in \left(H_0^1(\Omega)\right)^\perp. \\ \|\mathcal{I}^{-1}b\|_{1,2,\Omega} &= \|\ell b\|_{1,2,\Omega} \le C(\Gamma_N)\|b\|_{1/2,2,\Gamma_N}, \quad b \in H_{00}^{1/2}(\Gamma_N). \end{aligned}$$

Hence $\left(H_0^1(\Omega)\right)^\perp$ and $H_{00}^{1/2}(\Gamma_N)$ are isomorphic, $\left(H_0^1(\Omega)\right)^\perp \cong H_{00}^{1/2}(\Gamma_N)$, and therefore

$$U \cong H_0^1(\Omega) \oplus H_{00}^{1/2}(\Gamma_N).$$

Let $f \in H^{-1}(\Omega)$ and $g \in H^{-1/2}(\Gamma_N)$. Then the pair (f, g) induces an element $F \in U'$ via

$$\langle F, u\rangle_{U',U} = \langle f, Pu\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle g, \gamma_N u\rangle_{H^{-1/2}(\Gamma_N),H_{00}^{1/2}(\Gamma_N)} \quad (u \in U)$$

where P is the orthogonal projection $P : U \to H_0^1(\Omega)$. The mapping $(f, g) \mapsto F$ is linear, bounded and injective. To conclude that it is an isomorphism between $H^{-1}(\Omega) \oplus H^{-1/2}(\Gamma_N)$ and U', it remains to show surjectivity. Let $F \in U'$ then, for any $u \in U$, we have

$$\begin{aligned} \langle F, u\rangle_{U',U} &= \langle F, Pu\rangle_{U',U} + \langle F, (I_U - P)u\rangle_{U',U} \\ &= \left\langle F|_{H_0^1(\Omega)}, Pu\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \left\langle F|_{\left(H_0^1(\Omega)\right)^\perp} \circ \mathcal{I}^{-1}, \mathcal{I} \circ (I_U - P)u\right\rangle_{H^{-1/2}(\Gamma_N),H_{00}^{1/2}(\Gamma_N)} \\ &= \left\langle F|_{H_0^1(\Omega)}, Pu\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \left\langle F|_{\left(H_0^1(\Omega)\right)^\perp} \circ \mathcal{I}^{-1}, \gamma_N u\right\rangle_{H^{-1/2}(\Gamma_N),H_{00}^{1/2}(\Gamma_N)}. \end{aligned}$$

□

Given that $\boldsymbol{U}'$ can be identified with $\mathbf{H}^{-1}(\Omega) \oplus \mathbf{H}^{-\frac{1}{2}}(\Gamma_{\mathrm{N}})$, the force term in (2.5) will be selected to preserve the do-nothing boundary condition, that is, $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$. Consequently, there exits $\boldsymbol{f}^0$, $\boldsymbol{f}^1$, ..., $\boldsymbol{f}^n \in L^2(\Omega)^n$ such that

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\boldsymbol{U}',\boldsymbol{U}} = \int_\Omega \boldsymbol{f}^0 \cdot \boldsymbol{v} \mathrm{d}x + \sum_{i=1}^n \int_\Omega \boldsymbol{f}^i \cdot \frac{\partial \boldsymbol{v}}{\partial x_i} \mathrm{d}x.$$

Concerning the Dirichlet data, we will take $\boldsymbol{g} \in \boldsymbol{T}_D$.

In order to write the weak formulation in a compact form, we define

$$c : \boldsymbol{X} \times \boldsymbol{X} \times \boldsymbol{U} \to \mathbb{R}, \quad c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}) = \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{z} \mathrm{d}x$$

which satisfies the estimate

$$|c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z})| \le \|\boldsymbol{u}\|_{4,\Omega} \|\nabla \boldsymbol{v}\|_{2,\Omega} \|\boldsymbol{z}\|_{4,\Omega} \le C_c \|\boldsymbol{u}\|_{\boldsymbol{X}} \|\nabla \boldsymbol{v}\|_{2,\Omega} |\boldsymbol{z}|_{\boldsymbol{U}}. \tag{4.5}$$

In (4.5) we used $\|\boldsymbol{\varphi}\|_{4,\Omega} \le C_S \|\boldsymbol{\varphi}\|_{1,2,\Omega}$, where C_S is a constant associated with Sobolev embedding, and set $C_c := C_S^2 \sqrt{1 + C_P^2} = C_S^2 \lambda_P$.

We also introduce a new nonlinear operator associated with the RDDN condition:

$$d_\delta : \boldsymbol{X} \times \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{R}, \quad d_\delta(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}) = \frac{1}{2} \int_{\Gamma_N} \Psi_\delta(\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{v} \cdot \boldsymbol{z}) \mathrm{d}\sigma.$$

From (2.8) and (3.4) we get

$$\begin{aligned}
|d_\delta(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z})| &\le \frac{1}{2} \|C\delta + [\boldsymbol{u} \cdot \boldsymbol{n}]^- \|_{4,\Gamma_N} \|\boldsymbol{v}\|_{2,\Gamma_N} \|\boldsymbol{z}\|_{4,\Gamma_N} \\
&\le \frac{1}{2} \left(\tilde{c}\delta |\Gamma_N|^{1/4} + \|\boldsymbol{u}\|_{4,\Gamma_N} \right) \|\boldsymbol{v}\|_{2,\Gamma_N} \|\boldsymbol{z}\|_{4,\Gamma_N} \\
&\le \tilde{C}_S \left(C_{\Gamma_N} \delta + \|\boldsymbol{u}\|_{\boldsymbol{X}} \right) \|\boldsymbol{v}\|_{\boldsymbol{X}} \|\boldsymbol{z}\|_{\boldsymbol{X}}. \\
&\le C_d^0 \left(\delta + \|\boldsymbol{u}\|_{\boldsymbol{X}} \right) |\boldsymbol{v}|_{\boldsymbol{U}} |\boldsymbol{z}|_{\boldsymbol{U}}.
\end{aligned}$$

For later purposes, recalling (2.9), we set $C_d := \max\{C_d^0, C_d^0 M_{\Psi_\delta'}\}$, so that

$$|d_\delta(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z})| \le C_d(\delta + \|\boldsymbol{u}\|_{\boldsymbol{X}}) |\boldsymbol{v}|_{\boldsymbol{U}} |\boldsymbol{z}|_{\boldsymbol{U}} \tag{4.6}$$

and, for $\boldsymbol{w} \in \boldsymbol{X}$,

$$\left| \frac{1}{2} \int_{\Gamma_N} (\boldsymbol{w} \cdot \boldsymbol{n}) \Psi_\delta'(\boldsymbol{u} \cdot \boldsymbol{n}) \boldsymbol{v} \cdot \boldsymbol{z} \, \mathrm{d}\sigma \right| \le C_d \|\boldsymbol{w}\|_{\boldsymbol{X}} \|\boldsymbol{u}\|_{\boldsymbol{X}} |\boldsymbol{v}|_{\boldsymbol{U}} |\boldsymbol{z}|_{\boldsymbol{U}} \tag{4.7}$$

Note that the operator c is trilinear, but d_δ is not linear in the first argument. When $\delta \to 0$, we recover the operator $d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}) := \frac{1}{2} \int_{\Gamma_N} [\boldsymbol{u} \cdot \boldsymbol{n}]^- (\boldsymbol{v} \cdot \boldsymbol{z}) \mathrm{d}\sigma$ and the associated estimates for the DDN boundary condition.

Based on the notation introduced above, the weak formulation of problem (2.5) takes the form: find $(\boldsymbol{v}, p) \in \boldsymbol{X} \times Q$ such that

$$\begin{cases}
a(\boldsymbol{v}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, p) - b(\boldsymbol{v}, q) + c(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\varphi}) \\
\qquad + d_\delta(\boldsymbol{v}, \boldsymbol{v} - \boldsymbol{v}_r, \boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{\boldsymbol{U}', \boldsymbol{U}}, \ \forall (\boldsymbol{\varphi}, q) \in \boldsymbol{U} \times Q \\
\gamma_D \boldsymbol{v} = \boldsymbol{g}.
\end{cases} \tag{4.8}$$

The next lemma, which has used in a previous analysis of the state equations (see [24]), will also be important in several stages of the study of the optimal control problem. Recall that $C_d := \max\{C_d^0, C_d^0 M_{\Psi'_\delta}\}$ was already used in (4.6).

Lemma 4.3 *For $c : \boldsymbol{X} \times \boldsymbol{X} \times \boldsymbol{U} \to \mathbb{R}$ and $d_\delta : \boldsymbol{X} \times \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{R}$ defined above, it holds:*

1. *for $\boldsymbol{v}^{(i)} \in \boldsymbol{X}$, $\boldsymbol{u}^{(i)} \in \boldsymbol{U}$, $i = 1, 2$, and $\boldsymbol{\varphi} \in \boldsymbol{U}$,*

$$\begin{aligned} &c(\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(1)}, \boldsymbol{\varphi}) - c(\boldsymbol{v}^{(2)}, \boldsymbol{v}^{(2)}, \boldsymbol{\varphi}) \\ &\quad + d_\delta(\boldsymbol{v}^{(1)}, \boldsymbol{u}^{(1)}, \boldsymbol{\varphi}) - d_\delta(\boldsymbol{v}^{(2)}, \boldsymbol{u}^{(2)}, \boldsymbol{\varphi}) \\ &= c(\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}, \boldsymbol{v}^{(1)}, \boldsymbol{\varphi}) + c(\boldsymbol{v}^{(2)}, \boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}, \boldsymbol{\varphi}) \\ &\quad + d_\delta(\boldsymbol{v}^{(2)}, \boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}, \boldsymbol{\varphi}) + d_\delta(\boldsymbol{v}^{(1)}, \boldsymbol{u}^{(1)}, \boldsymbol{\varphi}) - d_\delta(\boldsymbol{v}^{(2)}, \boldsymbol{u}^{(1)}, \boldsymbol{\varphi}), \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} &|d_\delta(\boldsymbol{v}^{(1)}, \boldsymbol{u}^{(1)}, \boldsymbol{\varphi}) - d_\delta(\boldsymbol{v}^{(2)}, \boldsymbol{u}^{(1)}, \boldsymbol{\varphi})| \\ &\le \frac{M_{\Psi'_\delta}}{2} \|\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}\|_{4,\Gamma_N} \|\boldsymbol{u}^{(1)}\|_{2,\Gamma_N} \|\boldsymbol{\varphi}\|_{4,\Gamma_N} \\ &\le C_d \left\| \boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)} \right\|_{\boldsymbol{X}} |\boldsymbol{u}^{(1)}|_{\boldsymbol{U}} |\boldsymbol{\varphi}|_{\boldsymbol{U}}, \end{aligned} \tag{4.10}$$

2. *if $\boldsymbol{v} \in \boldsymbol{X}$, $\boldsymbol{z} \in \boldsymbol{V}$ then*

$$c(\boldsymbol{v}, \boldsymbol{z}, \boldsymbol{z}) + d_\delta(\boldsymbol{v}, \boldsymbol{z}, \boldsymbol{z}) \ge \frac{1}{2} \int_{\Gamma_N} [\boldsymbol{v} \cdot \boldsymbol{n}]^+ |\boldsymbol{z}|^2 \mathrm{d}\sigma, \tag{4.11}$$

3. *for $\boldsymbol{v}^{(i)} \in \boldsymbol{X}$ with $\nabla \cdot \boldsymbol{v}^{(i)} \equiv 0$, $\boldsymbol{u}^{(i)} \in \boldsymbol{U}$, $\boldsymbol{r}^{(i)} := \boldsymbol{v}^{(i)} - \boldsymbol{u}^{(i)}$, $i = 1, 2$, and $\boldsymbol{z} := \boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}$ we have*

$$\begin{aligned} &c(\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(1)}, \boldsymbol{z}) - c(\boldsymbol{v}^{(2)}, \boldsymbol{v}^{(2)}, \boldsymbol{z}) \\ &\quad + d_\delta(\boldsymbol{v}^{(1)}, \boldsymbol{u}^{(1)}, \boldsymbol{z}) - d_\delta(\boldsymbol{v}^{(2)}, \boldsymbol{u}^{(2)}, \boldsymbol{z}) \\ &\ge c(\boldsymbol{z}, \boldsymbol{v}^{(1)}, \boldsymbol{z}) + c(\boldsymbol{r}^{(1)} - \boldsymbol{r}^{(2)}, \boldsymbol{v}^{(1)}, \boldsymbol{z}) \\ &\quad + c(\boldsymbol{v}^{(2)}, \boldsymbol{r}^{(1)} - \boldsymbol{r}^{(2)}, \boldsymbol{z}) + d_\delta(\boldsymbol{v}^{(1)}, \boldsymbol{u}^{(1)}, \boldsymbol{z}) - d_\delta(\boldsymbol{v}^{(2)}, \boldsymbol{u}^{(1)}, \boldsymbol{z}). \end{aligned} \tag{4.12}$$

The well-posedness of the Navier–Stokes equations with RDDN condition was proved in [24] using the above properties of c and d_δ.

Theorem 4.4 *Let $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{T}_D$. If $\|\boldsymbol{g}\|_{\boldsymbol{T}_D} < \frac{\nu}{2\kappa C_c}$ then problem (4.8) has at least a solution. Given $\varepsilon \in (0, 1)$, if $\|\boldsymbol{g}\|_{\boldsymbol{T}_D} \le \frac{(1-\varepsilon)\nu}{2\kappa C_c}$ then the following estimates hold:*

$$\|\boldsymbol{v}\|_{\boldsymbol{X}} \leq 2\kappa \|\boldsymbol{g}\|_{\boldsymbol{T}_D} + \left(\frac{1}{\varepsilon} - 1\right) \kappa \lambda_P \|\boldsymbol{g}\|_{\boldsymbol{T}_D} + \frac{\lambda_P}{\varepsilon \nu} \|\boldsymbol{f}\|_{-1,\Omega},$$

$$\|p\|_Q \leq \kappa \tilde{c} \delta C_d \left(\frac{1-\varepsilon}{\varepsilon} 2\kappa \|\boldsymbol{g}\|_{\boldsymbol{T}_D} + \frac{1}{\varepsilon \nu} \|\boldsymbol{f}\|_{-1,\Omega}\right) + \kappa \|\boldsymbol{f}\|_{-1,\Omega} + \kappa C_c \|\boldsymbol{v}\|_{\boldsymbol{X}}^2$$
$$+ \nu\kappa \|\boldsymbol{v}\|_{\boldsymbol{X}} + \kappa C_d \left(\frac{1-\varepsilon}{\varepsilon} 2\kappa \|\boldsymbol{g}\|_{\boldsymbol{T}_D} + \frac{1}{\varepsilon \nu} \|\boldsymbol{f}\|_{-1,\Omega}\right) \|\boldsymbol{v}\|_{\boldsymbol{X}}.$$

Moreover, if, for the given $\varepsilon \in (0,1)$, the data satisfy

$$\|\boldsymbol{g}\|_{\boldsymbol{T}_D} + \frac{(C_c + C_d)}{2\nu\kappa \left[C_c + C_d(1-\varepsilon)\right]} \|\boldsymbol{f}\|_{-1,\Omega} < \frac{\varepsilon \nu}{2\kappa \lambda_P \left[C_c + C_d(1-\varepsilon)\right]}$$

then the solution $(\boldsymbol{v}, p)$ of (4.8) is unique in the class of all weak solutions corresponding to the same data $\boldsymbol{g}$ and $\boldsymbol{f}$.

Remark 4.5 In the proof of Theorem 4.4, the velocity is obtained in the form $\boldsymbol{v} = \boldsymbol{v}_r + \boldsymbol{u}$. The estimate obtained for $\boldsymbol{u} \in \boldsymbol{V}$ is:

$$|\boldsymbol{u}|_{\boldsymbol{U}} \leq \left(\frac{1}{\varepsilon} - 1\right) 2\kappa \|\boldsymbol{g}\|_{\boldsymbol{T}_D} + \frac{1}{\varepsilon \nu} \|\boldsymbol{f}\|_{-1,\Omega}.$$

5 Formulation of the Control Problem. Existence of an Optimal Solution

We will assume that $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$ is fixed, and that, together with $\boldsymbol{g} \in \boldsymbol{T}_D$, it satisfies

$$\begin{cases} \|\boldsymbol{g}\|_{\boldsymbol{T}_D} \leq \eta_{1,\varepsilon}, \quad \eta_{1,\varepsilon} := \dfrac{(1-\varepsilon)\nu}{2\kappa C_c}, \quad \varepsilon \in (0,1) \\ \dfrac{(C_c + C_d)\|\boldsymbol{f}\|_{-1,\Omega}}{2\kappa \nu \left[C_c + C_d(1-\varepsilon)\right]} + \|\boldsymbol{g}\|_{\boldsymbol{T}_D} < \eta_{2,\varepsilon} := \dfrac{\varepsilon \nu}{2\kappa \lambda_P [C_c + C_d(1-\varepsilon)]}, \end{cases} \tag{5.1}$$

which, by Theorem 4.4, is sufficient for the existence and uniqueness of a solution $(\boldsymbol{v}, p)$ of (4.8). Motivated by the restrictions (5.1), we take

$$0 < \widehat{\eta} < \min\left\{\eta_{1,\varepsilon}, \eta_{2,\varepsilon} - \frac{\left(C_c + C_d\right)\|\boldsymbol{f}\|_{-1,\Omega}}{2\nu\kappa \left[C_c + C_d(1-\varepsilon)\right]}\right\} \tag{5.2}$$

and define the set of admissible boundary controls as

$$\mathcal{U}_{ad} = \left\{\boldsymbol{g} \in \boldsymbol{T}_D : \|\boldsymbol{g}\|_{\boldsymbol{T}_D} \leq \widehat{\eta}\right\}. \tag{5.3}$$

Now, we introduce the reference Stokes operator $\mathcal{S}_r : \mathcal{U}_{ad} \to \boldsymbol{X} \times Q$, defined by $\mathcal{S}_r(\boldsymbol{g}) = (\boldsymbol{v}_r, p_r)$, and, since, for the next results, the velocity component $\boldsymbol{v}_r$ is the most relevant part of the solution, we also define the velocity operator $\mathcal{S}_{r,v} : \mathcal{U}_{ad} \to \boldsymbol{X}$ by $\mathcal{S}_{r,v}(\boldsymbol{g}) = \boldsymbol{v}_r$. Analogously, for the Navier-Stokes problem, the operator $\mathcal{S} : \mathcal{U}_{ad} \to$

$\boldsymbol{X} \times Q$ is defined by $\mathcal{S}(\boldsymbol{g}) = (\boldsymbol{v}, p)$, where $(\boldsymbol{v}, p)$ solves (4.8), and the mapping $\mathcal{S}_v : \mathcal{U}_{ad} \to \boldsymbol{X}$ gives the velocity component of the solution to the Navier–Stokes problem.

In this framework, the minimization problem consists in finding $\widehat{\boldsymbol{g}} \in \mathcal{U}_{ad}$ such that

$$\mathcal{J}(\boldsymbol{v}(\widehat{\boldsymbol{g}}), \widehat{\boldsymbol{g}}) = \min_{\substack{\boldsymbol{g} \in \mathcal{U}_{ad}, \\ \boldsymbol{v}(\boldsymbol{g}) = \mathcal{S}_v(\boldsymbol{g})}} \mathcal{J}(\boldsymbol{v}(\boldsymbol{g}), \boldsymbol{g}) \tag{5.4}$$

where the cost functional $\mathcal{J}$ is of target velocity form

$$\mathcal{J}(\boldsymbol{v}, \boldsymbol{g}) = \frac{1}{2}\|\boldsymbol{v} - \boldsymbol{v}_\Omega\|_{2,\Omega}^2 + \frac{\tau}{2}\|\boldsymbol{g}\|_{\boldsymbol{T}_D}^2, \tag{5.5}$$

for a small parameter $\tau \geq 0$, and a given velocity field $\boldsymbol{v}_\Omega \in L^2(\Omega)^n$.

For what follows, it is convenient to introduce additional notations. If, for $\boldsymbol{g} \in \mathcal{U}_{ad}$, we define

$$\beta_r(\|\boldsymbol{g}\|_{\boldsymbol{T}_D}) := 2\kappa\|\boldsymbol{g}\|_{\boldsymbol{T}_D},\ \beta_u(\|\boldsymbol{g}\|_{\boldsymbol{T}_D}) := \left(\frac{1}{\varepsilon} - 1\right) 2\kappa\|\boldsymbol{g}\|_{\boldsymbol{T}_D} + \frac{1}{\varepsilon\nu}\|\boldsymbol{f}\|_{-1,\Omega}, \tag{5.6}$$

$$\beta_v(\|\boldsymbol{g}\|_{\boldsymbol{T}_D}) := 2\kappa\|\boldsymbol{g}\|_{\boldsymbol{T}_D} + \left(\frac{1}{\varepsilon} - 1\right)\kappa\lambda_P\|\boldsymbol{g}\|_{\boldsymbol{T}_D} + \frac{\lambda_P}{\varepsilon\nu}\|\boldsymbol{f}\|_{-1,\Omega}, \tag{5.7}$$

$$\begin{aligned}\beta_p(\|\boldsymbol{g}\|_{\boldsymbol{T}_D}) := {} & \delta\kappa\tilde{c}C_d\left(\frac{1-\varepsilon}{\varepsilon}2\kappa\|\boldsymbol{g}\|_{\boldsymbol{T}_D} + \frac{1}{\varepsilon\nu}\|\boldsymbol{f}\|_{-1,\Omega}\right) + \kappa\|\boldsymbol{f}\|_{-1,\Omega} \\ & + \kappa C_c\beta_v^2 + \nu\kappa\beta_v + \kappa C_d\left(\frac{1-\varepsilon}{\varepsilon}2\kappa\|\boldsymbol{g}\|_{\boldsymbol{T}_D} + \frac{1}{\varepsilon\nu}\|\boldsymbol{f}\|_{-1,\Omega}\right)\beta_v,\end{aligned} \tag{5.8}$$

then, from Theorem 4.4 and Remark 4.5, the estimates for the forward problem can be written as

$$|\boldsymbol{u}|_{\boldsymbol{U}} \leq \beta_u(\|\boldsymbol{g}\|_{\boldsymbol{T}_D}), \tag{5.9}$$

$$\|\boldsymbol{v}_r\|_{\boldsymbol{X}} \leq \beta_r(\|\boldsymbol{g}\|_{\boldsymbol{T}_D}), \quad \|\boldsymbol{v}\|_{\boldsymbol{X}} \leq \beta_v(\|\boldsymbol{g}\|_{\boldsymbol{T}_D}), \quad \|p\|_Q \leq \beta_p(\|\boldsymbol{g}\|_{\boldsymbol{T}_D}). \tag{5.10}$$

Moreover, let

$$\alpha(\|\boldsymbol{g}\|_{\boldsymbol{T}_D}) := \nu - \frac{2\kappa\lambda_P[C_c + C_d(1-\varepsilon)]\|\boldsymbol{g}\|_{\boldsymbol{T}_D}}{\varepsilon} - \frac{\lambda_P\big(C_c + C_d\big)\|\boldsymbol{f}\|_{-1,\Omega}}{\nu\varepsilon}$$

and observe that the second condition in (5.1), a restriction on the data that was used in Theorem 4.4 to establish uniqueness of solutions for problem (4.8), is equivalent to

$$\alpha(\|\boldsymbol{g}\|_{\boldsymbol{T}_D}) > 0. \tag{5.11}$$

The first main result for the control problem is the following.

Theorem 5.1 *Under the assumption* (5.2) *there exists* $\widehat{\boldsymbol{g}} \in \mathcal{U}_{ad}$ *such that* $\widehat{\boldsymbol{g}}$ *and the associated state* $\widehat{\boldsymbol{v}} = \mathcal{S}_v(\widehat{\boldsymbol{g}})$ *are a solution of the minimization problem* (5.4) *with the cost functional* (5.5).

Proof Let

$$\mathcal{E} := \left\{(\boldsymbol{g}, \boldsymbol{v}_r, \boldsymbol{v}, p) \in \mathcal{U}_{ad} \times \boldsymbol{X} \times \boldsymbol{X} \times Q;\ \boldsymbol{v}_r = \mathcal{S}_{r,v}(\boldsymbol{g}),\ (\boldsymbol{v}, p) = \mathcal{S}(\boldsymbol{g})\right\}.$$

If $\boldsymbol{g} \in \mathcal{U}_{ad}$ then Lemma 4.1 and Theorem 4.4 yield existence and uniqueness of a state $(\mathcal{S}_r(\boldsymbol{g}), \mathcal{S}(\boldsymbol{g}))$ for (4.1)–(4.8). Hence, $\mathcal{E} \neq \emptyset$.

In what follows, in order to simplify the notation, we will write $\boldsymbol{R} = \boldsymbol{v}_r$.

Since $J := \inf_{(\boldsymbol{g},\boldsymbol{R},\boldsymbol{v},p)\in\mathcal{E}} \mathcal{J}(\boldsymbol{v}, \boldsymbol{g}) \geq 0$, there exists a minimizing sequence $\left\{(\boldsymbol{g}_k, \boldsymbol{R}_k, \boldsymbol{v}_k, p_k)\right\}_{k\in\mathbb{N}} \subset \mathcal{E}$, such that $\mathcal{J}(\boldsymbol{v}_k, \boldsymbol{g}_k) \to J$ when $k \to \infty$.

The sequence $\{(\boldsymbol{g}_k, \boldsymbol{R}_k, \boldsymbol{v}_k, p_k)\}_{k\in\mathbb{N}}$ is bounded. Indeed, since $\{\boldsymbol{g}_k\}_{k\in\mathbb{N}} \subset \mathcal{U}_{ad}$, we have $\|\boldsymbol{g}_k\|_{\boldsymbol{T}_D} \leq \widehat{\eta}$, and from the estimates for the solutions obtained in Lemma 4.1 and Theorem 4.4 and (5.10), it follows that

$$\|\boldsymbol{R}_k\|_{\boldsymbol{X}} \leq \beta_r(\widehat{\eta}), \quad \|\boldsymbol{v}_k\|_{\boldsymbol{X}} \leq \beta_v(\widehat{\eta}), \quad \|p_k\|_Q \leq \beta_p(\widehat{\eta}), \quad \forall k \in \mathbb{N}. \tag{5.12}$$

We conclude that there exists a subsequence and $(\widehat{\boldsymbol{g}}, \widehat{\boldsymbol{R}}, \widehat{\boldsymbol{v}}, \widehat{p}) \in \mathcal{U}_{ad} \times \boldsymbol{X} \times \boldsymbol{X} \times Q$ such that

$$(\boldsymbol{g}_{k'}, \boldsymbol{R}_{k'}, \boldsymbol{v}_{k'}, p_{k'}) \rightharpoonup (\widehat{\boldsymbol{g}}, \widehat{\boldsymbol{R}}, \widehat{\boldsymbol{v}}, \widehat{p}) \text{ in } \mathcal{U}_{ad} \times \boldsymbol{X} \times \boldsymbol{X} \times Q, \text{ as } k' \to \infty.$$

The subsequence $\{(\boldsymbol{g}_{k'}, \boldsymbol{v}_{k'}, p_{k'})\}_{k'\in\mathbb{N}}$, which, for simplicity, we will continue to denote by $\{(\boldsymbol{g}_k, \boldsymbol{v}_k, p_k)\}_{k\in\mathbb{N}}$, satisfies

$$\begin{cases} a(\boldsymbol{v}_k, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, p_k) - b(\boldsymbol{v}_k, q) + c(\boldsymbol{v}_k, \boldsymbol{v}_k, \boldsymbol{\varphi}) \\ \qquad + d_\delta(\boldsymbol{v}_k, \boldsymbol{v}_k - \boldsymbol{R}_k, \boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\boldsymbol{U}',\boldsymbol{U}}, \ \forall(\boldsymbol{\varphi}, q) \in \boldsymbol{U} \times Q \\ \gamma_D \boldsymbol{v}_k = \boldsymbol{g}_k. \end{cases}$$

It is immediate to conclude that $a(\boldsymbol{v}_k, \boldsymbol{\varphi}) \to a(\widehat{\boldsymbol{v}}, \boldsymbol{\varphi})$, $b(\boldsymbol{\varphi}, p_k) \to b(\boldsymbol{\varphi}, \widehat{p})$, for all $\boldsymbol{\varphi} \in \boldsymbol{X}$, and $b(\boldsymbol{v}_k, \phi) \to b(\widehat{\boldsymbol{v}}, \phi)$, as $k \to \infty$, for all $\phi \in Q$.

From Lemma 4.3, (4.5) and (4.6), setting $\boldsymbol{u}_k := \boldsymbol{v}_k - \boldsymbol{R}_k$ and $\widehat{\boldsymbol{u}} := \widehat{\boldsymbol{v}} - \widehat{\boldsymbol{R}}$, we deduce

$$\begin{aligned} &|c(\boldsymbol{v}_k, \boldsymbol{v}_k, \boldsymbol{\varphi}) + d_\delta(\boldsymbol{v}_k, \boldsymbol{u}_k, \boldsymbol{\varphi}) - c(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{v}}, \boldsymbol{\varphi}) - d_\delta(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{u}}, \boldsymbol{\varphi})| \\ &\quad \leq |c(\boldsymbol{v}_k - \widehat{\boldsymbol{v}}, \boldsymbol{v}_k, \boldsymbol{\varphi})| + |c(\widehat{\boldsymbol{v}}, \boldsymbol{v}_k - \widehat{\boldsymbol{v}}, \boldsymbol{\varphi}) + |d_\delta(\boldsymbol{v}_k, \boldsymbol{u}_k - \widehat{\boldsymbol{u}}, \boldsymbol{\varphi})| \\ &\qquad + |d_\delta(\boldsymbol{v}_k, \widehat{\boldsymbol{u}}, \boldsymbol{\varphi}) - d_\delta(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{u}}, \boldsymbol{\varphi})| \\ &\quad \leq \|\boldsymbol{v}_k\|_{2,\Omega} \|\boldsymbol{\varphi}\|_{4,\Omega} \|\boldsymbol{v}_k - \widehat{\boldsymbol{v}}\|_{4,\Omega} + |c(\widehat{\boldsymbol{v}}, \boldsymbol{v}_k - \widehat{\boldsymbol{v}}, \boldsymbol{\varphi})| \\ &\qquad + \frac{1}{2} \|C\delta + [\boldsymbol{v}_k \cdot \boldsymbol{n}]^-\|_{4,\Gamma_N} \|\boldsymbol{u}_k - \widehat{\boldsymbol{u}}\|_{2,\Gamma_N} \|\boldsymbol{\varphi}\|_{4,\Gamma_N} \end{aligned}$$

$$+\frac{M_{\Psi'_\delta}}{2}\|\boldsymbol{v}_k-\widehat{\boldsymbol{v}}\|_{2,\Gamma_N}\|\widehat{\boldsymbol{u}}\|_{4,\Gamma_N}\|\boldsymbol{\varphi}\|_{4,\Gamma_N}\to 0,$$

where, to pass to the limit $k\to\infty$, we used the uniform estimates (5.12), the compact embedding $X \hookrightarrow\hookrightarrow L^4(\Omega)^n$, the compactness of the trace operator from $\boldsymbol{X}$ into $L^2(\Gamma_N)^n$, and the fact that, for fixed $\widehat{\boldsymbol{v}}$ and $\boldsymbol{\varphi}$, $c(\widehat{\boldsymbol{v}},\cdot,\boldsymbol{\varphi})\in\boldsymbol{U}'$.

Thus, the weak limit $(\widehat{\boldsymbol{g}},\widehat{\boldsymbol{v}_r},\widehat{\boldsymbol{v}},\widehat{p})$ is also in $\mathcal{E}$. Since the cost functional $\mathcal{J}$ is convex and continuous, we conclude that it is weakly lower semicontinuous. Therefore,

$$\mathcal{J}(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{g}})\le\liminf_{k\in\mathbb{N}}\mathcal{J}(\boldsymbol{v}_k,\boldsymbol{g}_k)=J,$$

and $(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{g}})$ is a solution to the minimization problem (5.4). □

6 Lipschitz Continuity of the Control-to-State Operator

In this section, the assumptions on the data and the admissible controls are the same of Sect. 5, in particular, $\boldsymbol{f}\in\boldsymbol{H}^{-1}(\Omega)$ is fixed.

For $i=1,2$, suppose $\boldsymbol{g}^{(i)}\in\mathcal{U}_{ad}$, and let $\boldsymbol{v}_r{}^{(i)}:=\mathcal{S}_{r,v}(\boldsymbol{g}^{(i)})$, $(\boldsymbol{v}^{(i)},p^{(i)})=\mathcal{S}(\boldsymbol{g}^{(i)})$. Then

$$\begin{cases} a(\boldsymbol{v}^{(1)}-\boldsymbol{v}^{(2)},\boldsymbol{\varphi})+b\big(\boldsymbol{\varphi},p^{(1)}-p^{(2)}\big)-b(\boldsymbol{v}^{(1)}-\boldsymbol{v}^{(2)},\phi)\\ \quad +c(\boldsymbol{v}^{(1)},\boldsymbol{v}^{(1)},\boldsymbol{\varphi})-c(\boldsymbol{v}^{(2)},\boldsymbol{v}^{(2)},\boldsymbol{\varphi})\\ \quad +d_\delta(\boldsymbol{v}^{(1)},\boldsymbol{v}^{(1)}-\boldsymbol{v}_r{}^{(1)},\boldsymbol{\varphi})-d_\delta(\boldsymbol{v}^{(2)},\boldsymbol{v}^{(2)}-\boldsymbol{v}_r{}^{(2)},\boldsymbol{\varphi})\\ =0,\ \forall(\boldsymbol{\varphi},\phi)\in\boldsymbol{U}\times Q,\ \gamma_D(\boldsymbol{v}^{(1)}-\boldsymbol{v}^{(2)})=\boldsymbol{g}^{(1)}-\boldsymbol{g}^{(2)}.\end{cases}$$

Firstly, we consider the velocity component and show that the mappings $\mathcal{S}_{r,v},\mathcal{S}_v:\mathcal{U}_{ad}\to\boldsymbol{X}$ are Lipschitz continuous. It is convenient to recall Remark 4.5: for $i=1,2$, we have $\boldsymbol{v}^{(i)}=\boldsymbol{u}^{(i)}+\boldsymbol{v}_r^{(i)}$, where $\boldsymbol{u}^{(i)}\in\boldsymbol{V}$.

Theorem 6.1 *For the velocity component of the reference flow there holds*

$$\|\mathcal{S}_{r,v}(\boldsymbol{g}^{(1)})-\mathcal{S}_{r,v}(\boldsymbol{g}^{(2)})\|_{\boldsymbol{X}}\le 2\kappa\|\boldsymbol{g}^{(1)}-\boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D}\tag{6.1}$$

and there exists a constant $L_v>0$, such that

$$\|\mathcal{S}_v(\boldsymbol{g}^{(1)})-\mathcal{S}_v(\boldsymbol{g}^{(2)})\|_{\boldsymbol{X}}\le L_v\|\boldsymbol{g}^{(1)}-\boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D}\tag{6.2}$$

for all $\boldsymbol{g}^{(1)},\boldsymbol{g}^{(2)}\in\mathcal{U}_{ad}$. The constant L_v is independent of $\boldsymbol{g}^{(1)}$ and $\boldsymbol{g}^{(2)}$.

Proof Estimate (6.1) is a simple consequence of the linearity of the Stokes problem.

Let $\boldsymbol{u}^{(i)}:=\mathcal{S}_v(\boldsymbol{g}^{(i)})-\mathcal{S}_{r,v}(\boldsymbol{g}^{(i)})=\boldsymbol{v}^{(i)}-\boldsymbol{v}_r^{(i)}$, $i=1,2$. Our aim is to derive estimates for $\|\boldsymbol{u}^{(1)}-\boldsymbol{u}^{(2)}\|_{\boldsymbol{X}}$ of the form

$$\|\boldsymbol{u}^{(1)}-\boldsymbol{u}^{(2)}\|_{\boldsymbol{X}}\le L_u\|\boldsymbol{g}^{(1)}-\boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D},\tag{6.3}$$

where the constant $L_u > 0$ is independent of $\boldsymbol{g}^{(1)}$ and $\boldsymbol{g}^{(2)}$, and then

$$\|\mathcal{S}_v(\boldsymbol{g}^{(1)}) - \mathcal{S}_v(\boldsymbol{g}^{(2)})\|_{\boldsymbol{X}} \le \|\boldsymbol{v}_r{}^{(1)} - \boldsymbol{v}_r{}^{(2)}\|_{\boldsymbol{X}} + \|\boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}\|_{\boldsymbol{X}} \le L_v\|\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D},$$

where $L_v = 2\kappa + L_u$. If $\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)} \in \boldsymbol{V}$ are related with the boundary data $\boldsymbol{g}^{(1)}$ and $\boldsymbol{g}^{(2)}$ via the definition in the beginning of the proof, then $\boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}$ satisfies

$$\begin{aligned} &a(\boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}, \boldsymbol{\varphi}) + c(\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(1)}, \boldsymbol{\varphi}) - c(\boldsymbol{v}^{(2)}, \boldsymbol{v}^{(2)}, \boldsymbol{\varphi}) \\ &\quad + d_\delta(\boldsymbol{v}^{(1)}, \boldsymbol{u}^{(1)}, \boldsymbol{\varphi}) - d_\delta(\boldsymbol{v}^{(2)}, \boldsymbol{u}^{(2)}, \boldsymbol{\varphi}) = 0, \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}. \end{aligned}$$

Choosing $\boldsymbol{\varphi} = \boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)} =: \boldsymbol{z}$ in the above equation, since $\nu|\boldsymbol{z}|_{\boldsymbol{U}}^2 = a(\boldsymbol{z}, \boldsymbol{z})$, we get

$$\nu|\boldsymbol{z}|_{\boldsymbol{U}}^2 = c(\boldsymbol{v}^{(2)}, \boldsymbol{v}^{(2)}, \boldsymbol{z}) - c(\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(1)}, \boldsymbol{z}) + d_\delta(\boldsymbol{v}^{(2)}, \boldsymbol{u}^{(2)}, \boldsymbol{z}) - d_\delta(\boldsymbol{v}^{(1)}, \boldsymbol{u}^{(1)}, \boldsymbol{z})$$

and from (4.12), it follows that

$$\begin{aligned} &c(\boldsymbol{v}^{(2)}, \boldsymbol{v}^{(2)}, \boldsymbol{z}) - c(\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(1)}, \boldsymbol{z}) + d_\delta(\boldsymbol{v}^{(2)}, \boldsymbol{u}^{(2)}, \boldsymbol{z}) - d_\delta(\boldsymbol{v}^{(1)}, \boldsymbol{u}^{(1)}, \boldsymbol{z}) \\ &\quad \le c(\boldsymbol{v}_r{}^{(2)} - \boldsymbol{v}_r{}^{(1)}, \boldsymbol{v}^{(1)}, \boldsymbol{z}) - c(\boldsymbol{z}, \boldsymbol{v}^{(1)}, \boldsymbol{z}) + c(\boldsymbol{v}^{(2)}, \boldsymbol{v}_r{}^{(2)} - \boldsymbol{v}_r{}^{(1)}, \boldsymbol{z}) \\ &\quad\quad + d_\delta(\boldsymbol{v}^{(2)}, \boldsymbol{u}^{(1)}, \boldsymbol{z}) - d_\delta(\boldsymbol{v}^{(1)}, \boldsymbol{u}^{(1)}, \boldsymbol{z}). \end{aligned}$$

According to (5.1)–(5.3), if $\|\boldsymbol{g}^{(i)}\|_{\boldsymbol{T}_D} \le \widehat{\eta}$, $i = 1, 2$, then the estimates

$$\|\boldsymbol{v}_r{}^{(i)}\|_{\boldsymbol{X}} \le \beta_r(\widehat{\eta}), \quad \|\boldsymbol{v}^{(i)}\|_{\boldsymbol{X}} \le \beta_v(\widehat{\eta}), \quad |\boldsymbol{u}^{(i)}|_{\boldsymbol{U}} \le \beta_u(\widehat{\eta})$$

are valid. Thus

$$|-c(\boldsymbol{z}, \boldsymbol{v}^{(1)}, \boldsymbol{z})| \le C_c|\boldsymbol{v}^{(1)}|_{\boldsymbol{U}}|\boldsymbol{z}|_{\boldsymbol{U}}^2 \le C_c\lambda_P\,[\beta_r(\widehat{\eta}) + \beta_u(\widehat{\eta})]\,|\boldsymbol{z}|^2$$

and

$$\begin{aligned} &|c(\boldsymbol{v}_r{}^{(2)} - \boldsymbol{v}_r{}^{(1)}, \boldsymbol{v}^{(1)}, \boldsymbol{z})| + |c(\boldsymbol{v}^{(2)}, \boldsymbol{v}_r{}^{(2)} - \boldsymbol{v}_r{}^{(1)}, \boldsymbol{z})| \\ &\quad \le C_c\left(\|\boldsymbol{v}^{(1)}\|_{\boldsymbol{X}} + \|\boldsymbol{v}^{(2)}\|_{\boldsymbol{X}}\right)\|\boldsymbol{v}_r{}^{(1)} - \boldsymbol{v}_r{}^{(2)}\|_{\boldsymbol{X}}|\boldsymbol{z}|_{\boldsymbol{U}} \\ &\quad \le 4\kappa C_c\beta_v(\widehat{\eta})\|\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D}|\boldsymbol{z}|_{\boldsymbol{U}}. \end{aligned}$$

For the integrals on Γ_N, by (4.10) and (6.1), we obtain

$$\begin{aligned} &|d_\delta(\boldsymbol{v}^{(2)}, \boldsymbol{u}^{(1)}, \boldsymbol{z}) - d_\delta(\boldsymbol{v}^{(1)}, \boldsymbol{u}^{(1)}, \boldsymbol{z})| \\ &\quad \le C_d\|\boldsymbol{v}^{(2)} - \boldsymbol{v}^{(1)}\|_{\boldsymbol{X}}|\boldsymbol{u}^{(1)}|_{\boldsymbol{U}}|\boldsymbol{z}|_{\boldsymbol{U}} \\ &\quad \le C_d\lambda_P|\boldsymbol{u}^{(1)}|_{\boldsymbol{U}}|\boldsymbol{z}|_{\boldsymbol{U}}^2 + C_d|\boldsymbol{u}^{(1)}|_{\boldsymbol{U}}\|\boldsymbol{v}_r{}^{(1)} - \boldsymbol{v}_r{}^{(2)}\|_{\boldsymbol{X}}|\boldsymbol{z}|_{\boldsymbol{U}} \\ &\quad \le C_d\lambda_P\beta_u(\widehat{\eta})|\boldsymbol{z}|_{\boldsymbol{U}}^2 + 2\kappa C_d\beta_u(\widehat{\eta})\|\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D}|\boldsymbol{z}|_{\boldsymbol{U}}. \end{aligned}$$

We end up with the estimate

$$|z|_{U} \leq \frac{2C_c\beta_v(\widehat{\eta})\beta_r(\widehat{\eta}) + 2\kappa C_d\beta_u(\widehat{\eta})}{\alpha(\widehat{\eta})}\|\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D},$$

where, by (5.11), $\alpha(\widehat{\eta}) > 0$. Hence, we take $L_u := \frac{2C_c\beta_v(\widehat{\eta})\beta_r(\widehat{\eta})+2\kappa C_d\beta_u(\widehat{\eta})}{\alpha(\widehat{\eta})}$, which yields

$$\begin{aligned}\|\mathcal{S}_v(\boldsymbol{g}^{(1)}) - \mathcal{S}_v(\boldsymbol{g}^{(2)})\|_{\boldsymbol{X}} &\leq \|\boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}\|_{\boldsymbol{X}} + \|\boldsymbol{v}_r{}^{(1)} - \boldsymbol{v}_r{}^{(2)}\|_{\boldsymbol{X}} \\ &\leq (L_u + 2\kappa)\,\|\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D}\end{aligned}$$

and therefore, (6.2) holds true with $L_v := L_u + 2\kappa$. □

The result of Theorem 6.1 for the velocity part of the solution is enough for the purposes of the next sections. However, for completeness, we provide a Lipschitz estimate which also includes the pressure component of the solution.

Theorem 6.2 *There exists a constant $L_{\mathcal{S}} > 0$ such that*

$$\|\mathcal{S}(\boldsymbol{g}^{(1)}) - \mathcal{S}(\boldsymbol{g}^{(2)})\|_{\boldsymbol{X}\times Q} \leq L_{\mathcal{S}}\|\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D},$$

for all $\boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)} \in \mathcal{U}_{ad}$. The constant $L_{\mathcal{S}}$ is independent of $\boldsymbol{g}^{(1)}$ and $\boldsymbol{g}^{(2)}$.

Proof Let $\boldsymbol{F} \in \boldsymbol{V}^{\circ} \subset \boldsymbol{U}'$ be given by

$$\begin{aligned}\langle \boldsymbol{F}, \boldsymbol{\varphi}\rangle_{\boldsymbol{U}',\boldsymbol{U}} :=\,& a(\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}, \boldsymbol{\varphi}) + c(\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(1)}, \boldsymbol{\varphi}) - c(\boldsymbol{v}^{(2)}, \boldsymbol{v}^{(2)}, \boldsymbol{\varphi}) \\ &+ d_\delta(\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(1)} - \boldsymbol{v}_r{}^{(1)}, \boldsymbol{\varphi}) - d_\delta(\boldsymbol{v}^{(2)}, \boldsymbol{v}^{(2)} - \boldsymbol{v}_r{}^{(2)}, \boldsymbol{\varphi}).\end{aligned}$$

By Lemma 3.2 (ii), there exits a unique $q \in Q$ such that $B^*(q) = -\boldsymbol{F}$ and therefore $q = p^{(1)} - p^{(2)}$. Again, by Lemma 3.2 (ii),

$$\|p^{(1)} - p^{(2)}\|_Q \leq \kappa_b\|\boldsymbol{F}\|_{\boldsymbol{U}'}.$$

It remains to bound the norm $\|\boldsymbol{F}\|_{\boldsymbol{U}'}$:

$$|a(\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}, \boldsymbol{\varphi})| \leq \nu\|\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}\|_{\boldsymbol{X}}|\boldsymbol{\varphi}|_{\boldsymbol{U}} \leq \nu L_v\|\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D}|\boldsymbol{\varphi}|_{\boldsymbol{U}}$$

and, by Lemma 4.3 and Theorem 6.1,

$$\begin{aligned}&\Big|c(\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(1)}, \boldsymbol{\varphi}) - c(\boldsymbol{v}^{(2)}, \boldsymbol{v}^{(2)}, \boldsymbol{\varphi}) \\ &\quad + d_\delta(\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(1)} - \boldsymbol{v}_r{}^{(1)}, \boldsymbol{\varphi}) - d_\delta(\boldsymbol{v}^{(2)}, \boldsymbol{v}^{(2)} - \boldsymbol{v}_r{}^{(2)}, \boldsymbol{\varphi})\Big| \\ &\leq C_c(\|\boldsymbol{v}^{(1)}\|_{\boldsymbol{X}} + \|\boldsymbol{v}^{(2)}\|_{\boldsymbol{X}})\|\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}\|_{\boldsymbol{X}}|\boldsymbol{\varphi}|_{\boldsymbol{U}} \\ &\quad + C_d\|\boldsymbol{v}^{(2)}\|_{\boldsymbol{X}}\|\boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}\|_{\boldsymbol{X}}|\boldsymbol{\varphi}|_{\boldsymbol{U}} + C_d\|\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}\|_{\boldsymbol{X}}|\boldsymbol{u}^{(1)}|_{\boldsymbol{U}}\|\boldsymbol{\varphi}|_{\boldsymbol{U}} \\ &\leq 2C_c\beta_v(\hat{\eta})L_v\|\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D}|\boldsymbol{\varphi}|_{\boldsymbol{U}}\end{aligned}$$

$$
\begin{aligned}
&+ C_d \beta_v(\hat{\eta}) L_u \|\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D} |\boldsymbol{\varphi}|_{\boldsymbol{U}} \\
&+ C_d L_v \beta_u(\hat{\eta}) \|\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D} |\boldsymbol{\varphi}|_{\boldsymbol{U}} \\
= &\left(2C_c \beta_v(\hat{\eta}) L_v + C_d \beta_v(\hat{\eta}) L_u + C_d \beta_u(\hat{\eta}) L_v\right) \|\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D} |\boldsymbol{\varphi}|_{\boldsymbol{X}}.
\end{aligned}
$$

We conclude that

$$
\|p^{(1)} - p^{(2)}\|_Q \leq L_p \|\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D} \tag{6.4}
$$

with

$$
L_p := \kappa_b \left[\nu L_v + 2C_c \beta_v(\hat{\eta}) L_v + C_d \beta_v(\hat{\eta}) L_u + C_d \beta_u(\hat{\eta}) L_v\right].
$$

Combining (6.4) and (6.2) yields

$$
\|\mathcal{S}(\boldsymbol{g}^{(1)}) - \mathcal{S}(\boldsymbol{g}^{(2)})\|_{\boldsymbol{X} \times Q} \leq \sqrt{L_v^2 + L_p^2} \|\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}\|_{\boldsymbol{T}_D}.
$$

□

7 Gâteaux Differentiability of the Control-to-State Operator

Let $\boldsymbol{g}_a, \boldsymbol{g} \in \mathcal{U}_{ad}$. Our aim is to compute the Gâteaux derivatives of $\mathcal{S}_r$ and $\mathcal{S}$ at $\boldsymbol{g} \in \mathcal{U}_{ad}$ in the direction $\boldsymbol{g}_a - \boldsymbol{g}$, by taking $\boldsymbol{g}_h := \boldsymbol{g} + h(\boldsymbol{g}_a - \boldsymbol{g})$, $0 < h < 1$.

Due to the linearity of the Stokes problem, the Gâteaux derivative corresponding to the reference flow $\mathcal{S}_r'(\boldsymbol{g})(\boldsymbol{g}_a - \boldsymbol{g}) =: (\boldsymbol{w}_r, q_r) \in \boldsymbol{X} \times Q$ does exist and satisfies

$$
\begin{cases}
a(\boldsymbol{w}_r, \boldsymbol{\varphi}) + b\big(\boldsymbol{\varphi}, q_r\big) - b\big(\boldsymbol{w}_r, \phi\big) = 0, & \forall (\boldsymbol{\varphi}, \phi) \in \boldsymbol{U} \times Q \\
\gamma_D \boldsymbol{w}_r = \boldsymbol{g}_a - \boldsymbol{g},
\end{cases} \tag{7.1}
$$

which in strong form reads

$$
\begin{cases}
\nabla \cdot \widetilde{\mathbb{T}}(\boldsymbol{w}_r, q_r) = 0 & \text{in } \Omega \\
\nabla \cdot \boldsymbol{w}_r = 0 & \text{in } \Omega \\
\boldsymbol{w}_r = \boldsymbol{g}_a - \boldsymbol{g} & \text{in } \Gamma_D \\
\boldsymbol{n} \cdot \widetilde{\mathbb{T}}(\boldsymbol{w}_r, q_r) = \boldsymbol{0} & \text{on } \Gamma_N.
\end{cases} \tag{7.2}
$$

Now, our aim is to find the main component of the Gâteaux derivative of the control-to-state map, $\mathcal{S}'(\boldsymbol{g})(\boldsymbol{g}_a - \boldsymbol{g})$. Under the assumptions of Lemma 4.1 and Theorem 4.4, let $(\boldsymbol{v}, p) := \mathcal{S}(\boldsymbol{g})$, $\boldsymbol{v}_r := \mathcal{S}_{r,v}(\boldsymbol{g})$ and let $(\boldsymbol{v}_h, p_h) \in \boldsymbol{X} \times Q$ be the solution to (4.8) corresponding to $\boldsymbol{g}_h \in \mathcal{U}_{ad}$, that is, $(\boldsymbol{v}_h, p_h) := \mathcal{S}(\boldsymbol{g}_h)$. Then

$$
(\boldsymbol{w}_h, q_h) := \frac{(\boldsymbol{v}_h, p_h) - (\boldsymbol{v}, p)}{h} = \frac{\mathcal{S}(\boldsymbol{g}_h) - \mathcal{S}(\boldsymbol{g})}{h}
$$

satisfies

$$\begin{cases} a(\boldsymbol{w}_h, \boldsymbol{\varphi}) + b\big(\boldsymbol{\varphi}, q_h\big) - b(\boldsymbol{w}_h, \phi) + c(\boldsymbol{v}, \boldsymbol{w}_h, \boldsymbol{\varphi}) + c(\boldsymbol{w}_h, \boldsymbol{v}, \boldsymbol{\varphi}) \\ +h\, c(\boldsymbol{w}_h, \boldsymbol{w}_h, \boldsymbol{\varphi}) + d_\delta(\boldsymbol{v}, \boldsymbol{w}_h - \boldsymbol{w}_r, \boldsymbol{\varphi}) \\ +\dfrac{1}{h}\big[d_\delta(\boldsymbol{v}_h, \boldsymbol{v}_h - \boldsymbol{v}_{rh}, \boldsymbol{\varphi}) - d_\delta(\boldsymbol{v}, \boldsymbol{v}_h - \boldsymbol{v}_{rh}, \boldsymbol{\varphi})\big] = 0,\ \forall(\boldsymbol{\varphi}, \phi) \in \boldsymbol{U} \times Q \\ \gamma_D \boldsymbol{w}_h = \boldsymbol{g}_a - \boldsymbol{g}. \end{cases} \tag{7.3}$$

The next step is to pass to the limit $h \to 0^+$ in $(\boldsymbol{w}_h, q_h)$ and find the Gâteaux derivative $\mathcal{S}'(\boldsymbol{g})(\boldsymbol{g}_a - \boldsymbol{g})$.

Theorem 7.1 *Under the assumption (5.11), let* $(\boldsymbol{v}_r, \boldsymbol{v}) := (\mathcal{S}_{r,v}(\boldsymbol{g}), \mathcal{S}_v(\boldsymbol{g})) \in \boldsymbol{X} \times \boldsymbol{X}$ *and* $\boldsymbol{w}_r := \mathcal{S}'_{r,v}(\boldsymbol{g})(\boldsymbol{g}_a - \boldsymbol{g}) \in \boldsymbol{X}$. *Then the linearized system*

$$\begin{cases} a(\boldsymbol{w}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, q) - b\big(\boldsymbol{w}, \phi\big) \\ \quad + c(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\varphi}) + c(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{\varphi}) + d_\delta(\boldsymbol{v}, \boldsymbol{w} - \boldsymbol{w}_r, \boldsymbol{\varphi}) \\ \quad + \dfrac{1}{2}\displaystyle\int_{\Gamma_N} (\boldsymbol{w} \cdot \boldsymbol{n}) \Psi'_\delta(\boldsymbol{v} \cdot \boldsymbol{n})(\boldsymbol{v} - \boldsymbol{v}_r) \cdot \boldsymbol{\varphi}\, \mathrm{d}\sigma = 0,\ \forall(\boldsymbol{\varphi}, \phi) \in \boldsymbol{U} \times Q \\ \gamma_D \boldsymbol{w} = \boldsymbol{g}_a - \boldsymbol{g} \end{cases} \tag{7.4}$$

has a unique solution $(\boldsymbol{w}, q) \in \boldsymbol{X} \times Q$ *and* $\mathcal{S}'(\boldsymbol{g})(\boldsymbol{g}_a - \boldsymbol{g}) = (\boldsymbol{w}, q)$.

Proof Let $\boldsymbol{u} := \boldsymbol{v} - \boldsymbol{v}_r \in \boldsymbol{U}$. Firstly, we consider the problem of finding $\boldsymbol{\vartheta} \in \boldsymbol{V}$ such that

$$\begin{aligned} a(\boldsymbol{\vartheta}, \boldsymbol{\varphi}) + c(\boldsymbol{\vartheta}, \boldsymbol{v}, \boldsymbol{\varphi}) + c(\boldsymbol{v}, \boldsymbol{\vartheta}, \boldsymbol{\varphi}) + d_\delta(\boldsymbol{v}, \boldsymbol{\vartheta}, \boldsymbol{\varphi}) \\ + \frac{1}{2}\int_{\Gamma_N} (\boldsymbol{\vartheta} \cdot \boldsymbol{n}) \Psi'_\delta(\boldsymbol{v} \cdot \boldsymbol{n}) \boldsymbol{u} \cdot \boldsymbol{\varphi} \mathrm{d}\sigma \\ = -a(\boldsymbol{w}_r, \boldsymbol{\varphi}) - c(\boldsymbol{v}, \boldsymbol{w}_r, \boldsymbol{\varphi}) - c(\boldsymbol{w}_r, \boldsymbol{v}, \boldsymbol{\varphi}) \\ amp; \qquad -\frac{1}{2}\int_{\Gamma_N} (\boldsymbol{w}_r \cdot \boldsymbol{n}) \Psi'_\delta(\boldsymbol{v} \cdot \boldsymbol{n}) \boldsymbol{u} \cdot \boldsymbol{\varphi}\, \mathrm{d}\sigma, \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V} \end{aligned} \tag{7.5}$$

We are assuming that $\boldsymbol{u}$ and $\boldsymbol{v}$ are fixed. For $\boldsymbol{\vartheta}, \boldsymbol{\varphi} \in \boldsymbol{V}$, set

$$\begin{aligned} \mathcal{A}(\boldsymbol{v}, \boldsymbol{u}; \boldsymbol{\vartheta}, \boldsymbol{\varphi}) :=& a(\boldsymbol{\vartheta}, \boldsymbol{\varphi}) + c(\boldsymbol{v}, \boldsymbol{\vartheta}, \boldsymbol{\varphi}) + c(\boldsymbol{\vartheta}, \boldsymbol{v}, \boldsymbol{\varphi}) \\ &+ d_\delta(\boldsymbol{v}, \boldsymbol{\vartheta}, \boldsymbol{\varphi}) + \frac{1}{2}\int_{\Gamma_N} (\boldsymbol{\vartheta} \cdot \boldsymbol{n}) \Psi'_\delta(\boldsymbol{v} \cdot \boldsymbol{n}) \boldsymbol{u} \cdot \boldsymbol{\varphi} \mathrm{d}\sigma, \\ \langle \mathcal{F}(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}_r), \boldsymbol{\varphi} \rangle_{\boldsymbol{V}', \boldsymbol{V}} :=& - c(\boldsymbol{v}, \boldsymbol{w}_r, \boldsymbol{\varphi}) - c(\boldsymbol{w}_r, \boldsymbol{v}, \boldsymbol{\varphi}) \\ &- \frac{1}{2}\int_{\Gamma_N} (\boldsymbol{w}_r \cdot \boldsymbol{n}) \Psi'_\delta(\boldsymbol{v} \cdot \boldsymbol{n}) \boldsymbol{u} \cdot \boldsymbol{\varphi} \mathrm{d}\sigma \end{aligned}$$

where we used $a(\boldsymbol{w}_r, \boldsymbol{\varphi}) = 0$. Using (2.9), (4.5)–(4.7), we get

$$|\mathcal{A}(\boldsymbol{v}, \boldsymbol{u}; \boldsymbol{\vartheta}, \boldsymbol{\varphi})| \leq \big[\nu + C_d\delta + ((1 + \lambda_P)C_c + C_d)\|\boldsymbol{v}\|_{\boldsymbol{X}}$$

$$
\begin{aligned}
&+C_d\lambda_P|\boldsymbol{u}|_{\boldsymbol{U}}\big]|\boldsymbol{\vartheta}|_{\boldsymbol{U}}|\boldsymbol{\varphi}|_{\boldsymbol{U}},\ \forall\boldsymbol{\vartheta},\boldsymbol{\varphi}\in\boldsymbol{V},\\
|\langle\mathcal{F}(\boldsymbol{v},\boldsymbol{u},\boldsymbol{w}_r),\boldsymbol{\varphi}\rangle_{\boldsymbol{V}',\boldsymbol{V}}| &\le \big[2C_c\|\boldsymbol{v}\|_{\boldsymbol{X}}+C_d\|\boldsymbol{u}\|_{\boldsymbol{X}}\big]\|\boldsymbol{w}_r\|_{\boldsymbol{X}}|\boldsymbol{\varphi}|_{\boldsymbol{X}},\ \forall\boldsymbol{\varphi}\in\boldsymbol{V}.
\end{aligned}
$$

From

$$
c(\boldsymbol{v},\boldsymbol{\vartheta},\boldsymbol{\vartheta})+d_\delta(\boldsymbol{v},\boldsymbol{\vartheta},\boldsymbol{\vartheta})\ge\frac{1}{2}\int_{\Gamma_N}[\boldsymbol{v}\cdot\boldsymbol{n}]^+\,|\boldsymbol{\vartheta}|^2\mathrm{d}\sigma\ge 0
$$

and (5.9), (5.10), it follows that

$$
\begin{aligned}
\mathcal{A}(\boldsymbol{v},\boldsymbol{u};\boldsymbol{\vartheta},\boldsymbol{\vartheta}) &\ge \nu|\boldsymbol{\vartheta}|_{\boldsymbol{U}}^2-\Big|c(\boldsymbol{\vartheta},\boldsymbol{v},\boldsymbol{\vartheta})\Big|-\frac{1}{2}\left|\int_{\Gamma_N}(\boldsymbol{\vartheta}\cdot\boldsymbol{n})\Psi_\delta'(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{u}\cdot\boldsymbol{\vartheta}\,\mathrm{d}\sigma\right|\\
&\ge\big(\nu-C_c\lambda_P\|\nabla\boldsymbol{v}\|_{2,\Omega}-C_d\lambda_P|\boldsymbol{u}|_{\boldsymbol{U}}\big)\,|\boldsymbol{\vartheta}|_{\boldsymbol{U}}^2\\
&\ge\alpha(\|\boldsymbol{g}\|_{\boldsymbol{T}_D})|\boldsymbol{\vartheta}|_{\boldsymbol{U}}^2.
\end{aligned}
$$

If $\alpha=\alpha(\|\boldsymbol{g}\|_{\boldsymbol{T}_D})$ satisfies (5.11) then $\mathcal{A}(\boldsymbol{v},\boldsymbol{u};\boldsymbol{\vartheta},\boldsymbol{\vartheta})\ge\alpha|\boldsymbol{\vartheta}|^2$. By the Lax-Milgram Theorem, if (5.11) holds then problem (7.5) has a unique solution $\boldsymbol{\vartheta}\in\boldsymbol{V}$. Then $\boldsymbol{w}$ is given by $\boldsymbol{w}=\boldsymbol{\vartheta}+\boldsymbol{w}_r$ and the pressure q is obtained and estimated by the same reasoning used in Theorem 4.4, Step 4 (see [24]) and Theorem 6.2 of the previous section.

Now, we pass to the limit $h\to 0^+$. Let $\boldsymbol{u}_h:=\boldsymbol{v}_h-\boldsymbol{v}_{rh}$ and $(\boldsymbol{\xi}_h,\theta_h):=(\boldsymbol{w}_h,q_h)-(\boldsymbol{w},q)\in\boldsymbol{V}\times Q$, which satisfies (notice that $\boldsymbol{w}_{rh}=\boldsymbol{w}_r$, for all h)

$$
\begin{aligned}
a(\boldsymbol{\xi}_h,\boldsymbol{\varphi})&+b\big(\boldsymbol{\varphi},\theta_h\big)-b(\boldsymbol{\xi}_h,\phi)\\
&+c(\boldsymbol{v},\boldsymbol{\xi}_h,\boldsymbol{\varphi})+d_\delta(\boldsymbol{v},\boldsymbol{\xi}_h,\boldsymbol{\varphi})+c(\boldsymbol{\xi}_h,\boldsymbol{v},\boldsymbol{\varphi})\\
&+\frac{1}{h}\big[d_\delta(\boldsymbol{v}_h,\boldsymbol{u},\boldsymbol{\varphi})-d_\delta(\boldsymbol{v},\boldsymbol{u},\boldsymbol{\varphi})\big]\\
&+\frac{1}{h}\big[d_\delta(\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{\varphi})-d_\delta(\boldsymbol{v},\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{\varphi})\big]\\
&-\frac{1}{2}\int_{\Gamma_N}(\boldsymbol{w}\cdot\boldsymbol{n})\Psi_\delta'(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}\sigma\\
&=-h\,c(\boldsymbol{w}_h,\boldsymbol{w}_h,\boldsymbol{\varphi}),\quad\forall(\boldsymbol{\varphi},\phi)\in\boldsymbol{U}\times Q.
\end{aligned}
\tag{7.6}
$$

By Lemma 6.1, the following estimate holds for $\boldsymbol{w}_h$

$$
\|\boldsymbol{w}_h\|_{\boldsymbol{X}}=\frac{1}{h}\|\mathcal{S}_v(\boldsymbol{g}_h)-\mathcal{S}_v(\boldsymbol{g})\|_{\boldsymbol{X}}\le L_v\|\boldsymbol{g}_a-\boldsymbol{g}\|_{\boldsymbol{T}_D},\quad 0<h<1,
$$

and recalling (6.3), we have

$$
\|\boldsymbol{u}_h-\boldsymbol{u}\|_{\boldsymbol{X}}\le L_u\|\boldsymbol{g}_h-\boldsymbol{g}\|_{\boldsymbol{T}_D}=hL_u\|\boldsymbol{g}_a-\boldsymbol{g}\|_{\boldsymbol{T}_D},\quad 0<h<1.
$$

Taking $\boldsymbol{\varphi} = \boldsymbol{\xi}_h$ in (7.6) yields

$$
\begin{aligned}
&(\nu - C_c\lambda_P|\boldsymbol{v}|_{\boldsymbol{U}})\,|\boldsymbol{\xi}_h|^2_{\boldsymbol{U}} \\
&\quad \le \frac{1}{h}\left|d_\delta(\boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{\xi}_h) - d_\delta(\boldsymbol{v}, \boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{\xi}_h)\right| + h\,\left|c(\boldsymbol{w}_h, \boldsymbol{w}_h, \boldsymbol{\xi}_h)\right| \\
&\qquad + \left|\frac{1}{h}\left[d_\delta(\boldsymbol{v}_h, \boldsymbol{u}, \boldsymbol{\xi}_h) - d_\delta(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\xi}_h)\right] - \frac{1}{2}\int_{\Gamma_N}(\boldsymbol{w}\cdot\boldsymbol{n})\Psi'_\delta(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{u}\cdot\boldsymbol{\xi}_h \mathrm{d}\sigma\right|
\end{aligned}
$$

where

$$
\begin{aligned}
|h\,c(\boldsymbol{w}_h, \boldsymbol{w}_h, \boldsymbol{\xi}_h)| &\le h\,C_c\lambda_P\|\boldsymbol{w}_h\|^2_{\boldsymbol{X}}|\boldsymbol{\xi}_h|_{\boldsymbol{U}} \le h\,C_c\lambda_P L_v^2\|\boldsymbol{g}_a - \boldsymbol{g}\|^2_{\boldsymbol{T}_D}|\boldsymbol{\xi}_h|_{\boldsymbol{U}}, \\
&\frac{1}{h}\left[d_\delta(\boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{\xi}_h) - d_\delta(\boldsymbol{v}, \boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{\xi}_h)\right] \\
&\le C_d\lambda_P\,\|\boldsymbol{w}_h\|_{\boldsymbol{X}}\,|\boldsymbol{u}_h - \boldsymbol{u}|_{\boldsymbol{U}}|\boldsymbol{\xi}_h|_{\boldsymbol{U}} \le hC_d\lambda_P L_u L_v\|\boldsymbol{g}_a - \boldsymbol{g}\|^2_{\boldsymbol{T}_D}|\boldsymbol{\xi}_h|_{\boldsymbol{U}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\frac{1}{h}\left[d_\delta(\boldsymbol{v}_h, \boldsymbol{u}, \boldsymbol{\xi}_h) - d_\delta(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\xi}_h)\right] - \frac{1}{2}\int_{\Gamma_N}(\boldsymbol{w}\cdot\boldsymbol{n})\Psi'_\delta(\boldsymbol{v}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{\xi}_h)\mathrm{d}\sigma\right| \\
&\quad = \frac{1}{2}\left|\int_{\Gamma_N}\left[\frac{\Psi_\delta(\boldsymbol{v}_h\cdot\boldsymbol{n}) - \Psi_\delta(\boldsymbol{v}\cdot\boldsymbol{n})}{h} - (\boldsymbol{w}\cdot\boldsymbol{n})\Psi'_\delta(\boldsymbol{v}\cdot\boldsymbol{n})\right](\boldsymbol{u}\cdot\boldsymbol{\xi}_h)\mathrm{d}\sigma\right| \\
&\quad = \frac{1}{2}\left|\int_{\Gamma_N}\left[(\boldsymbol{w}_h\cdot\boldsymbol{n})\int_0^1 \Psi'_\delta\left((\boldsymbol{v}_h - \boldsymbol{v})\cdot\boldsymbol{n}\theta + \boldsymbol{v}\cdot\boldsymbol{n}\right)\mathrm{d}\theta\right.\right. \\
&\qquad\qquad \left.\left. -(\boldsymbol{w}\cdot\boldsymbol{n})\Psi'_\delta(\boldsymbol{v}\cdot\boldsymbol{n})\right](\boldsymbol{u}\cdot\boldsymbol{\xi}_h)\mathrm{d}\sigma\right| \\
&\quad \le \frac{1}{2}\left|\int_{\Gamma_N}\int_0^1 \Psi'_\delta\left((\boldsymbol{v}_h - \boldsymbol{v})\cdot\boldsymbol{n}\theta + \boldsymbol{v}\cdot\boldsymbol{n}\right)\mathrm{d}\theta(\boldsymbol{\xi}_h\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{\xi}_h)\mathrm{d}\sigma\right| \\
&\qquad + \frac{1}{2}\left|\int_{\Gamma_N}\int_0^1\left[\Psi'_\delta(h\boldsymbol{w}_h\cdot\boldsymbol{n}\theta + \boldsymbol{v}\cdot\boldsymbol{n}) - \Psi'_\delta(\boldsymbol{v}\cdot\boldsymbol{n})\right]\mathrm{d}\theta(\boldsymbol{w}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{\xi}_h)\mathrm{d}\sigma\right|.
\end{aligned}
$$

Analogously to (4.6), by the property (iii) of the function Ψ_δ,

$$
\left|\int_{\Gamma_N}\int_0^1 \Psi'_\delta\left((\boldsymbol{v}_h - \boldsymbol{v})\cdot\boldsymbol{n}\theta + \boldsymbol{v}\cdot\boldsymbol{n}\right)\mathrm{d}\theta(\boldsymbol{\xi}_h\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{\xi}_h)\mathrm{d}\sigma\right| \le C_d\lambda_P|\boldsymbol{u}|_{\boldsymbol{U}}|\boldsymbol{\xi}_h|^2_{\boldsymbol{U}}
$$

and, again by property (iii) and (2.9),

$$
\begin{aligned}
&\left|\int_{\Gamma_N}\int_0^1\left[\Psi'_\delta(h\boldsymbol{w}_h\cdot\boldsymbol{n}\theta + \boldsymbol{v}\cdot\boldsymbol{n}) - \Psi'_\delta(\boldsymbol{v}\cdot\boldsymbol{n})\right]\mathrm{d}\theta(\boldsymbol{w}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{\xi}_h)\mathrm{d}\sigma\right| \\
&\quad \le C_{\Gamma_N}C_d h L_{\Psi'_\delta}\|\boldsymbol{w}_h\|_{\boldsymbol{X}}\|\boldsymbol{w}\|_{\boldsymbol{X}}|\boldsymbol{u}|_{\boldsymbol{U}}|\boldsymbol{\xi}_h|_{\boldsymbol{U}} \\
&\quad \le C_{\Gamma_N}C_d h L_{\Psi'_\delta}L_v\|\boldsymbol{g}_a - \boldsymbol{g}\|_{\boldsymbol{T}_D}\|\boldsymbol{w}\|_{\boldsymbol{X}}|\boldsymbol{u}|_{\boldsymbol{U}}|\boldsymbol{\xi}_h|_{\boldsymbol{U}}.
\end{aligned}
$$

We conclude that

$$\left|\frac{1}{h}\left[d_\delta(\boldsymbol{v}_h,\boldsymbol{u},\boldsymbol{\xi}_h)-d_\delta(\boldsymbol{v},\boldsymbol{u},\boldsymbol{\xi}_h)\right]-\frac{1}{2}\int_{\Gamma_N}(\boldsymbol{w}\cdot\boldsymbol{n})\Psi'_\delta(\boldsymbol{v}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{\xi}_h)\mathrm{d}\sigma\right|$$
$$\leq C_d\lambda_P|\boldsymbol{u}|_U|\boldsymbol{\xi}_h|_{\boldsymbol{U}}^2+C_d h L_{\Psi'_\delta}L_v\|\boldsymbol{g}_a-\boldsymbol{g}\|_{\boldsymbol{T}_D}|\boldsymbol{u}|_U\|\boldsymbol{w}\|_{\boldsymbol{X}}|\boldsymbol{\xi}_h|_{\boldsymbol{U}}$$

Combining all the above estimates, we conclude that, under the assumption (5.11), $\boldsymbol{\xi}_h$ satisfies

$$\begin{aligned}\alpha|\boldsymbol{\xi}_h|_{\boldsymbol{U}} \leq & h\, C_c\lambda_P L_v^2\|\boldsymbol{g}_a-\boldsymbol{g}\|_{\boldsymbol{T}_D}^2+h\, C_d\lambda_P L_u L_v\|\boldsymbol{g}_a-\boldsymbol{g}\|_{\boldsymbol{T}_D}^2\\ &+h(1+C_{\Gamma_N})C_d L_{\Psi'_\delta}L_v\|\boldsymbol{g}_a-\boldsymbol{g}\|_{\boldsymbol{T}_D}|\boldsymbol{u}|_{\boldsymbol{U}}\|\boldsymbol{w}\|_{\boldsymbol{X}}\end{aligned}$$

and letting $h\to 0^+$ we conclude that $|\boldsymbol{\xi}_h|_{\boldsymbol{U}}\to 0$. Consequently, $\boldsymbol{w}_h\to\boldsymbol{w}$ in $\boldsymbol{X}$. To see that θ_h also goes to zero one can use, similarly to what was done in the proof of Theorem 4.4 (see [24]), an inf-sup argument, showing that $\|\theta_h\|_Q\leq M|\boldsymbol{\xi}_h|_{\boldsymbol{U}}\to 0$.

8 Adjoint System

Assume that $v_\Omega\in L^2(\Omega)^n$. Let $\boldsymbol{g}\in\mathcal{U}_{ad}$ and $(\boldsymbol{v}_r,\boldsymbol{v}):=(\mathcal{S}_{r,v}(\boldsymbol{g}),\mathcal{S}_v(\boldsymbol{g}))$, so that (5.11) holds true [27].

The first part of the adjoint system is the problem of finding $(z,\pi)\in\boldsymbol{U}\times Q$ which satisfy the linearized equations

$$\begin{aligned}&a(\boldsymbol{\varphi},z)-b(\boldsymbol{\varphi},\pi)+b(z,\phi)+c(\boldsymbol{v},\boldsymbol{\varphi},z)+c(\boldsymbol{\varphi},\boldsymbol{v},z)\\&\qquad+d_\delta(\boldsymbol{v},\boldsymbol{\varphi},z)+\frac{1}{2}\int_{\Gamma_N}(\boldsymbol{\varphi}\cdot\boldsymbol{n})\Psi'_\delta(\boldsymbol{v}\cdot\boldsymbol{n})\left[(\boldsymbol{v}-\boldsymbol{v}_r)\cdot z\right]\mathrm{d}\sigma\\&\quad=(\boldsymbol{v}-\boldsymbol{v}_\Omega,\boldsymbol{\varphi})_\Omega,\ \forall(\boldsymbol{\varphi},\phi)\in\boldsymbol{U}\times Q.\end{aligned}\tag{8.1}$$

Once z is obtained from (8.1), the following linear problem can be solved for $(z_r,\pi_r)\in\boldsymbol{U}\times Q$:

$$a(\boldsymbol{\varphi},z_r)+b(\boldsymbol{\varphi},\pi_r)-b(z_r,\phi)=d_\delta(\boldsymbol{v},\boldsymbol{\varphi},z),\ \forall(\boldsymbol{\varphi},\phi)\in\boldsymbol{U}\times Q.\tag{8.2}$$

Theorem 8.1 *For $\boldsymbol{g}\in\mathcal{U}_{ad}$, the adjoint problem (8.1), (8.2) has a unique solution.*

Proof Firstly, we consider the problem of finding $z\in V$ such that

$$\begin{aligned}&a(\boldsymbol{\varphi},z)+c(\boldsymbol{v},\boldsymbol{\varphi},z)+c(\boldsymbol{\varphi},\boldsymbol{v},z)+d_\delta(\boldsymbol{v},\boldsymbol{\varphi},z)\\&\quad+\frac{1}{2}\int_{\Gamma_N}(\boldsymbol{\varphi}\cdot\boldsymbol{n})\Psi'_\delta(\boldsymbol{v}\cdot\boldsymbol{n})(\boldsymbol{v}-\boldsymbol{v}_r)\cdot z\,\mathrm{d}\sigma=(\boldsymbol{v}-\boldsymbol{v}_\Omega,\boldsymbol{\varphi})_\Omega,\ \forall\boldsymbol{\varphi}\in\boldsymbol{V},\end{aligned}\tag{8.3}$$

where $\boldsymbol{v},\boldsymbol{v}_r,\boldsymbol{v}_\Omega$ are fixed and $\boldsymbol{v}-\boldsymbol{v}_\Omega\in L^2(\Omega)^n$.

Let $\boldsymbol{u} := \boldsymbol{v} - \boldsymbol{v}_r$, and, for $\boldsymbol{z}, \boldsymbol{\varphi} \in \boldsymbol{V}$, define

$$\mathcal{A}(\boldsymbol{v}, \boldsymbol{u}; \boldsymbol{z}, \boldsymbol{\varphi}) := a(\boldsymbol{\varphi}, \boldsymbol{z}) + c(\boldsymbol{v}, \boldsymbol{\varphi}, \boldsymbol{z}) + c(\boldsymbol{\varphi}, \boldsymbol{v}, \boldsymbol{z}) + d_\delta(\boldsymbol{v}, \boldsymbol{\varphi}, \boldsymbol{z}) + \frac{1}{2}\int_{\Gamma_N} (\boldsymbol{\varphi} \cdot \boldsymbol{n})\Psi'_\delta(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{u} \cdot \boldsymbol{z} \mathrm{d}\sigma.$$

From now on the reasoning is the same as in the proof of Theorem 7.1, problem (7.5). In particular, $\mathcal{A}$ satisfies

$$\begin{aligned}
\mathcal{A}(\boldsymbol{v}, \boldsymbol{u}; \boldsymbol{z}, \boldsymbol{z}) &:= a(\boldsymbol{z}, \boldsymbol{z}) + c(\boldsymbol{v}, \boldsymbol{z}, \boldsymbol{z}) + c(\boldsymbol{z}, \boldsymbol{v}, \boldsymbol{z}) + d_\delta(\boldsymbol{v}, \boldsymbol{z}, \boldsymbol{z}) \\
&\quad + \frac{1}{2}\int_{\Gamma_N} (\boldsymbol{z} \cdot \boldsymbol{n})\Psi'_\delta(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{u} \cdot \boldsymbol{z} \mathrm{d}\sigma \\
&\geq \nu|\boldsymbol{z}|^2 - \left|c(\boldsymbol{z}, \boldsymbol{v}, \boldsymbol{z})\right| - \frac{1}{2}\left|\int_{\Gamma_N} (\boldsymbol{z} \cdot \boldsymbol{n})\Psi'_\delta(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{u} \cdot \boldsymbol{z} \mathrm{d}\sigma\right| \\
&\geq \left(\nu - C_c\lambda_P\|\nabla \boldsymbol{v}\|_{2,\Omega} - C_d\lambda_P|\boldsymbol{u}|_{\boldsymbol{U}}\right)|\boldsymbol{z}|^2_{\boldsymbol{U}}
\end{aligned}$$

and id (5.11) holds true then

$$\mathcal{A}(\boldsymbol{v}, \boldsymbol{u}; \boldsymbol{z}, \boldsymbol{z}) \geq \alpha|\boldsymbol{z}|^2, \quad \forall \boldsymbol{z} \in \boldsymbol{V}.$$

By the Lax-Milgram Theorem, problem (8.3) has a unique solution $\boldsymbol{z} \in \boldsymbol{V}$. The pressure π is obtained by the same reasoning used in Theorem 4.4, see [24].

With existence of $\boldsymbol{z}$ established, existence and uniqueness for the Stokes problem (8.2) is immediate. □

Notice that, in strong form, problem (8.1), (8.2) takes the form: first solve

$$\begin{cases}
-\nabla \cdot \widetilde{\mathbb{T}}^*(\boldsymbol{z}, \pi) + (\nabla \boldsymbol{v})\boldsymbol{z} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{z} = \boldsymbol{v} - \boldsymbol{v}_\Omega & \text{in } \Omega \\
\nabla \cdot \boldsymbol{z} = 0 & \text{in } \Omega \\
\boldsymbol{z} = \boldsymbol{0} & \text{on } \Gamma_D \\
\boldsymbol{n} \cdot \widetilde{\mathbb{T}}^*(\boldsymbol{z}, \pi) + \frac{1}{2}\Psi_\delta(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{z} & \\
\quad + \frac{1}{2}\Psi'_\delta(\boldsymbol{v} \cdot \boldsymbol{n})\left[(\boldsymbol{v} - \boldsymbol{v}_r) \cdot \boldsymbol{z}\right]\boldsymbol{n} = -(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{z} & \text{on } \Gamma_N
\end{cases} \tag{8.4}$$

and then, using $\boldsymbol{z}$, solve

$$\begin{cases}
\nabla \cdot \widetilde{\mathbb{T}}^*(\boldsymbol{z}_r, \pi_r) = 0 & \text{in } \Omega \\
\nabla \cdot \boldsymbol{z}_r = 0 & \text{in } \Omega \\
\boldsymbol{z}_r = \boldsymbol{0} & \text{on } \Gamma_D \\
\boldsymbol{n} \cdot \widetilde{\mathbb{T}}^*(\boldsymbol{z}_r, \pi_r) = \frac{1}{2}\Psi_\delta(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{z} & \text{on } \Gamma_N.
\end{cases} \tag{8.5}$$

9 First Order Optimality Conditions

Let $(\widehat{\boldsymbol{v}}_r, \widehat{\boldsymbol{v}}, \widehat{p}, \widehat{\boldsymbol{g}})$ be an optimal solution, in accordance to Theorem 5.1, and let $\boldsymbol{g}_a \in \mathcal{U}_{ad}$. Consider $\boldsymbol{v}_h = \mathcal{S}_v(\boldsymbol{g}_h)$ where $\boldsymbol{g}_h := \widehat{\boldsymbol{g}} + h(\boldsymbol{g}_a - \widehat{\boldsymbol{g}})$, $0 < h < 1$, and $\boldsymbol{w}_h = (\boldsymbol{v}_h - \widehat{\boldsymbol{v}})/h$. By Theorem 7.1, we obtain

$$\begin{aligned}
&\lim_{h\to 0} \frac{\mathcal{J}(\boldsymbol{v}_h, \boldsymbol{g}_h) - \mathcal{J}(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{g}})}{h} \\
&\quad = \lim_{h\to 0} \left[\int_\Omega \boldsymbol{w}_h \cdot (\widehat{\boldsymbol{v}} - \boldsymbol{v}_\Omega) \mathrm{d}x + \frac{h}{2} \|\boldsymbol{w}_h\|_2^2 + \tau(\widehat{\boldsymbol{g}}, \boldsymbol{g}_a - \widehat{\boldsymbol{g}})_{\Gamma_D} + \frac{h\tau}{2} \|\boldsymbol{g}_a - \widehat{\boldsymbol{g}}\|_{\Gamma_D}^2 \right] \\
&\quad = \int_\Omega (\widehat{\boldsymbol{v}} - \boldsymbol{v}_\Omega) \cdot \widehat{\boldsymbol{w}} \mathrm{d}x + \tau(\widehat{\boldsymbol{g}}, \boldsymbol{g}_a - \widehat{\boldsymbol{g}})_{\Gamma_D}.
\end{aligned}$$

where $\widehat{\boldsymbol{w}} := \lim_{h\to 0}(\boldsymbol{v}_h - \widehat{\boldsymbol{v}})/h$.

From the fact that $(\widehat{\boldsymbol{v}}, \widehat{p}, \widehat{\boldsymbol{g}})$ is optimal, we know that $\mathcal{J}(\boldsymbol{v}_h, \boldsymbol{g}_h) - \mathcal{J}(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{g}}) \geq 0$ for all $h \in (0, 1)$, and therefore,

$$\int_\Omega (\widehat{\boldsymbol{v}} - \boldsymbol{v}_\Omega) \cdot \widehat{\boldsymbol{w}} \, \mathrm{d}x + \tau(\widehat{\boldsymbol{g}}, \boldsymbol{g}_a - \widehat{\boldsymbol{g}})_{\Gamma_D} \geq 0. \tag{9.1}$$

We can dot-multiply the first equation of (8.4) by a test function $\boldsymbol{\varphi} \in \boldsymbol{W} = \{\boldsymbol{\varphi} \in \boldsymbol{X} : \gamma_D \boldsymbol{\varphi} \in \boldsymbol{T}_D\}$ and integrate in Ω. Using the boundary conditions of (8.4) and defining

$$\boldsymbol{\sigma} := \boldsymbol{n} \cdot \widetilde{\mathbb{T}}^*(z, \pi),$$

we find

$$\begin{aligned}
&a(\boldsymbol{\varphi}, z) - b(\boldsymbol{\varphi}, \pi) + c(\boldsymbol{v}, \boldsymbol{\varphi}, z) + c(\boldsymbol{\varphi}, \boldsymbol{v}, z) + d_\delta(\boldsymbol{v}, \boldsymbol{\varphi}, z) - \langle \boldsymbol{\sigma}, \boldsymbol{\varphi} \rangle_{\Gamma_D} \\
&\quad + \frac{1}{2} \int_{\Gamma_N} (\boldsymbol{\varphi} \cdot \boldsymbol{n}) \Psi_\delta'(\boldsymbol{v} \cdot \boldsymbol{n}) \left[(\boldsymbol{v} - \boldsymbol{v}_r) \cdot z \right] \mathrm{d}\sigma = (\boldsymbol{v} - \boldsymbol{v}_\Omega, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in \boldsymbol{W}
\end{aligned} \tag{9.2}$$

Analogously, defining

$$\boldsymbol{\sigma}_r := \boldsymbol{n} \cdot \widetilde{\mathbb{T}}^*(z_r, \pi_r),$$

from (8.5), we obtain

$$a(\boldsymbol{\varphi}, z_r) - b(\boldsymbol{\varphi}, \pi_r) - \langle \boldsymbol{\sigma}_r, \boldsymbol{\varphi} \rangle_{\Gamma_D} = d_\delta(\boldsymbol{v}, \boldsymbol{\varphi}, z), \quad \forall \boldsymbol{\varphi} \in \boldsymbol{W}. \tag{9.3}$$

Now, let $(\widehat{z}, \widehat{\pi})$ be the solution to the adjoint problem (8.1) associated with the optimal solution $(\widehat{\boldsymbol{v}}_r, \widehat{\boldsymbol{v}}) := (\mathcal{S}_{r,v}(\widehat{\boldsymbol{g}}), \mathcal{S}_v(\widehat{\boldsymbol{g}}))$.

Choosing, in (9.2) and in the second equation of (8.4), $\boldsymbol{\varphi} = \widehat{\boldsymbol{w}}$ and $\phi = \widehat{q}$, we obtain

$$\begin{cases} a(\widehat{\boldsymbol{w}}, \widehat{\boldsymbol{z}}) - b(\widehat{\boldsymbol{w}}, \widehat{\pi}) + c(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{w}}, \widehat{\boldsymbol{z}}) + c(\widehat{\boldsymbol{w}}, \widehat{\boldsymbol{v}}, \widehat{\boldsymbol{z}}) + d_\delta(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{w}}, \widehat{\boldsymbol{z}}) - \langle \widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{w}} \rangle_{\Gamma_D} \\ \quad + \frac{1}{2} \int_{\Gamma_N} (\widehat{\boldsymbol{w}} \cdot \boldsymbol{n}) \Psi'_\delta(\widehat{\boldsymbol{v}} \cdot \boldsymbol{n})(\widehat{\boldsymbol{v}} - \widehat{\boldsymbol{v}}_r) \cdot \widehat{\boldsymbol{z}} \, \mathrm{d}\sigma = (\widehat{\boldsymbol{v}} - \boldsymbol{v}_\Omega, \widehat{\boldsymbol{w}})_\Omega, \\ b(\widehat{\boldsymbol{z}}, q) = 0 \end{cases} \tag{9.4}$$

and taking $\boldsymbol{\varphi} = \widehat{\boldsymbol{z}}$ and $\phi = \widehat{\pi}$ in (7.4) yields

$$\begin{cases} a(\widehat{\boldsymbol{w}}, \widehat{\boldsymbol{z}}) + b(\widehat{\boldsymbol{z}}, q) + c(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{w}}, \widehat{\boldsymbol{z}}) + c(\widehat{\boldsymbol{w}}, \widehat{\boldsymbol{v}}, \widehat{\boldsymbol{z}}) \\ \quad + d_\delta(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{w}} - \widehat{\boldsymbol{w}_r}, \widehat{\boldsymbol{z}}) + \frac{1}{2} \int_{\Gamma_N} (\widehat{\boldsymbol{w}} \cdot \boldsymbol{n}) \Psi'_\delta(\widehat{\boldsymbol{v}} \cdot \boldsymbol{n})(\widehat{\boldsymbol{v}} - \widehat{\boldsymbol{v}}_r) \cdot \widehat{\boldsymbol{z}} \, \mathrm{d}\sigma = 0, \\ b\big(\widehat{\boldsymbol{w}}, \widehat{\pi}\big) = 0. \end{cases} \tag{9.5}$$

Comparing (9.4) with (9.5), we conclude that

$$(\widehat{\boldsymbol{v}} - \boldsymbol{v}_\Omega, \widehat{\boldsymbol{w}})_\Omega = d_\delta(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{w}_r}, \widehat{\boldsymbol{z}}) - \langle \widehat{\boldsymbol{\sigma}}, \boldsymbol{g}_a - \widehat{\boldsymbol{g}} \rangle_{\Gamma_D}$$

and therefore (9.1) takes the form

$$d_\delta(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{w}_r}, \widehat{\boldsymbol{z}}) - \langle \widehat{\boldsymbol{\sigma}}, \boldsymbol{g}_a - \widehat{\boldsymbol{g}} \rangle_{\Gamma_D} + \tau \langle \widehat{\boldsymbol{g}}, \boldsymbol{g}_a - \widehat{\boldsymbol{g}} \rangle_{\Gamma_D} \geq 0, \ \forall \boldsymbol{g}_a \in \mathcal{U}_{ad}. \tag{9.6}$$

In order to simplify the first term in (9.6), we use (7.1) with $(\boldsymbol{\varphi}, \phi) = (\widehat{\boldsymbol{z}}_{\boldsymbol{r}}, \widehat{\pi}_r)$ and (9.3) and the second equation of (8.2) with $(\boldsymbol{\varphi}, \phi) = (\widehat{\boldsymbol{w}_r}, \widehat{q}_r)$, to obtain

$$-\langle \widehat{\boldsymbol{\sigma}}_r, \boldsymbol{g}_a - \widehat{\boldsymbol{g}} \rangle_{\Gamma_D} + d_\delta(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{w}_r}, \widehat{\boldsymbol{z}}) = 0$$

so that

$$\mathcal{J}'(\widehat{\boldsymbol{g}})(\boldsymbol{g}_a - \widehat{\boldsymbol{g}}) = \langle (\widehat{\boldsymbol{\sigma}}_r - \widehat{\boldsymbol{\sigma}}) + \tau \widehat{\boldsymbol{g}}, \boldsymbol{g}_a - \widehat{\boldsymbol{g}} \rangle_{\Gamma_D} \geq 0, \ \forall \boldsymbol{g}_a \in \mathcal{U}_{ad}. \tag{9.7}$$

Our main result concerning the characterization of the optimal solution can then be stated as follows.

Theorem 9.1 *Let* $(\widehat{\boldsymbol{v}}_r, \widehat{p}_r, \widehat{\boldsymbol{v}}, \widehat{p}, \widehat{\boldsymbol{g}})$ *be an optimal solution of* (5.4) *for* $\boldsymbol{v}_\Omega \in L^2(\Omega)^n$*. Then, there exists a unique* $(\widehat{\boldsymbol{z}}, \widehat{\pi}) \in \boldsymbol{X} \times Q$ *and* $(\widehat{\boldsymbol{z}}_r, \widehat{\pi}_r) \in \boldsymbol{X} \times Q$ *satisfying* (8.1)*,* (8.2) *and the inequality* (9.7)*.*

Remark 9.2 If $\mathcal{U}_{ad} \subset \boldsymbol{H}_0^1(\Gamma_D)$ then we can compute the whole gradient

$$\mathcal{J}'(\widehat{\boldsymbol{g}}) = (\widehat{\boldsymbol{\sigma}}_r - \widehat{\boldsymbol{\sigma}}) + \tau \widehat{\boldsymbol{g}} \in \boldsymbol{H}^{-1/2}(\Gamma_D) = (\boldsymbol{H}_{00}^{1/2}(\Gamma_D))'$$

as an element of $\boldsymbol{H}_0^1(\Gamma_D)$. In fact, taking $\boldsymbol{r} = (\widehat{\boldsymbol{\sigma}}_r - \widehat{\boldsymbol{\sigma}}) \in \boldsymbol{H}^{-1/2}(\Gamma_D)$ we have

$$\langle \boldsymbol{r}, \tilde{\boldsymbol{g}} \rangle_{\Gamma_D} = \langle \imath \boldsymbol{r}, \tilde{\boldsymbol{g}} \rangle_{\boldsymbol{H}^{-1}(\Gamma_D) \times \boldsymbol{H}_0^1(\Gamma_D)}$$

where $i : \boldsymbol{H}^{-1/2}(\Gamma_D) \rightarrow \boldsymbol{H}^{-1}(\Gamma_D)$ is the canonical identifier.

We may then look for the unique Riesz representative $\widehat{\boldsymbol{\eta}} = R(i\boldsymbol{r}) \in \boldsymbol{H}_0^1(\Gamma_D)$ such that

$$(\widehat{\boldsymbol{\eta}}, \tilde{\boldsymbol{g}})_{\boldsymbol{H}_0^1(\Gamma_D)} = \langle i\boldsymbol{r}, \tilde{\boldsymbol{g}} \rangle_{\boldsymbol{H}^{-1}(\Gamma_D)\times \boldsymbol{H}_0^1(\Gamma_D)} \quad \forall \tilde{\boldsymbol{g}} \in \boldsymbol{H}_0^1(\Gamma_D) \tag{9.8}$$

and $\nabla \mathcal{J}(\widehat{\boldsymbol{g}}) = \widehat{\boldsymbol{\eta}} + \tau \widehat{\boldsymbol{g}} \in \boldsymbol{H}_0^1(\Gamma_D)$.

10 Numerical Experiments

In this section, we present some numerical experiments related to a common problem in different fields such as atmospheric sciences or cardiovascular modeling, which is the data assimilation problem (DAP). For artificially truncated computational domains, the DAP consists of identifying the boundary condition which is able to produce a solution minimizing the mismatch with respect to data that has been acquired. Mathematically, this can be considered as the minimization of a fitting functional, having the form (5.5), with respect to a noisy target solution, denoted by $\boldsymbol{v}_\Omega$, which has been corrupted by random errors. To solve the optimization problem, we opted for a steepest descent method, with an optimize-then-discretize approach. As explained in Remark 9.2, assuming some extra regularity, the gradient of the cost functional can be represented as

$$\nabla \mathcal{J}(\widehat{\boldsymbol{g}}) = \tau \widehat{\boldsymbol{g}} + \widehat{\boldsymbol{\eta}}, \tag{10.1}$$

where $\widehat{\boldsymbol{\eta}} \in \boldsymbol{H}_0^1(\Gamma_{in})$ solves (9.8), which corresponds to solving the weak formulation of the 1D Poisson equation

$$-\Delta \widehat{\boldsymbol{\eta}} = \widehat{\boldsymbol{\sigma}}_r - \widehat{\boldsymbol{\sigma}} \quad \text{in } \Gamma_{in}. \tag{10.2}$$

See, for instance, [6, 7] where this approach was also used. We implemented Algorithm 1 in FreeFem++. To simplify the calculations, a fixed step $\varrho = 0.05$ was used in every iteration. As stopping criteria, a cost threshold can be used instead of the gradient norm threshold.

Algorithm 1 Steepest Descent for NS-RDDN

0. Define an initial guess $\boldsymbol{g}^0$ for the optimization procedure;
while (niter<nmax and norm > tol) **do**
 1. Compute the Reference Flow $(\boldsymbol{v}_r^k, p_r^k)$ for the given control $\boldsymbol{g}^k$;
 2. Compute the NS-RDDN Flow $(\boldsymbol{v}^k, p^k)$ for $\boldsymbol{g}^k$, and the cost $\mathcal{J}_k = \mathcal{J}(\boldsymbol{v}^k, \boldsymbol{g}^k)$;
 3. Using $(\boldsymbol{v}^k, p^k)$, compute $(\boldsymbol{z}^k, \boldsymbol{q}^k)$, solution to the Adjoint Problem (8.1);
 4. Compute the Reference Adjoint (z_r^k, q_r^k), using (z^k, q^k) and $(\boldsymbol{v}^k, \pi^k)$;
 5. Solve the 1D equation (10.2) and determine the gradient $\nabla \mathcal{J}_k$ from (10.1);
 6. Update the control $\boldsymbol{g}^{k+1} = \boldsymbol{g}^k - \varrho \nabla \mathcal{J}_k$;
 7. Set norm= $\|\nabla \mathcal{J}_k\|_{2,\Gamma_{in}}$ and niter++
end while

For the numerical approximation of the state and adjoint equations, we employ $\boldsymbol{P}_2/P_1$ finite elements, which are known to be LBB-stable.

10.1 Example 1

Inspired by biomedical applications in computational hemodinamics, we consider the problem of adjusting an artificial Dirichlet boundary condition in order to reproduce a flow profile from which we only have noisy measurements available. In the tests that follow we use the cost functional (5.5) with $\tau = 10^{-5}$.

We consider the stenotic domain represented in Fig. 2a. The inlet is located at $\Gamma_{in} := \{0\} \times (0, 1)$, a RDDN condition with (2.6) and $\delta = 10^{-6}$ is prescribed at $\Gamma_N = \{3.5\} \times (0, 1)$, and the highest depth of the stenosis is $\ell = 0.45$. A parabolic inflow $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}(x_1, x_2) = (V(1 - x_2)x_2, 0)$ is considered at Γ_{in}, and $\boldsymbol{v} = \boldsymbol{0}$ on $\Gamma \setminus (\Gamma_{in} \cup \Gamma_N)$. The parabolic profile presents a mean velocity $V_{mean} = V/6$ and therefore the Reynolds number can be defined as $\mathrm{Re} = 2\ell V_{mean}/\nu = 0.15V/\nu$. The target velocity $\boldsymbol{v}_\Omega$ is the solution to the system (2.4), (2.5) with $\nu = 10^{-3}$, RDDN condition defined above and parabolic inflow with $V = 1.2$, which corresponds to $\mathrm{Re} = 180$. At first, we consider the basic situation in which the target velocity $\boldsymbol{v}_\Omega$ is perfectly known. It serves the purpose of verifying our solution approach.

As previously stated, we use $\boldsymbol{P}_2/P_1$ finite elements, defined in a mesh with diameter $h = 0.0405875$, 33443 triangles, 67666 velocity nodes and 17112 pressure nodes. Moreover, 241 nodes are placed at Γ_{in}, where the control is applied. All numerical results have been obtained by Newton's method with a stabilization in the linearized Navier–Stokes systems. It is important to highlight that with the CDN, for $\mathrm{Re} = 180$, a solution cannot be computed. In fact, the CDN produces solutions until $\mathrm{Re} = 120$. The RDDN condition was tested in simulations of the Navier-Stokes equations until reaching $\mathrm{Re} = 225$.

We tested this minimization scheme assuming as initial guess $\boldsymbol{g}^{(0)} = \boldsymbol{0}$.

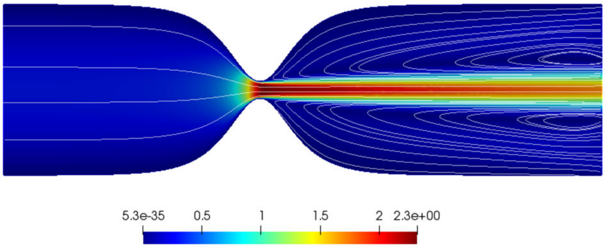
After 100 iterations, we arrived at the result shown in Fig. 2a, with the following relative errors for the velocity in Ω:

$$\frac{\|\boldsymbol{v} - \boldsymbol{v}_\Omega\|_2}{\|\boldsymbol{v}\|_2} = 0.000763927.$$

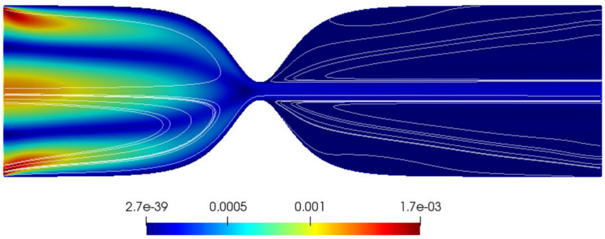
10.2 Example 2

In this example, we simulate a realistic scenario where only noisy data is available. To do so, we revisit the previous example, but now assume that the target velocity is corrupted by white noise (see Fig. 3). The target has a discrepancy of 33.8 % from the "exact" solution, which we recall is given by $\mathcal{S}_\delta(\boldsymbol{g})$, with

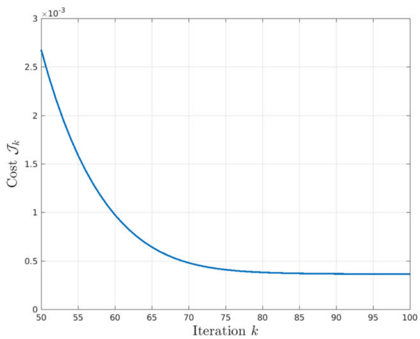
$$\boldsymbol{g}(x_1, x_2) = \begin{cases} (1.2x_2 \times (1 - x_2), 0), & x_1 = 1 \\ (0, 0), & \text{otherwise} \end{cases} \tag{10.3}$$



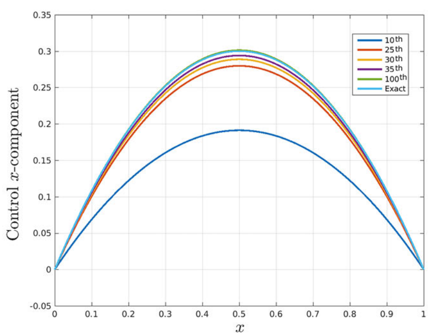
(a) Velocity magnitude and streamlines.



(b) Magnitude of the velocity error.



(c)



(d)

Fig. 2 Results from Example 1. **a** Numerical solution attained after 100 iteration. **b** Shows the difference $\boldsymbol{v} - \boldsymbol{v}_\Omega$. **c** Shows the cost functional value along iteration and **d** illustrates the progression of the controls profiles at Γ_{in} for some selected cases

and δ being the regularization parameter and equal to 10^{-6}. Using the same model parameters and discretization assumptions as in the previous example, and after performing 50 iterations of the descent method (Algorithm 1), we obtained the numerical solution shown in Fig. 3b.

The relative error (L^2-norm) for the optimal state velocity is approximately 1% when compared to the ground truth solution obtained prior to the introduction of

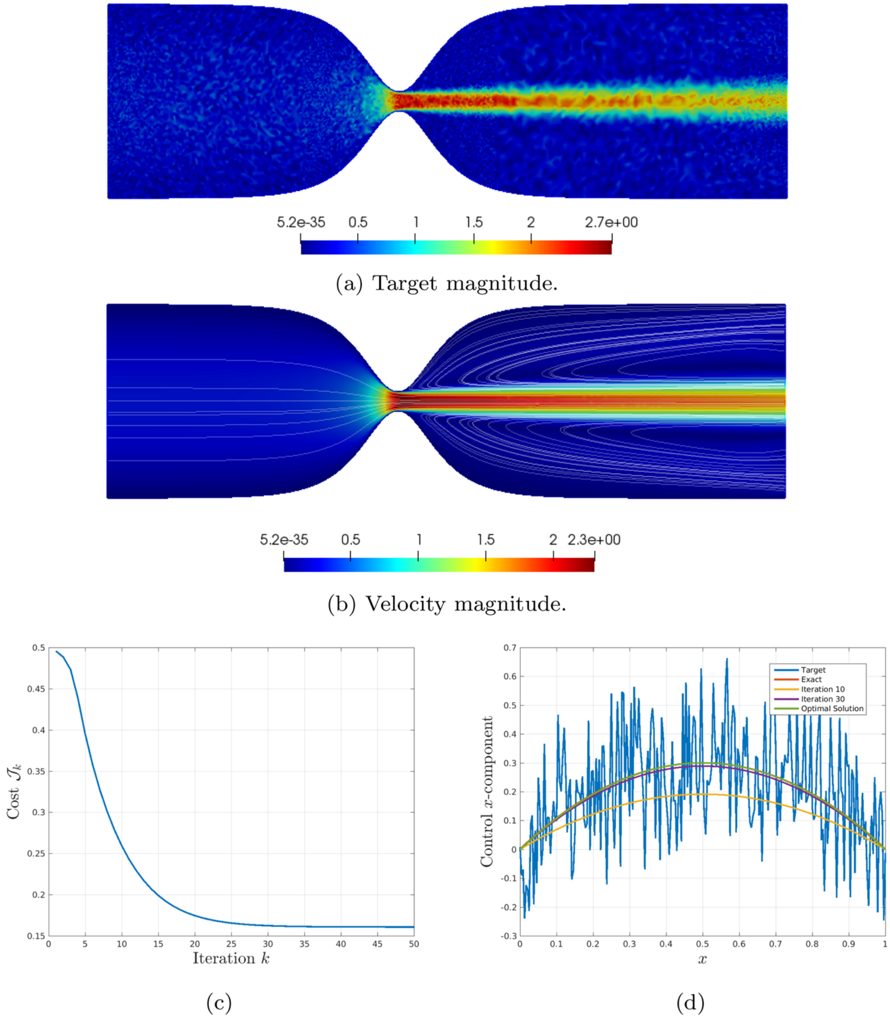


Fig. 3 Results from Example 2: **a** represents the target. After 50 iterations, the flow depicted in **b** is obtained. **c** Illustrates the evolution of the cost functional over the iterations. Finally, **d** presents the trace profiles at Γ_{in} for selected cases

noise. In Fig. 3d, we represent the control variable (Dirichlet boundary condition) after several iterations and compare it with the (exact) profile which was used to obtain the ground truth solution, as well as the noisy solution at the controlled boundary.The optimal solution obtained after 50 iterations closely matches the exact solution. In Fig. 3c, a consistent decrease of the cost functional can be observed.

10.3 Example 3

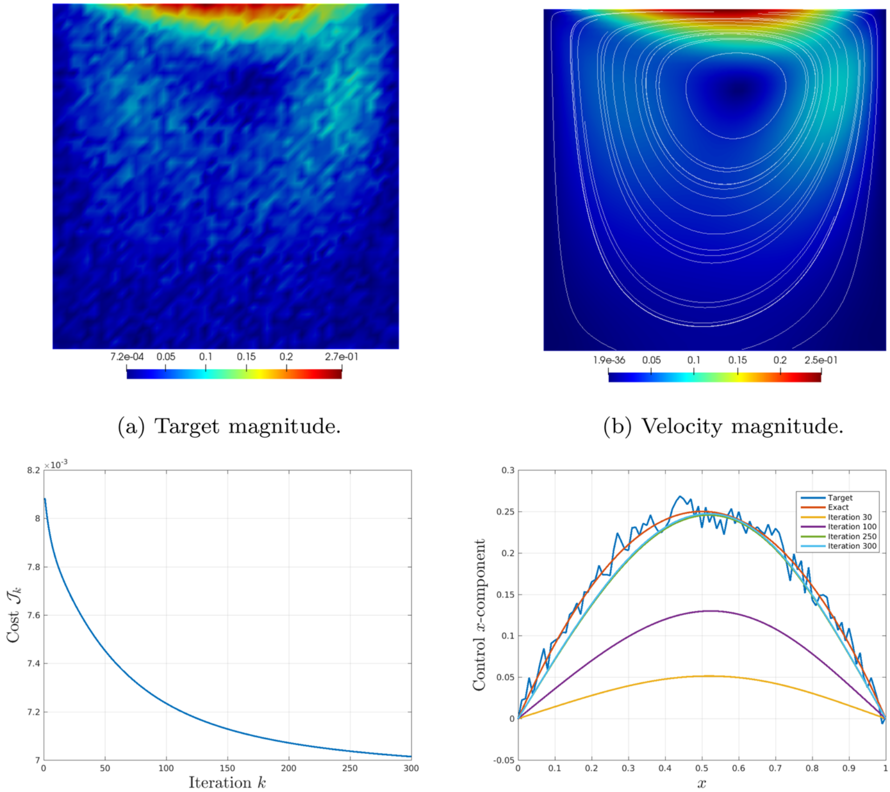
Finally, we test the algorithm for an open cavity configuration. Modeling cavity flows is of major importance in several engineering applications. Again, we consider the problem of flow reconstruction from noisy measurements. Concerning the computational domain, we consider $\Omega = (0, 1)^2$, $\Gamma_N = (0, 1) \times \{0\}$, $\Gamma_D = \Gamma \setminus \Gamma_N$ and we control de Dirichlet boundary condition at $\Gamma_{in} = (0, 1) \times \{1\}$. The mesh is composed by 5000 uniform triangles, 10201 velocity nodes, 2601 pressure nodes and grid size $h = 0.02828$. To mimic the acquired data, we started by generating a ground truth solution as $\mathcal{S}_\delta(\boldsymbol{g})$, where $\delta = 10^{-6}$ and

$$\boldsymbol{g}(x_1, x_2) = \begin{cases} (x_1(1 - x_1), 0), & x_2 = 1 \\ (0, 0), & \text{otherwise.} \end{cases} \tag{10.4}$$

We then perturbed the ground truth solution until reaching a target solution corresponding to 25% of relative error (in the L^2-norm), with respect to the exact solution. After the minimization process, which took 300 iterations, we achieved a relative error of only 4%. Figure 4a and b show the reconstructed (optimized) solution compared with the data (target). In the bottom row we can appreciate the cost function reduction as well as the progression of the control variable at several iterations of the algorithm.

11 Conclusions

Velocity tracking by means of localized boundary controls has been investigated for the Navier–Stokes equations with mixed Dirichlet and RDDN outflow boundary conditions. The state equations are actually a coupled problem which requires to obtain a suitable reference flow whose velocity component is subsequently used to define the outflow boundary condition for the Navier–Stokes equations. Therefore, a two stage analysis was carried out for the control-to-state mapping, as the reference flow, which we defined as a Stokes flow, also depends on the control. The regularity of the RDDN condition is quite relevant when computing the Gâteaux derivative of the control-to-state mapping, as seen throughout the proof of Theorem 7.1. This justifies studying the RDDN condition before the DDN condition in the context of optimal control of Navier–Stokes flows. The first order optimality conditions, stated in Theorem 9.1, are described, as usual, in terms of the solution of the adjoint problem. This is a nontrivial coupled problem, (8.4), (8.5) in strong form, consisting of a linearized Navier–Stokes system, followed by a Stokes system. Our study was supplemented with numerical simulations to demonstrate the advantages of the RDDN condition over the CDN, as well as applications involving cases with only noisy measurements available. Another important reason for preferring the RDDN condition over the DDN condition is the possibility of applying the classical Newton method in the numerical simulations of the state equations.



(a) Target magnitude. (b) Velocity magnitude.

(c) (d)

Fig. 4 Results from Example 3. **a** Is the target. After 50 iterations, the flow in **b** is achieved. **c** Illustrates the cost functional development along iterations. **d** Shows the trace profiles at Γ_{in} for a selection of cases

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Declarations

Competing interests The authors declare no competing interest.

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