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The SU(5) Grand Unification Theory Revisited

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Resumo

Revemos o Modelo Padrão da Física de Partículas (SM) e discutimos as suas limitações e desafios ainda não resolvidos. Propomos então uma extensão através do uso do grupo SU(5) no âmbito de uma Teoria de Grande Unificação (GUT). Desenvolvemos o modelo mínimo, onde por mínimo entendemos como tendo os mesmos campos de matéria do SM, e estudamos as suas consequências - nomeadamente a previsão de decaimento do protão devido a processos que violam os números leptónico e bariónico - e a realização de unificação neste cenário. Construímos ferramentas fenomenológicas para estudar sistematicamente os limites e restrições. O modelo mínimo é posteriormente estendido através da inclusão de outros campos e/ou termos não-renormalizáveis, a fim de salvá-lo de previsões erradas como a que iguala as massas dos quarks down com a dos leptões carregados. Vemos algumas extensões que são escolhidas com o propósito de transformar o modelo mínimo em teorias de massa do neutrino através dos três diferentes mecanismos de seesaw. Concluímos com uma discussão sobre a viabilidade dos modelos GUT baseados em SU(5).

Palavras-chave: Teoria de Grande Unificação (GUT), SU(5), Decaimento do Protão, Unificação, B-Test, Mass de Neutrinos, Tipo-II e Tipo-I+III Mechanismos de Seesaw

Abstract

We review the Standard Model of Particle Physics (SM) and discuss its limitations and challenges left unsolved. We then propose an extension through the use of the group $\mathsf{SU}(5)$ in the context of Grand Unified Theory (GUT). We develop the minimal model, where minimal is understood as having the same matter fields as the SM, and study the consequences, namely the proton decay prediction through Baryon and Lepton number violating processes, and the achievement of unification within the minimal framework. We construct phenomenological tools to systematically study the bounds and constraints. The minimal model is later extended through the inclusion of other fields and/or non-renormalizable terms in order to save it from wrong predictions such as the one which equals the down quarks masses with the charged leptons. Extensions are chosen in order to transform the minimal model into neutrino mass theories through the three different seesaw mechanisms. We conclude with a discussion on the viability of GUT models based on $\mathsf{SU}(5)$.

Keywords: Grand Unification Theory (GUT), $\mathsf{SU}(5)$, Proton Decay, Unification, B-Test, Neutrino Mass, Type-II and Type-I+III Seesaw Mechanism

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Chapter 1

Introduction and Motivation

The Standard Model of Particle Physics (SM) is usually considered the third great scientific discovery of the XXth century, following Quantum Mechanics and the Theory of Relativity, as a consistent, predictive and renormalizable implementation of gauge theories in a quantum field theory (QFT) framework. Even so it is not without its flaws and it does not account for many phenomena in Particle Physics. It is well understood nowadays that New Physics Beyond the Standard Model (BSM) is necessary to explain all the loose ends of the SM. One set of BSM theories is of the Grand Unified Theories (GUTs) which extend the structure of the gauge symmetries of the SM in order to unify the three known gauge couplings at some scale.

In GUTs we embed the SM gauge group, $\mathsf{G_{SM}}$, in a larger group. This larger group may be a single simple group or a product of identical simple groups in order to achieve a coupling unification at some scale where the unified group is the effective gauge group. Just like the SM this scale is such that a Higgs like mechanism breaks the larger group symmetry through a scalar field multiplet.

This work is organized as follows: in the rest of this chapter we will resume the SM in a brief review in order to understand its flaws and limitations. The review will be carried out having the GUT framework in mind and so we will focus great part of our attention on the gauge and interaction features of the SM. In Chapter 2 we will develop the minimal model based on an $\mathsf{SU(5)}$ gauge group, this will have two main purposes which are the identification of problems present in the minimal model, and to develop a coherent notation and working tools to study systematically this kind of models. In Chapter 3 we introduce some extensions of the minimal $\mathsf{SU(5)}$ model, as we will see they can save the theory from its main problems and at the same time provide new and exciting predictions, the chapter is made as an organized review of the more realistic and interesting models that are being studied nowadays. Finally in Chapter 4 we draw our conclusions and state our final remarks on the GUT models based on the $\mathsf{SU(5)}$ group.

As we shall see GUTs are very predictive theories and can be accommodated easily with other BSM extensions such as Supersymmetry. Besides being predictive, which makes them good physical theories, they arise naturally when one enquires the gauge structure of the SM.

In order to understand the motivation behind these theories we will briefly review the SM and account its flaws and unexplained features. This will be a *tour de force* and will let undiscussed some of the SM's aspects. For a full recent review we recommend [1], for a canonical text on gauge theories [2], for a pedagogical text in Portuguese [3] and for full textbooks on the SM and QFT [4–6], for more advanced QFT techniques such as renormalization we also used [7], which is written in a superposition of Portuguese and English. The review that follows was highly based on the previous references and will also serve as an introduction to the notation and conventions that will be used through the rest of this work.

1.1 The Standard Model of Particle Physics

Recall that SM is a gauge theory that consists of a gauge group, roughly speaking responsible for the interactions, and a Higgs Mechanism, that ultimately is responsible for the mass particle generation. These are the two main ingredients of the SM and will also be the main ingredients of any GUT.

Historically, the SM started to be shaped as Glashow [8] in 1961 constructed a gauge theory responsible for the Weak and Electromagnetic interactions. Later, 1967 and '68, Weinberg [9] and Salam [10] developed the model into a consistent theory with masses being originated by a mechanism developed by Brout-Englert-Higgs-Kibble around 1964, that we will refer to as Higgs mechanism [11].

The SM gauge group is constituted by a product of three different groups, each responsible for a different type of interaction, i.e.

$$\mathsf{G_{SM}} \ : \ \mathsf{SU(3)_c} \times \mathsf{SU(2)_L} \times \mathsf{U(1)_Y} \ , \tag{1.1}$$

where the first group is for strong interactions, the quantum numbers associated to it are the so called colours; the second group is the weak interaction group, historically the isospin group; and the last group is the hypercharge group.

The gauge groups account for a set of symmetries under which the Lagrangian is locally invariant. We understand as local symmetries those whose transformations upon the Lagrangian terms have an explicit space-time dependence. If we demand the Lagrangian to be invariant under those transformations we will need to account for vector bosons who will be the mediators of the interactions. This is roughly the definition of a gauge theory.

The particle content is made of three different types of particle: vector bosons (spin 1) responsible for the mediation of the interactions; Dirac fermions (spin 1/2) usually called the matter fields since most of the bound states in nature are naturally constituted by them; and scalar bosons (spin 0) responsible for the breaking of the gauge group into a smaller one.

Table 1.1: SM Vector Bosons Fields, the quantum numbers are regarding ($\mathsf{SU(3)}$,$\mathsf{SU(2)}$,$\mathsf{U(1)}$)

Vector Fields	Quantum Numbers
G^a_μ	($\mathbf{8}$,$\mathbf{1}$,0)
W^a_μ	($\mathbf{1}$,$\mathbf{3}$,0)
B_μ	($\mathbf{1}$,$\mathbf{1}$,0)

The SM vector bosons and fermion fields are listed in Tables 1.1 and 1.2 in the usual SM basis. We note that each fermion field is repeated threefold, indicating that we have three families. Recall that **3** and **2** are the fundamental representations of $\mathsf{SU(3)}$ and $\mathsf{SU(2)}$ respectively, which are the smallest non trivial representation of an $\mathsf{SU(n)}$ group and an overline means conjugated. The hypercharge quantum number being an eigenvalue of the hypercharge operator is represented solely by a number.

Table 1.2: SM Fermionic Fields, the quantum numbers are regarding (SU(3),SU(2),U(1))

Quark Fields	Quantum Numbers	Lepton Fields	Quantum Numbers
$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	($\mathbf{3}$,$\mathbf{2}$,1/3)	$L_L = \begin{pmatrix} \nu \\ e^- \end{pmatrix}$	($\mathbf{1}$,$\mathbf{2}$,-1)
u_R	($\mathbf{3}$,$\mathbf{1}$,4/3)	e_R	($\mathbf{1}$,$\mathbf{1}$,-2)
d_R	($\mathbf{3}$,$\mathbf{1}$,-2/3)		

Note that in the SM all fermions are either on a fundamental representation or in a singlet state, so a field with non vanishing SM charges has the covariant derivative

$$D_\mu = \partial_\mu + ig_3 \sum_{a=1}^{8} G_\mu^a \frac{\lambda^a}{2} + ig_w \sum_{a=1}^{3} W_\mu^a \frac{\sigma^a}{2} + ig_y \frac{Y}{2} B_\mu \ , \tag{1.2}$$

where g_3, g_w and g_y are the couplings for each three interactions, λ^a are the Gell-Mann matrices for SU(3), σ^a are the Pauli matrices for SU(2)[1] and Y is the hypercharge operator.

The covariant derivatives generate the interactions between the gauge bosons and other fields with non vanishing gauge quantum numbers. We also need to add the kinetic term for the gauge boson fields, which is normally called the field strength tensor or curvature tensor. For an abelian gauge symmetry of the type U(1), the respective gauge field strength tensor is

$$A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{1.3}$$

while for an non-abelian gauge symmetry is

$$A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \ , \tag{1.4}$$

where f^{abc} are the group structure constants for each simple subgroup. These terms not only account for the kinematics of the respective fields but they are also necessary for the gauge invariance and also imply self-interactions for the non-abelian cases. In a GUT with only a simple group unifying the gauge sector we will have only a term of this kind while in the SM the gauge sector has three different contributions from the three different interactions

$$\mathcal{L}_{gauge} = -\frac{1}{4} B^{\mu\nu} B_{\mu\nu} - \frac{1}{4} W^{a\mu\nu} W^a_{\mu\nu} - \frac{1}{4} G^{a\mu\nu} G^a_{\mu\nu} \ . \tag{1.5}$$

The gauge bosons in the theory described by the previous lagrangean are massless. A mass term for a gauge field that has the form

$$m^2 A^\mu A_\mu \ , \tag{1.6}$$

[1] Both Gell-Mann and Pauli matrices represent the group generators (apart from a normalization factor) in the fundamental representation. For a general SU(n) these are normally called the generalized Gell-Mann matrices.

would explicitly break the gauge symmetry. This would not be a problem if the gauge bosons were to be massless, but the fields responsible for the weak interactions are observed to be massive.

The Higgs mechanism solves this problem, in it an interacting scalar field acquires a non vanishing vacuum expectation value (vev) which produces a spontaneous symmetry breaking of the theory's gauge symmetries. The vev will also be responsible for the particle mass generation through the coupling of this scalar field with the other fields, without spoiling the unitarity.

In the SM the Higgs mechanism spontaneously breaks the $\mathsf{SU(2)_L} \times \mathsf{U(1)_Y}$ group into the $\mathsf{U(1)_Q}$, the electromagnetism group.

We consider the scalar ϕ which is a doublet of $\mathsf{SU(2)_L}$ with hypercharge $+1$, we shall represent it as

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} , \tag{1.7}$$

the signs in superscript will be explained as we develop the model. The most general renormalizable Lagrangian terms for this field are

$$\mathcal{L} = (D^\mu \phi)^\dagger (D_\mu \phi) - V(\phi^\dagger \phi) , \tag{1.8}$$

where $V(\phi^\dagger\phi)$ is the potential which includes all the non kinetic terms allowed by the symmetries of the theory and the covariant derivative is to respect only to the weak and hypercharge gauge groups, since it is an $\mathsf{SU(3)}$ colour singlet. Noting that a scalar field has mass dimension equal to one and we wish to preserve the renormalizability of the theory, the most general potential has the form

$$V(\phi^\dagger\phi) = \mu^2(\phi^\dagger\phi) + \lambda(\phi^\dagger\phi)^2 . \tag{1.9}$$

Due the scalar nature of this field it can acquire a non vanishing vev, which can be parametrised as

$$\phi_0 = \langle\phi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} , \tag{1.10}$$

with v being a real constant value. The parametrisation accounts for the fact that there is always the freedom to apply a global $\mathsf{SU(2)_L}$ transformation in order to rotate the doublet in the chosen direction.

We now compute v so that the potential (1.9) has an absolute minimum, we start by noticing that only when $\mu^2 < 0$ the potential will have a non zero vev. This is the case we want, or else the value of v would be zero which would lead to no consequences, i.e. to a unbroken gauge symmetry and massless particles. The minimization of the potential leads us to the solution

$$v^2 = -\frac{\mu^2}{\lambda} . \tag{1.11}$$

Before proceeding to the masses of the particles, we note that the when the Higgs doublet acquires the vev it still transforms under two generators of $\mathsf{SU(2)_L}$

$$T^1\phi_0 \neq 0 \ , \ T^2\phi_0 \neq 0 , \tag{1.12}$$

but it does not transform, i.e. is a singlet, of the linear combination of the other two

$$\left(T^3 + \frac{1}{2}Y\right)\phi_0 = 0 . \tag{1.13}$$

We identify this new generator as

$$Q = T^3 + \frac{1}{2} Y \ , \tag{1.14}$$

and the quantum numbers regarding it are just the eigenvalues of the fields, since it is diagonal, and correspond to the quanta of electric charge of the field. It is now clear the choice of signs in superscript when we introduced the scalar ϕ previously in (1.7). Also by being diagonal the gauge group is an $\mathsf{U(1)}$, this is the unbroken (residual) symmetry of the original group.

We now consider the consequences of the non vanishing vev. For that we consider the small oscillations near the vev which we will write with the parametrisation also known as the unitary gauge

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H \end{pmatrix} \ , \tag{1.15}$$

and through its covariant derivative we conclude that three of the four gauge bosons acquire mass due the non vanishing vev. The final physical fields are defined by the relations

$$Z_\mu = \cos\theta_W B_\mu - \sin\theta_W W^3_\mu \tag{1.16}$$

$$A_\mu = \cos\theta_W B_\mu + \sin\theta_W W^3_\mu \tag{1.17}$$

$$W^\pm_\mu = \frac{1}{\sqrt{2}} \left(W^1_\mu \mp i W^2_\mu \right) \ , \tag{1.18}$$

where θ_W is the weak angle, a crucial parameter of the SM, namely

$$\sin^2\theta_W = \frac{g_y^2}{g_w^2 + g_y^2} \ , \tag{1.19}$$

the final mass spectrum of the fields is

$$M_W^2 = \frac{1}{4} v^2 g_w^2 \ , \ M_Z^2 = \frac{1}{4} v^2 (g_w^2 + g_y^2) \ , \ M_A = 0 \ , \tag{1.20}$$

where one can also induce the following important relations[2]

$$M_W = M_Z \cos\theta_W \ , \ \alpha = \alpha_w \sin^2\theta_W = \alpha_y \cos^2\theta_W \ . \tag{1.21}$$

Through the potential we discover that the small oscillations around the vev are massive and represent a boson field with mass

$$m_H^2 = -2\mu^2 = 2\lambda v \ . \tag{1.22}$$

This mechanism is highly predictive. Lets consider the four-fermion Fermi effective theory for the β decay, one can relate the SM parameters with the easily observable and known Fermi's constant. We get the

[2]Recall we can write the couplings in fine structure notation $\alpha^2 = g^2/(4\pi)$.

predictions

$$v = 2M_W/g \simeq (\sqrt{2}G_F)^{-1/2} \simeq 246 \text{ GeV} \ , \tag{1.23}$$

$$\sin^2\theta_W \simeq 0.23 \ , \tag{1.24}$$

$$M_W^{\text{theo}} \simeq 80 - 81 \text{ GeV} \ , \tag{1.25}$$

$$M_Z^{\text{theo}} \simeq 91 - 92 \text{ GeV} \ . \tag{1.26}$$

We note that the range on the prediction of the bosons masses range arises from the 1-loop and the 2-loop calculations. By recent experimental results the 1-loop predictions do not completely agree, but higher order corrections restore the agreement.

Also have in mind that the Z_μ boson was predicted to exist and its mass estimated as above before it was detected experimentally. The confirmation of the Z_μ boson existence with the estimated mass is an outstanding result of the SM. Nevertheless one has to remember that as of the time of this writing no unambiguous evidence for the existence of the Higgs boson H. So if it is mere coincidence or not only the LHC will tell. Of course one should be critical of coincidences, and remember that high precision coincides above all.

We now turn to the generation of fermion masses. We have already seen how the Higgs mechanism in the SM generates masses for the vector bosons, in the case of fermions one gets a rather similar procedure: the interacting Higgs doublet will couple to fermions with some coupling constant, these interaction terms will have such structure that when the scalar acquires its non vanishing vev they become mass terms for the fermions. These terms are the Yukawa terms and we will discuss them now.

A Dirac fermion has a mass term that can be written in chirality states as

$$-\mathcal{L} = m(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R) \ , \tag{1.27}$$

the problem arises when we require R fields to be singlets of $\mathsf{SU(2)_L}$ and so these terms break explicitly the gauge symmetry. To understand this one has not to compute a transformation, but just notice we have a field with non zero $\mathsf{SU(2)_L}$ quantum number, ψ_L, coupled with another without quantum number, ψ_R, so it is impossible to construct an overall group singlet/invariant using only these fields.

Thankfully it is easy to solve this problem. For the sake of the model we had already introduced a scalar doublet with $\mathsf{SU(2)_L}$ charge and non zero hypercharge in (1.7). At that time we imposed the hypercharge of this field to be $+1$ with no apparent reason, but was chosen to be able to construct invariants as we will now discuss. Consider for example the electron, with is a $\mathsf{SU(2)_L}$ doublet and it has hypercharge -1 in order for the electric charge assignment be in agreement with experiment. Since the right-handed electron has Hypercharge -2 for reasons already discussed one can construct a $\mathsf{SU(2)_L} \times \mathsf{U(1)_Y}$ invariant of the form

$$Y\overline{L}_L\phi e_R \ , \tag{1.28}$$

where Y is a Yukawa coupling. Remember that as we have three matter generations, more generally one can give a matrix structure for this couplings: if the fields have well defined masses for different generations then this matrix is diagonal, if not there is mixing and then the matrix is not diagonal. We have then the SM Yukawa sector:

$$-\mathcal{L}_{yuk} = (Y_u)_{mn}(\overline{q}_L)_m\tilde{\phi}(u_R)_n + (Y_d)_{mn}(\overline{q}_L)_m\phi(d_R)_n + (Y_e)_{mn}(\overline{L}_L)_m\phi(e_R)_n + \text{ h.c.} \ , \tag{1.29}$$

where m, n are family indexes.

We note now that we can use left handed charge conjugated fields instead of the right-handed neutrinos via the charge conjugation matrix

$$(\psi^c)_L = C\overline{\psi}_R^T \, , \tag{1.30}$$

the properties of the C matrix and the relations between the spinors are listed and studied at Appendix B. In this notation, which is equivalent to the previous one, we list our particles according to Table 1.3.

Table 1.3: SM Fermionic Fields, the quantum numbers are regarding (SU(3),SU(2),U(1)) in charge conjugation notation

Quark Fields	Quantum Numbers	Lepton Fields	Quantum Numbers
$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\mathbf{3},\mathbf{2},1/3)$	$L_L = \begin{pmatrix} \nu \\ e^- \end{pmatrix}$	$(\mathbf{1},\mathbf{2},-1)$
$(u^c)_L$	$(\overline{\mathbf{3}},\mathbf{1},-4/3)$	$(e^c)_L$	$(\mathbf{1},\mathbf{1},2)$
$(d^c)_L$	$(\overline{\mathbf{3}},\mathbf{1},2/3)$		

Take for example a down quark Yukawa (mass) term, we can then rewrite it as

$$Y_d \overline{q}_L \phi d_R + \text{ h.c.} \rightarrow Y_d q_L^T C \phi^* (u^c)_L + \text{ h.c.} \, , \tag{1.31}$$

which is easier to read the quantum numbers structure and hence to build a group invariant while preserving Lorentz invariance. This is the main reason why this notation is more convenient when studying GUT, since usually one has to incorporate SM R and L fields in the same group multiplet. We finish this introduction to the new notation by noting two things: 1) usually the notation is condensed into a more symbolic way where the T (of transpose) and the C matrix are omitted and regarded as implicit; 2) the kinetic terms are written in the standard notation since it is more straightforward to derive Feynman rules and propagators.

Problems, limitations and less elegant features of the Standard Model

In the beginning of this text we stated the SM not being without its flaws, we will now enumerate them and discuss why they can be problematic.

- Dark Energy and Vacuum Energy

  It was firstly pointed out by Zel'dovich [12] that the scalar potential of the SM below the spontaneous symmetry breaking must be interpreted as a vacuum energy density. One can then compute it as a contribution for the cosmological constant by

$$\Lambda_{SM} = \frac{8\pi}{c^4} G_N V(v) = -\left(\frac{2\pi G_N v^2}{c^4}\right) |\mu|^2 \simeq -(2.5525 \times 10^{-33}) |\mu|^2 \, , \tag{1.32}$$

  and so for a Higgs mass of about 100 GeV one gets

$$\Lambda_{SM} \simeq -1.3 \times 10^{-29} \text{ GeV}^2 \ . \tag{1.33}$$

The current experimental (indirect) measurement of the vacuum density, the overall cosmological constant, is [13]

$$\Lambda_{exp} \simeq 3.9 \times 10^{-84} \text{ GeV}^2 \ . \tag{1.34}$$

The SM contribution for the vacuum energy density has 50 orders of magnitude more relevance than the observed and the sign is the opposite, as it was measured by Perlmutter et al [13]. Of course one might speculate about other contributions that will eventually explain the observations, but cancelling out so many orders of magnitude implies a naturality problem and this is commonly known as the worst in physics.

- Gravity

  The SM does not incorporate gravity. It is not even understood whether gravity is to be treated as a gauge theory since it has not been quantized properly as the other interactions. Theories that try to unify gravity with other interactions have failed to develop a finite QFT for gravity and so it might remain an open problem for some years to come.

  A quantum theory for gravity would eventually also explain the cosmological constant problem, but this also has failed: take for example string theory which worsens the prediction by predicting 100 orders of magnitude apart and keeping the opposite sign.

- Hierarchy problem

  When one computes the Higgs mass with its radiative corrections one gets the contribution

$$\lambda \int^{\Lambda} \frac{1}{k^2 - m_H^2} \ \sim \ \lambda \Lambda^2 \phi^\dagger \phi \ , \tag{1.35}$$

  where Λ is the cutoff scale. The Higgs mass would then be be corrected by

$$\mu_{phys}^2 = \mu^2 + \lambda \Lambda^2 \ . \tag{1.36}$$

  This means that the Higgs mass is radiatively corrected with a square dependence of the scale, and so if one goes to higher energies one finds a fine-tuning problem in order for the Higgs physical mass stay at the same order of magnitude, i.e. not to diverge.

  This is a problem because there is no problem, i.e. there is no formal constraint in fine-tuning the theory's parameters, albeit it is not natural the parameters to be this fine-tuned, this is the same to say it is a naturality problem.

  On the other hand, if we interpret the cut off scale as an energy scale where new physics come to be then we can speculate, by keeping a naturality argument, that there is new physics at ~ 1 TeV.

- No neutrino mass

The SM has no right-handed neutrino, ν_R, and so we can not assign a (Dirac) mass to the left handed neutrino, ν_L, through a (Dirac) mass term. As ν_R would be a singlet of the SM gauge group it was not introduced in the particle content of the theory.

Nevertheless neutrino oscillations are an experimental evidence for the massive nature of neutrinos. By experimental input we do know that at least two of the three neutrinos are massive. The bounds on neutrino masses are at about 1 eV which is a very small value comparing with the rest of the SM mass spectrum.

There is no natural way to explain this in the SM framework. However one can speculate if higher energy physics, BSM physics, might be responsible for neutrino mass generation. An higher energy effect can be introduced in the SM Lagrangian as an effective operator when integrating out the fields responsible for the new physics process, this is called the Weinberg operator [14]. For the SM the $(d = 5)$ operator that would hide the New Physics would be

$$\mathcal{L} = y_{ij}^{SM} \frac{1}{M} (L_i^T C \epsilon_2 H)(H^T \epsilon_2 L_j) + \text{ h.c.} \ , \tag{1.37}$$

where M would be the scale of the new physics, i.e. the mass of the field that was integrated out. Experimentally one can fit y_{ij}^{SM}/M in order to constrain new physics.

As Ma pointed out there are only three possible $(d = 5)$ $\mathsf{SU(2)_L} \times \mathsf{U(1)_Y}$ invariant operators bilinear in L [15], i.e. in the context of the SM a mass term for the neutrino come from a limited set of possible interactions.

The three possibilities for the responsible heavy field are: 1) a singlet field, like ν_R; 2) a scalar triplet; 3) a fermionic triplet. As it is clearer in (1.37), the heavier these fields the lighter will be the left handed neutrinos. These three possibilities are the so called the seesaw mechanisms of type I, II and III respectively.

Also from (1.37) it is immediate that the neutrino will have a contribution from a Majorana mass term

$$\nu_L^T C \nu_L \ , \tag{1.38}$$

which violates any charge. While this is not a problem for the electric charge it violates fermionic number, and so this is responsible for new physics clearly BSM. As we will see, seesaw mechanisms arise naturally in the context of some GUTs.

- Yukawa and Higgs Parameters

The Higgs potential parameters are all arbitrary[3], the only constraints come from viability arguments (parameter space constraints in order for the theory be valid), renormalization constraints (λ has to be positive for the potential be bounded from below, if one renormalize this condition one gets to limits on the Higgs mass).

Other arbitrary quantities in the SM are the Yukawas, whose constraints are only experimental. The Yukawas also impose mass hierarchies upon fermions, these hierarchies are not well understood and do not follow a clear reasoning, e.g. for the first generation the u quark is lighter than the d but for the other families the inverse is true.

[3] By arbitrary quantities we refer to parameters with no underlying physics that would control their values.

Also, the CP-violation parameters in the V_{CKM} are in good agreement with experiment, still the amount of CP violation is not sufficient to account cosmological requirements for baryogenesis to happen.

Any physics proposed for these parameters lies inevitably outside of the SM scope. Nevertheless we know the general structure of a mass matrix, for example for a quark mass matrix we know we have six different masses and four mixing angles accounting a total of 10 parameters. On the other hand in the leptonic sector we account for a total of 12, the higher number of parameters is counter-intuitive as we do not have charged leptons mixing but the possibility for Majorana neutrino mass allows more phases which we can not cancel out as in the case of Dirac masses.

- No family structure

  There are three generations of matter fields in the SM. The reason for this is unknown and impossible to explain in a minimal SM framework. It is fortunate however that for every lepton doublet we have a quark doublet because this makes the SM anomaly free. Apart from this we have no indication on why the gauge representations are the ones observed. On the other hand the large number of free and unconstrained parameters in the Yukawa sector makes the SM a difficult framework to workout family structure symmetries. It is expected that a more constrained Lagrangian in representations and Yukawas will ease the framework to study the family structure and the repetition of the gauge representations.

  Further New Physics to explain the repetition of the three families is necessary. This discussions is beyond the scope of this work, one can check [16] and references contained in [17, 18].

- Gauge group and couplings

  The SM has three different gauge couplings with no physics relating them, this means three different and unrelated parameters. We can always ask the question of why this is so and whether there is some physics that ultimately describes all the gauge interactions in a single structure, being the three different interactions a low energy consequence of that higher theory. As it was already said this is one of the motivations for GUT where the argument is intuitive: what if we can construct a large gauge group which embeds the SM's gauge group and returns the SM at low energies.

  The idea is such that a larger group will be broken through a Higgs-like mechanism into a smaller group. This smaller group will eventually be identified as the $\mathsf{G_{SM}}$, i.e. as a product of three different groups. The different SM's subgroups are subgroups of the larger group which before the breaking is responsible for one unified interaction. The breaking of the larger group will isolate subgroups of the larger group, this process will then define new separate quantum numbers and different interactions.

  When the breaking happens we then will need to redefine the couplings for each unbroken subgroup. The scenario of this work is such that we will want to study the case where we get three different subgroups, and so three different couplings, at the breaking

$$\alpha_U = \alpha_1 = \alpha_2 = \alpha_3 \ . \tag{1.39}$$

  One will want then to identify them as the SM's gauge couplings. But we must keep in mind that the different subgroup generators had their normalization constrained between each other before the breaking, since they formed a full Lie algebra. After the breaking the normalization factors can be hidden into the coupling constants and so one has

$$\alpha_U = k_1\alpha_y = k_2\alpha_w = k_3\alpha_s \ . \tag{1.40}$$

Obviously the k_i depend on the unified group and one can establish classes of unifying groups according to these k_i. One of those classes is called the canonical class of GUT groups, in it we have $k_i \propto (5/3, 1, 1)$ and one of the groups belonging to this class is the $\mathsf{SU}(5)$.

We now run the SM couplings as they depend on the energy scale using the Renormalization Group Equations (RGE). By evaluating how they evolve to high energy scales we can study if they unify at such scales into an $\mathsf{SU}(5)$ unified group using the k_i for the $\mathsf{SU}(5)$. As we want to study the possibility of unification within the SM we will consider only its minimal particle content, the running can be seen in picture 1.1 and we can see that the SM does not unify into $\mathsf{SU}(5)$ out-of-the-box.

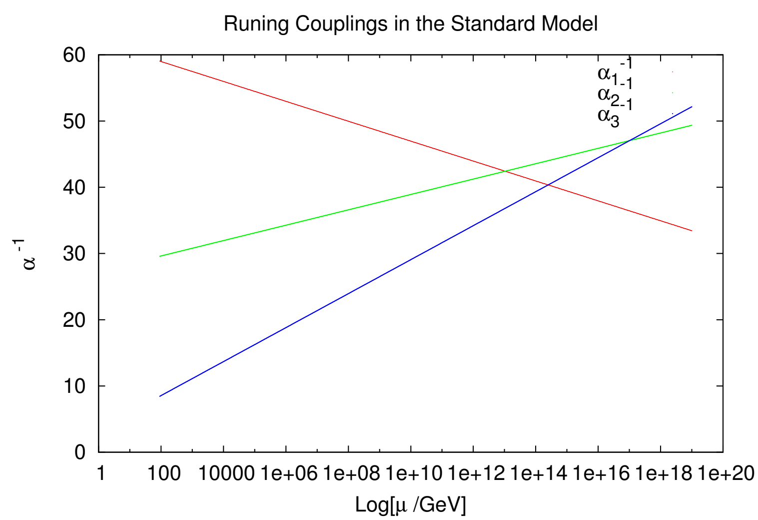


Figure 1.1: Running Couplings in the Standard Model.

Although the unification is not an immediate result we argue that it almost happens. Note that the weak and strong coupling unify at about $\sim 10^{17}$ GeV, if only we got the other coupling to decrease at a slower rate we might achieve unification. Of course one could demand the other couplings to meet α_1 earlier in the running, but as we will see in the next chapter a realistic unified picture involves a large unification scale of $\gtrsim 10^{15}$ GeV.

We finish the discussion on this result with a technical detail. By consulting the RGE in Appendix A one sees that is impossible to diminish the slop of α_1 by including new particles, this is because only vector bosons have positive contributions to the slope. But one can study what k_1 would be needed to achieve unification, the value is about 1.658 which is not very far from $5/3 = 1.65$ so again one might think that unifying the SM gauge couplings into an $\mathsf{SU}(5)$ group has some grounds to be considered.

- No electric charge quantization

The hypercharge is assigned in order for the electric charge to be the same as the one measured experimentally. The SM lacks an *a priori* assignment of the hypercharge and also can not explain the fractional charges of the quarks. A more structured gauge group, where the hypercharge generator would be constrained could cure this. As GUTs give structure to the gauge group we will see that this problem can be solved within a GUT setup.

Chapter 2

The Minimal SU(5) Grand Unification Theory

The prototypical GUTs are those based on the SU(5) gauge group. It was initially proposed by Georgi and Glashow in 1974 [19] and it is the simplest to work with and one can derive highly predictive and complete models with this gauge group as we shall see.

In this chapter we study the SU(5) GUT based theory and the minimal model. Our approach is meant to be a modern pedagogical introduction to the theory by fully constructing the minimal model and discussing all its features and predictions.

There are many references with similar purpose as this chapter available: for a full pedagogical textbook see the last chapter in [6], which follows a somehow different approach in the beginning but constructs rapidly a phenomenologically working model, we advise the reader to look out for some convention inconsistencies throughout the chapter and for some typos in formulae and algebraic results; we also recommend two lecture notes from two different authors and summer schools [20,21] that provide a great insight in the subject albeit being short notes; finally we must refer to the canonical review on the subject by Langacker [22], which offers a complete discussion on the phenomenological results of various GUT theories, unfortunately it dates from 1980 and many results, assumptions and conventions are outdated. Lastly, on group and representation theory we would recommend the textbooks [23,24] and for a complete review with important tables see [25].

2.1 SU(5) Group and Representations

As it was introduced in the previous chapter a GUT consists of a gauge theory where a larger group embeds the SM gauge group in such way that after a sequence of spontaneous symmetry breakings it returns the SM gauge group at low energy scale. If the larger group is simple, as SU(5) is, or a direct product of identical simple groups then when the larger group is effective the gauge coupling is unique, that is the couplings unify which is one of the main motivations for the study of GUTs.

The SM gauge group consists of a total of 12 generators, one by one correspondence with the gauge bosons, with four of them being simultaneously diagonal, i.e. $\text{rank}(G_{SM}) = 4$, the minimal GUT theory will then be based on a rank$=4$ simple group or direct product of identical simple groups. By looking at all possibilities, either classical or exceptional groups, we conclude we don't have much choice if we want to embed the SM group.

We have nine candidates: $\mathsf{SU(2)}^4$,$\mathsf{SO(5)}^2$, G_2^2,SO(8),SO(9),Sp(8),F_4,$\mathsf{SU(3)}^2$ and SU(5). The first two are ruled out because they don not contain an SU(3), the next five do not have complex representations and hence can not reproduce the SM particle content with the observed chiral structure, one can construct theories which would mimic the SM chiral structure by doubling the matter content and so it is not a minimal particle content theory nor a SM embedding. Finally, the $\mathsf{SU(3)}^2$ group would work fine but it is not possible to define an electric charge generator without adding an extensive list of non-SM matter fields. Finally $\mathsf{SU(5)}$ enables us to consider only SM matter fields and with the correct quantum numbers.

Thus, the group with the minimal particle content setup, i.e. the matter fields are the SM ones, that embeds the SM group and preserves a L/R structure for the matter fields is $\mathsf{SU(5)}$, in what follows we assume the breaking pattern

$$\mathsf{SU(5)} \to \mathsf{SU(3)_c} \times \mathsf{SU(2)_L} \times \mathsf{U(1)_Y} \to \mathsf{SU(3)_c} \times \mathsf{U(1)_Q} \ . \tag{2.1}$$

As it was said, the $\mathsf{SU(5)}$ group is a simple group. This means that when it is the effective group, i.e. above the scale M_X where it is broken, the couplings are unified

$$g_1(M_X) = g_2(M_X) = g_3(M_X) = g_5 \ , \tag{2.2}$$

and the covariant derivative for the fundamental representation is

$$\mathsf{SU(5)} : D_\mu = \partial_\mu + i g_5 \sum_{a=0}^{23} A_\mu^a \frac{\lambda^a}{2} \ , \tag{2.3}$$

where we have used the fact that $\mathsf{SU(5)}$ has a total of 24 generators, in contrast with the SM 12 generators. This means we have new vector bosons and so new interactions. These will play an huge part in this theory.

In Appendix C we explicitly define the generalized Gell-Mann matrices for $\mathsf{SU(5)}$ in a basis of interest, this basis is such that the $\mathsf{SU(3)}$ and $\mathsf{SU(2)}$ parts are not overlapped

$$[\lambda^i]^a{}_b \ , \ a, b = \begin{cases} 1,2,3 & \mathsf{SU(3)} \text{ Indexes} \\ 4,5 & \mathsf{SU(2)} \text{ Indexes,} \end{cases} \tag{2.4}$$

and so one can explicitly interpret the gauge indexes and relate them to the SM gauge quantum numbers. This is possible since the SM group is a maximal subgroup of $\mathsf{SU(5)}$ and so one can keep the generators of the different subalgebras of the SM separated by blocks in the new direct sum representation of the generators. The complex vectorial space where the group action acts can then be constructed by using the fundamental representations of $\mathsf{SU(3)}$ and $\mathsf{SU(2)}$ of the SM, namely

$$\mathbf{5} = \mathbf{3} \oplus \mathbf{2} = (\square, \mathbf{1}, -2/3) \oplus (\mathbf{1}, \square, 1) \ , \tag{2.5}$$

the first entry is for $\mathsf{SU(3)}$ quantum numbers, the second for $\mathsf{SU(2)}$ quantum numbers and the last is the SM hypercharge, where we used the hypercharge that a SM field, with the other quantum numbers configuration, has.

We turn our attention to the diagonal generator that does not belong to the Cartan sub algebras of $\mathsf{SU(3)}$ and $\mathsf{SU(2)}$

$$\lambda^{24} = \frac{3}{\sqrt{15}} \begin{pmatrix} 2/3 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} , \tag{2.6}$$

this is a very important generator because we want to read from it the hypercharge operator. This is so since the SM hypercharge generator is a diagonal generator that commutes with the $\mathsf{SU(3)}$ and $\mathsf{SU(2)}$ generators.

By looking at (2.6) we could identify the eigenvalues of it with the ones in (2.5), but it seems this fails badly, for once (2.6) seems to have an overall numerical factor wrong, and second the signs are wrong. Neither of this issues are a problem: first the hypercharge in the SM is a subgroup of a direct product group $\mathsf{G_{SM}}$ so its normalization is not constrained by any commutation relation to the others generators as in $\mathsf{SU(5)}$ and so we can redefine it; regarding the second issue we can always use the anti-fundamental representation

$$\overline{\mathbf{5}} = \overline{\mathbf{3}} \oplus \mathbf{2} = (\overline{\square}, \mathbf{1}, 2/3) \oplus (\mathbf{1}, \square, -1) \ , \tag{2.7}$$

whose SM quantum numbers lead us to immediately identify it with the matter fields

$$\overline{\mathbf{5}}_F = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ -\nu_e \end{pmatrix} . \tag{2.8}$$

The SM hypercharge will now have to be renormalized so we can compare it to the $\mathsf{SU(5)}$ eigenvalues for the diagonal generator, for that we make the identification

$$g_y Y = g_1 \lambda^{24} \ , \tag{2.9}$$

this leads us to

$$c g_1 Y = \sqrt{\frac{3}{5}} g_1 Y \ . \tag{2.10}$$

And we identify the inverse square of c with k_1 in (1.40). We can systematically define these k_i as the factor bewteen the SM generator norm and the GUT generators' normalization, i.e.

$$k_i = \frac{\mathrm{Tr}\{T_i^2\}}{\mathrm{Tr}\{T^2\}} \ , \tag{2.11}$$

where T_i are generators of the SM's subgroup i and T are the unified group's generators. Due to the fact $\mathsf{G_{SM}}$ is maximal subgroup of $\mathsf{SU(5)}$ the other generators will have $k_i = 1$ and so $\mathsf{SU(5)}$ has $k_i = (5/3, 1, 1)$.

It is easy to understand that an overall numerical factor will not change these weight factors between the different generators of the SM interactions and can be absorbed into the unified coupling, so there is an equivalence between groups with similar k_i. We then identify them together and form classes based on these weights structure, the class of groups with $k_i \propto (5/3, 1, 1)$ is the class of the canonical groups. There are nine groups that form the canonical class and they are [26]: $\mathsf{SU(5)}$, $\mathsf{SO(10)}$, $\mathsf{E_6}$, $\mathsf{SU(3)^3 \times Z_3}$, $\mathsf{SU(15)}$, $\mathsf{SU(16)}$, $\mathsf{SU(8) \times SU(8)}$, $\mathsf{E_8}$ and $\mathsf{SO(18)}$.

We still do not have all the SM particle content, the rest of the SM fields can be contained in an antisymmetric **10**-dimensional representation in the following way

$$\mathbf{10}_F = \left(\begin{array}{ccc|cc} 0 & u_3^c & -u_2^c & u_1 & d_1 \\ -u_3^c & 0 & u_1^c & u_2 & d_2 \\ u_2^c & -u_1^c & 0 & u_3 & d_3 \\ \hline -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{array}\right) . \tag{2.12}$$

All the explicit representation theory calculations are performed in Appendix C, in there one can also check the electric charges and the hypercharges of all $\mathsf{SU(5)}$ fields.

It is important to add that albeit we fit all the SM matter fields in a set of two representations we still have to consider three copies of this set in order to account for all families, this means we did not solve the family repetition problem of the SM. Also we do not have a reason for why these representations except for the fact they have the right SM quantum numbers. But not all is bad, we have retrieved the right charge quantization since hypercharge is quantized, remember that in the SM it is arbitrary and assigned only by experimental input.

2.2 Gauge Couplings' Running and Unification

Before we construct the full working model we will discuss now the unification issue. We have already pointed out in the previous chapter that a minimal $\mathsf{SU(5)}$ GUT model does not seem to unify the SM gauge couplings and therefore the theory seems to be of little interest due the definition of a Grand Unification Theory. When the theory was proposed in 1980 this was not a problem due to large experimental uncertainties on the gauge couplings, specially the strong coupling, which motivated the study of the minimal model. As other inconsistencies of the theory were discovered, as we shall derive and discuss them throughout this chapter, minimal extensions where proposed. In many cases these extensions, which will be studied in Chapter 3, can also save the unification and so the theory remains interesting albeit the loss of minimality. It is then important to comprehend the unification failure and to construct tools to systematically study the unification in an extended version of the theory.

As the couplings are dependent of the energy we can write (1.19) as a scale dependent parameter through

$$\sin^2\theta_W(\mu) = \frac{\alpha_y(\mu)}{\alpha_w(\mu)+\alpha_y(\mu)} . \tag{2.13}$$

At the unification scale, Λ_{GUT}, the SM couplings are unified

$$\alpha_5 = \frac{5}{3}\alpha_y = \alpha_w = \alpha_s , \tag{2.14}$$

which means that at Λ_{GUT} we have

$$\sin^2\theta_W(\Lambda_{GUT}) = \frac{1}{1+k_1/k_2} = \frac{3}{8} . \tag{2.15}$$

This result serves as an example on how GUTs reduce the parameters of the gauge structure of a theory: the value of the weak angle is solely determined at GUT scale by group theoretical factors. By knowing how the couplings run with the scale we can then predict the low energy value of the weak angle, which is experimentally well known.

The couplings are evaluated as scale dependent parameters through the Renormalization Group Equations (RGE). The running of a gauge coupling at 1-loop is given by (A.4), we write the result here for completeness

$$\alpha_i^{-1}(\mu_2) = \alpha_i^{-1}(\mu_1) - \frac{b_i}{4\pi} \ln\left(\frac{\mu_2^2}{\mu_1^2}\right) \ , \tag{2.16}$$

with $\mu_1 > \mu_2$. Making use of the low energy results as the integration constants,

$$\alpha^{-1}(M_Z) \simeq 128 \ , \tag{2.17}$$

$$\alpha_s(M_Z) = 0.1184(7) \ , \tag{2.18}$$

$$\sin^2\theta_W(M_Z) = 0.23116(13) \ , \tag{2.19}$$

we have that for energies above the SM the three different couplings' energy dependency is given by

$$\alpha_1^{-1}(\mu) = \alpha^{-1}(M_Z)\frac{3}{5}(1-\sin^2\theta(M_Z)) - \frac{b_1}{4\pi}\ln\left(\frac{\mu^2}{M_Z^2}\right) \ , \tag{2.20}$$

$$\alpha_2^{-1}(\mu) = \alpha^{-1}(M_Z)\sin^2\theta(M_Z) - \frac{b_2}{4\pi}\ln\left(\frac{\mu^2}{M_Z^2}\right) \ , \tag{2.21}$$

$$\alpha_3^{-1}(\mu) = \alpha_3^{-1}(M_Z) - \frac{b_3}{4\pi}\ln\left(\frac{\mu^2}{M_Z^2}\right) \ , \tag{2.22}$$

where $\alpha_3 = \alpha_s$ due to $k_3 = 1$, α is the fine structure constant from electromagnetism which relates to the hypercharge and weak couplings through the relations already presented. The b_i coefficients are group theoretically derived and arise from the gauge symmetries of the theory when computing the 1-loop contributions, they are calculated in Appendix A and read

$$b_1 = \frac{41}{10} \quad b_2 = -\frac{19}{6} \quad b_3 = -7 \ . \tag{2.23}$$

These functions are the ones we already used to study the unification of the SM into an $\mathsf{SU(5)}$ minimal model, which can be seen in Figure 1.1 where we concluded that the unification fails.

This same problem can be seen with a reverse reasoning: consider now the unification assumption, i.e. we impose the unification at some scale and we run the couplings down to the SM scale. The three running couplings down the energy scale are given by

$$\alpha_3^{-1}(\mu) = \alpha_5^{-1} + \frac{b_3}{4\pi}\ln\left(\frac{\Lambda_{GUT}^2}{\mu^2}\right) \ , \tag{2.24}$$

$$\alpha_2^{-1}(\mu) = \alpha^{-1}(\mu)\sin^2\theta_W(\mu) = \alpha_5^{-1} + \frac{b_2}{4\pi}\ln\left(\frac{\Lambda_{GUT}^2}{\mu^2}\right) \ , \tag{2.25}$$

$$\alpha_1^{-1}(\mu) = \alpha^{-1}(\mu)\cos^2\theta_W(\mu)\frac{3}{5} = \alpha_5^{-1} + \frac{b_1}{4\pi}\ln\left(\frac{\Lambda_{GUT}^2}{\mu^2}\right) \ . \tag{2.26}$$

We do not know the unified coupling, α_5, or the unification scale, M_X. They can be computed and this will be useful as it will give us an estimate on the values of these parameters. Keep in mind that although the model is wrong these estimates will be useful as an indication of the values of these parameters. By manipulating the equations, for example by doing $8/3(2.24) - ((2.25) + 5/3(2.26))$, we get

$$\ln\left(\frac{\Lambda_{GUT}^2}{\mu^2}\right) = \frac{12\pi}{-8b_3+3b_2+5b_1}\left[\frac{1}{\alpha(\mu)} - \frac{8}{3}\frac{1}{\alpha_3(\mu)}\right] \ , \tag{2.27}$$

and as the values of α and α_3 are well known at the SM scale we can not only compute M_X but use this result to arrive at other predictions. Namely by doing $(2.25) + 5/3(2.26)$ we get the scale dependence of the weak angle

$$\sin^2\theta_W(\mu) = \frac{3(b_2-b_3)}{5b_1+3b_2-8b_3} + \frac{5(b_1-b_2)}{5b_1+3b_2-8b_3}\frac{\alpha(\mu)}{\alpha_3(\mu)} \ , \tag{2.28}$$

which can be used to compute the SM's weak angle and therefore it is a good way to test the theory.

Finally, by similar algebraic manipulations we get

$$\alpha_5^{-1} = \alpha^{-1}(\mu)\frac{1}{-8b_3+3b_2+5b_1}\left[-3b_3+(5b_1+3b_2)\frac{\alpha(\mu)}{\alpha_3(\mu)}\right] \ . \tag{2.29}$$

We now compute the estimates for these parameters. Considering the b_i coefficients already listed and plugging in experimental values from [27]

$$\alpha^{-1}(M_Z) \simeq 128 \ , \tag{2.30}$$

$$\alpha_3(M_Z) = 0.1184(7) \ , \tag{2.31}$$

we get the following numerical values for the new parameters of the theory

$$\Lambda_{GUT} = 6.73 \times 10^{14} \text{ GeV} \ , \tag{2.32}$$

$$\alpha_5^{-1} = 41.5 \ , \tag{2.33}$$

and the running of the couplings can be seen in Figure 2.1 as they unify at Λ_{GUT}.

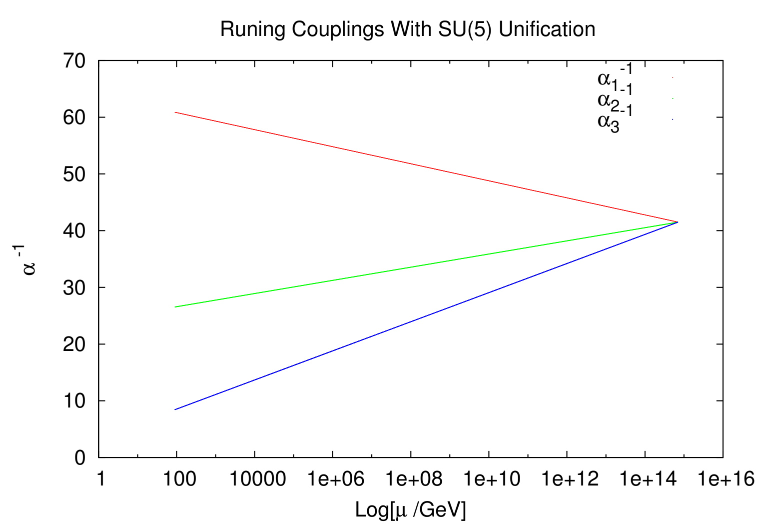


Figure 2.1: Running couplings with unification.

Despite this we have not tested this theory, i.e.we have not compared yet any prediction with an experimentally determined parameter. For this the simplest way is to use the result from (2.28) and to compute its prediction for the SM scale, one gets

$$\sin^2\theta_W(M_Z) = 0.208 \ , \tag{2.34}$$

and this fails, as the experimental value is about 0.231 and so this theory, with this particle content, does not return the SM.

Now that we have a deeper understanding on the running of the couplings we can construct an important phenomenological tool to study unification which is the B-Test which was first proposed by Giveon et al [28].

B-Test and Unification

The reasoning is simple: we want to have a way to easily compare the predictions of a GUT with extended particle content with the low energies experimental values.

Independently of the GUT the running down of the three SM related couplings from an unified gauge group is given by

$$\alpha_i^{-1}(\mu) = \alpha_U^{-1} + \frac{b_i}{4\pi}\ln\left(\frac{\Lambda_{GUT}^2}{\mu^2}\right) \ , \tag{2.35}$$

where obviously $M_Z < \mu < M_{GUT}$. Consider now we have an extended particle sector, each particle I that contributes for the running has a mass M_I with $M_I < \mu < M_{GUT}$. We would then need to consider carefully the particle spectrum and to run each coupling in different ranges where different fields contribute. But if we impose the unification and recalling that the contribution from each particle to a given coupling is highly sensible to the contribution for the b_i, we can then constrain the masses of these particles.

Consider then an overall contribution to the b_i given by

$$B_i = b_i + \sum_I b_i^I r^I \ , \ r^I = \frac{\ln(\Lambda_{GUT}/M_I)}{\ln(\Lambda_{GUT}/M_Z)} \ , \tag{2.36}$$

where b_i are te SM coefficients, b_i^I is the contribution from the particle I and r^I accounts for a relevance factor due to its mass: note for high masses r^I is very small but grows fast as the mass approaches the weak scale.

Now one can derive the B-Test [28] by defining $B_{ij} = B_i - B_j$

$$B = \frac{B_{23}}{B_{12}} = \frac{\sin^2\theta_W - \frac{k_2}{k_3}\frac{\alpha}{\alpha_s}}{\frac{k_2}{k_1} - \left(1 + \frac{k_2}{k_1}\right)\sin^2\theta_W} \ , \tag{2.37}$$

where the RHS is given by low energy values and group theory results and the LHS is given by the particle content of the theory. We have then constrained the new particles masses with both the unification requirement and with low energy values. This is an easy to use phenomenological tool to study unification and to make predictions.

A GUT model unified with the $\mathsf{SU(5)}$ has a B-Test constraint of

$$B = 0.718 \pm 0.003 \ , \tag{2.38}$$

while the SM B-Test value is

$$B^{SM} \simeq 0.53 \ , \tag{2.39}$$

and, as we already know, the unification fails within a minimal SM framework.

This test is really useful because if we want to make a unified model, and regarding the fact the SM is the low energy ($\sim M_Z$) effective theory we will need to add particles that either increases the B_{23} contribution or reduces the B_{12}, or contribute in such way the above fraction rises. This means we will want light particles with favourable $b_2 - b_3$ and $b_1 - b_2$ contributions, while the unfavourable particles remain heavy.

2.3 SU(5) Lagrangian

We now discuss the Lagrangian of the minimal SU(5) gauge theory which is eventually broken in the direction of the SM. We have shown already that the theory at its minimal value is not realistic, but any extension will be made upon the minimal configuration and so this section will be useful in order to understand the depth of the different problems SU(5) GUTs face and to establish the conventions and notations for the rest of the text.

Just like the SM, the SU(5) Lagrangian can be separated into different sectors

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{int}}^{F} + \mathcal{L}_{\text{int}}^{S} + \mathcal{L}_{\text{yuk}} - V \ , \tag{2.40}$$

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} A^{a\mu\nu} A^{a}_{\mu\nu} \ . \tag{2.41}$$

From now on we will shorten the notation by defining the matrix of all gauge fields

$$\underset{\sim}{A}_\mu = \sum_{a=1}^{24} A^a_\mu \frac{\lambda^a}{2} \ , \tag{2.42}$$

the quantum numbers are explicitly computed in Appendix C and the matrix form for the gauge bosons is

$$\underset{\sim}{A}_\mu = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|cc} G^1_{1\mu} + \frac{2B_\mu}{\sqrt{30}} & G^1_{2\mu} & G^1_{3\mu} & X^{1c}_\mu & Y^{1c}_\mu \\ G^2_{1\mu} & G^2_{2\mu} + \frac{2B_\mu}{\sqrt{30}} & G^2_{3\mu} & X^{2c}_\mu & Y^{2c}_\mu \\ G^3_{1\mu} & G^3_{2\mu} & G^3_{3\mu} + \frac{2B_\mu}{\sqrt{30}} & X^{3}_\mu & Y^{3c}_\mu \\ \hline X^1_\mu & X^2_\mu & X^3_\mu & \frac{Z_\mu}{\sqrt{2}} - \sqrt{\frac{3}{10}} B_\mu & {W_\mu}^+ \\ Y^1_\mu & Y^2_\mu & Y^3_\mu & {W_\mu}^- & -\frac{Z_\mu}{\sqrt{2}} - \sqrt{\frac{3}{10}} B_\mu \end{array} \right) . \tag{2.43}$$

Now we will study the covariant derivatives for the two fermionic representations. Again we refer to Appendix C for explicit computations on the representation theory algebra on how different representations transform.

Recall that a covariant derivative is obtained by computing the derivative term if one imposes a gauge invariance, i.e. if one demands the derivative term to transform as the field in which it is applied. For a Dirac field in the fundamental representation this means

$$(\not{D}\Psi)' = U\not{D}\Psi \ . \tag{2.44}$$

The covariant derivative is obtained by computing what is missing in the partial derivative term in order to the transformation rule be satisfied. For the fundamental representation of a $\mathsf{SU(n)}$ group this is very similar to the SM. We will do this generically now for the fundamental representations as it will lead us to the transformation rules of the gauge fields for our sign conventions.

We start by computing the extra term that appears in the derivative term when the field transforms under a gauge transformation

$$\partial_\mu \Psi' = \partial_\mu U \Psi = U \partial_\mu \Psi + \left(-i \partial_\mu \underset{\sim}{\alpha}\right) U \Psi \ , \tag{2.45}$$

Consider that for the fundamental, **5**, the covariant derivative has the form

$$D_\mu = \partial_\mu + i g_5 \underset{\sim}{A}_\mu \ , \tag{2.46}$$

the idea, just like in the SM, is that we included the vector fields to assimilate the terms that arose from the derivative when we transformed the field Ψ. We need now to study the transformation rules of these vector fields in order for the Lagrangian be invariant under $\mathsf{SU(5)}$ transformations.

Note that for an anti-fundamental, $\overline{\mathbf{5}}$, representation the transformation is complex conjugated of the one of the fundamental representation, so for the $\mathsf{SU(5)}$ anti-fundamental fermion field we will instead use

$$D_\mu = \partial_\mu - i g_5 \underset{\sim}{A}_\mu^T \ . \tag{2.47}$$

Continuing the computation of the transformation rules for the gauge fields we now have to use the covariant derivative in (2.44), explicitly

$$(D_\mu \Psi)' = \left(\partial_\mu + i g_5 \underset{\sim}{A}'_\mu\right) U \Psi = U \partial_\mu \Psi + (\partial_\mu U) \Psi + i g_5 \underset{\sim}{A}'_\mu U \Psi = U (D_\mu \Psi) \ , \tag{2.48}$$

this is true if

$$\underset{\sim}{A}'_\mu = U \left[\underset{\sim}{A}_\mu + \frac{1}{g_5} \partial_\mu \underset{\sim}{\alpha}\right] U^\dagger = U \left[\underset{\sim}{A}_\mu + \frac{1}{i g_5} \partial_\mu\right] U^\dagger \ . \tag{2.49}$$

What we have done is in fact the procedure for computing the transformation rules for any $\mathsf{SU(n)}$ gauge boson and to determine the covariant derivative for its fundamental representation.

For the antisymmetric **10** representations the covariant derivative will be different, in Appendix C we prove that a **10** transforms as

$$\Psi' = U \Psi U^T \ , \tag{2.50}$$

so the covariant derivative must be such as

$$(D_\mu \Psi)' = U (D_\mu \Psi) U^T \ . \tag{2.51}$$

The correct form of the covariant derivative is in fact

$$D_\mu \Psi = \partial_\mu \Psi + i g_5 \left\{\underset{\sim}{A}_\mu \Psi + \Psi \underset{\sim}{A}_\mu^T\right\} \ , \tag{2.52}$$

one can guess this by checking how the partial derivative acts on a transformed **10**

$$\partial_\mu \Psi' = \partial_\mu U \Psi U^T = U (\partial_\mu \Psi) U^T + (\partial_\mu U) \Psi U^T + U \Psi (\partial_\mu U^T) \tag{2.53}$$

Now we can construct the covariant derivative terms for the fermionic fields. We have

$$\mathcal{L}^F = i\frac{1}{2}\mathrm{Tr}\left\{\overline{\Psi}_{10}\not{D}\Psi_{10}\right\} + i\overline{\Psi}_{\overline{5}}\not{D}\Psi_{\overline{5}} = \frac{1}{2}i\mathrm{Tr}\left\{\overline{\Psi}_{10}\not{\partial}\Psi_{10}\right\} + i\overline{\Psi}_{\overline{5}}\not{\partial}\Psi_{\overline{5}} + \mathcal{L}^F_{\mathrm{int}} \ , \tag{2.54}$$

$$\mathcal{L}^F_{\mathrm{int}} = -g_5\mathrm{Tr}\left\{\overline{\Psi}_{10}\gamma^\mu \underset{\sim}{A}_\mu\Psi_{10}\right\} + g_5\overline{\Psi}_{\overline{5}}\gamma^\mu \underset{\sim}{A}^T_\mu\Psi_{\overline{5}} \ . \tag{2.55}$$

We will work with this result latter on. For now note that the SM gauge bosons will add nothing new to the SM interactions, on the other hand X_μ and Y_μ constitute an $\mathsf{SU(2)}$ doublet and $\mathsf{SU(3)}$ triplet and so it can connect a lepton line with a quark line, which does not happen in the SM. These interactions will be responsible for (a type of) proton decay which was never observed experimentally and one can speculate this problem to be somehow a general problem of GUT theories since they enlarge the interactions possibilities.

It will be useful to separate the gauge bosons matrix in SM and new interactions

$$\underset{\sim}{A}_\mu = \underset{\sim}{A}^{\mathrm{SM}}_\mu + \frac{1}{\sqrt{2}}\begin{pmatrix} & & & X^{1c}_\mu & Y^{1c}_\mu \\ & & & X^{2c}_\mu & Y^{2c}_\mu \\ & & & X^{3c}_\mu & Y^{3c}_\mu \\ X^1_\mu & X^2_\mu & X^3_\mu & & \\ Y^1_\mu & Y^2_\mu & Y^3_\mu & & \end{pmatrix} \equiv \underset{\sim}{A}^{\mathrm{SM}}_\mu + \underset{\sim}{A}^X_\mu \ . \tag{2.56}$$

Higgs Sector and Potential

It was not mentioned so far, but the scalar sector of the $\mathsf{SU(5)}$ is more extended than the one of the SM if one wants to break the gauge in a realistic fashion. This means that the minimal $\mathsf{SU(5)}$ theory has in fact more fields than the SM just because one has to break the symmetry at some point. One can easily check that these fields do not save the unification problem as their masses, as we will see when the spontaneous symmetry breaking of the $\mathsf{SU(5)}$ group is discussed, are expected to be at the GUT scale.

The minimal Higgs sector will be constituted by two scalar representations: a **24** and a **5**, which will be denoted as $\mathbf{24}_H$ and $\mathbf{5}_H$ respectively. The $\mathbf{24}_H$ will be used to break $\mathsf{SU(5)}$ while $\mathbf{5}_H$ has the SM Higgs doublet and so it will break the SM into the electromagnetism.

The $\mathbf{24}_H$ is in the adjoint representation and so it does not break the rank of the group [29], recall $\mathsf{SU(5)}$ has the same rank as the SM group. This will also be explicitly shown as we study further below the spontaneous symmetry breaking.

We construct the $\mathbf{24}_H$ just like we constructed the gauge boson matrix. We have then

$$\mathbf{24}_H = \sum_{a=1}^{24}\phi^a\lambda^a \ , \tag{2.57}$$

and so its $\mathsf{SU(5)}$ indices are $(\mathbf{24}_H)^i{}_j$, one can check in Appendix C for explicit calculations, quantum numbers, and transformation rules.

The $\mathbf{5}_H$ is in the fundamental representation and so we write it down as

$$\mathbf{5}_H = \begin{pmatrix} T \\ H \end{pmatrix} \ , \tag{2.58}$$

where T denotes an $\mathsf{SU(3)}$ colour triplet.

The covariant derivative term for the $\mathbf{5}_H$ field is trivial and was already deduced, while for the $\mathbf{24}_H$ we note that it must transform such as

$$(D_\mu \Phi)' = U D_\mu \Phi U^\dagger \ . \tag{2.59}$$

The correct expression for the covariant derivative can be easily proven to be

$$D_\mu \mathbf{24}_H = \partial_\mu \mathbf{24}_H + i g_5 \left[\underset{\sim}{A}_\mu, \mathbf{24}_H \right] \ . \tag{2.60}$$

The two derivative terms are then

$$\mathcal{L}^S = \frac{1}{2} \mathrm{Tr} \left\{ (D_\mu \mathbf{24}_H)^\dagger (D^\mu \mathbf{24}_H) \right\} + (D_\mu \mathbf{5}_H)^\dagger (D^\mu \mathbf{5}_H) \ . \tag{2.61}$$

All the $\mathbf{24}_H$ non-derivative terms which respects the gauge symmetry form the potential

$$V(\mathbf{24}_H) = -\frac{\mu^2}{2} \ \mathrm{Tr} \left\{ \mathbf{24}_H^2 \right\} + \frac{a}{4} \ \mathrm{Tr} \left\{ \mathbf{24}_H^2 \right\}^2 + \frac{b}{4} \ \mathrm{Tr} \left\{ \mathbf{24}_H^4 \right\} + \frac{c}{3} \ \mathrm{Tr} \left\{ \mathbf{24}_H^3 \right\} \ , \tag{2.62}$$

and all $\mathbf{5}_H$ non-derivative terms form the potential

$$V(\mathbf{5}_H) = -\frac{\mu_5^2}{2} \mathbf{5}_H^\dagger \mathbf{5}_H + \frac{a_5}{4} (\mathbf{5}_H^\dagger \mathbf{5}_H)^2 \ , \tag{2.63}$$

one still needs to consider all the terms with both H and Φ, these are

$$V(\mathbf{24}_H, \mathbf{5}_H) = \alpha \mathbf{5}_H^\dagger \mathbf{5}_H \ \mathrm{Tr} \left\{ \mathbf{24}_H^2 \right\} + \beta \mathbf{5}_H^\dagger \mathbf{24}_H^2 \mathbf{5}_H + c_1 \mathbf{5}_H^\dagger \mathbf{24}_H \mathbf{5}_H \ , \tag{2.64}$$

putting all together we have then the potential

$$V = V(\mathbf{24}_H) + V(\mathbf{5}_H) + V(\mathbf{24}_H, \mathbf{5}_H) \ . \tag{2.65}$$

The Yukawa sector is simpler, there are only two $\mathsf{SU}(5)$ and Lorentz invariant terms one can build with the already studied fermion and scalar representations

$$\mathcal{L}_Y = \overline{\mathbf{5}}_F Y_5 \mathbf{10}_F \mathbf{5}_H^* + \frac{1}{8} \epsilon_5 \mathbf{10}_F Y_{10} \mathbf{10}_F \mathbf{5}_H + \text{ h.c.} \ , \tag{2.66}$$

note this is a symbolic way to put these terms, Lorentz invariance is implicit as

$$\mathcal{L}_Y = \overline{\mathbf{5}}_F^T C Y_5 \mathbf{10}_F \mathbf{5}_H^* + \frac{1}{8} \epsilon_5 \mathbf{10}_F^T C Y_{10} \mathbf{10}_F \mathbf{5}_H + \text{ h.c.} \ , \tag{2.67}$$

and in either way the family indexes are omitted. For the next discussion we will not need the explicit notation and we can work with the more symbolic (2.66). In fact one can work in a even more symbolic approach by working with 3×3, 2×2 and 3×2 blocks for the $\mathsf{SU}(3)$, $\mathsf{SU}(2)$ and mixed quantum numbers indexes respectively. We then compute the Yukawa terms relevant for the SM Yukawa sector

$$\overline{\mathbf{5}}_F Y_5 \mathbf{10}_F \mathbf{5}_H^* = \begin{pmatrix} d^c & \epsilon_2 L \end{pmatrix} Y_5 \begin{pmatrix} \epsilon_3 u^c & q \\ -q^T & \epsilon_2 e^c \end{pmatrix} \begin{pmatrix} T^* \\ H^* \end{pmatrix} = \epsilon_2 L Y_5 \epsilon_2 e^c H^* + q Y_5^T d^c H^* + \text{ (T terms)} \ , \tag{2.68}$$

by the usual definition of the Yukawas in (1.29) one concludes that

$$Y_e = Y_d^T \ , \tag{2.69}$$

which means that at Λ_{GUT} one has $m_e = m_d$, and equivalently for the other generations.

Concerning the Y_{10} part is more tricky since we have a ϵ_5 to work out, with all $\mathsf{SU(5)}$ indices explicit we have then

$$\epsilon_5 \mathbf{10}_F^T Y_{10} \mathbf{10}_F \mathbf{5}_H = \epsilon_{ijklm} (\mathbf{10}_F^T)^{ij} Y_{10} \mathbf{10}_F^{kl} \mathbf{5}_H^m \ , \tag{2.70}$$

and we will only retrieve the SM Yukawa terms, for that we have to consider only the last two entries in H which corresponds to limiting the index m to the range $m = 4, 5$. In order to manipulate the expression we will order the $\mathsf{SU(5)}$ indices such way we will have three $\mathsf{SU(3)}$ indices ranging $\alpha, \beta, \gamma = 1, .., 3$ and two $\mathsf{SU(2)}$ indexes with range $a, b = 1, 2$.

When ordering the indices one will have to be careful not to forget terms, for example when interchanging ordered $\gamma a \leftrightarrow a\gamma$ we have the same term, but before the ordering it would correspond for two different terms and so a factor of two must be added. Having this in mind we obtain

$$-\epsilon_{ijklm} \mathbf{10}_F^{ij} Y_{10} \mathbf{10}_F^{kl} \mathbf{5}_H^m \to -2\epsilon_{\alpha\beta\gamma}\epsilon_{ab} \mathbf{10}_F^{\alpha\beta} (Y_{10} + Y_{10}^T) \mathbf{10}_F^{\gamma a} \mathbf{5}_H^b \ , \tag{2.71}$$

and one finally gets

$$-2\epsilon_{\alpha\beta\gamma}\epsilon_{ab} \mathbf{10}_F^{\alpha\beta} (Y_{10} + Y_{10}^T) \mathbf{10}_F^{\gamma a} \mathbf{5}_H^b = -4q(Y_{10} + Y_{10}^T) u^c \epsilon_2 H \ . \tag{2.72}$$

This means

$$Y_u = Y_u^T \ , \tag{2.73}$$

which is not as restrictive as (2.69)[1] but still is more restrictive than what one gets in the SM framework, recall that the SM has no physical restrictions on the Yukawa matrices.

The SM Yukawa sector is then given by

$$\mathcal{L}_Y^H = L Y_5 e^c H^* + q Y_5^T d^c H^* - \frac{1}{2} q (Y_{10} + Y_{10}^T) u^c \epsilon_2 H + \text{ h.c.} \ , \tag{2.74}$$

which means we have only two Yukawa matrices instead of the three in the SM. These predictions and constraints are powerful and seem exciting, but we must ask the question: are they right? For that remind that (2.69) and (2.73) are valid at Λ_{GUT}, so we must run the masses from the SM scale into the unification scale and see if the down quark masses coincide with the charged letpons masses, for that we use the RGE which are computed Appendix A.

We have then to evaluate numerically

$$\begin{aligned} \log\left(\frac{m_{d_i}(t)}{m_{e_i}(t)}\right) &= \log\left(\frac{m_{d_i}(0)}{m_{e_i}(0)}\right) + \frac{2}{b_1}\log\left(\frac{g_1(t)}{g_1(0)}\right) - \frac{8}{b_3}\left(\frac{g_3(t)}{g_3(0)}\right) \\ &= \log\left(\frac{m_{d_i}(0)}{m_{e_i}(0)}\right) + \frac{1}{b_1}\log\left(\frac{\alpha_1(t)}{\alpha_1(0)}\right) - \frac{4}{b_3}\left(\frac{\alpha_3(t)}{\alpha_3(0)}\right) \ , \end{aligned} \tag{2.75}$$

for that consider for example what seems to us to be a reasonable estimate for the unified coupling

[1] In the sense that (2.69) restricts the masses of the down quarks and charged leptons, which is a constraint that does not exist naturally in the SM.

$$\alpha_5 \sim 0.025 \ , \tag{2.76}$$

and note that at low energies, i.e. at the SM (lightest) particles mass scale, the strong coupling varies throughout a wide spectrum [27]

$$\alpha_3(\Lambda_d) \gg 0.3 \ , \ \alpha_3(\Lambda_s) > 0.3 \ , \alpha_3(\Lambda_b) \sim 0.2 \ , \tag{2.77}$$

plus the fact the quark masses are a very difficult to measure physical parameter this makes this exercise highly imprecise in such a rough approach. For the heaviest family, where the masses and the couplings are better understood, we have

$$\frac{m_b}{m_\tau} \sim 0.79 \ , \tag{2.78}$$

which means the prediction fails even for the heaviest family where it is less problematic. As we will see in the next chapter this prediction is not present in some extensions and by doing so the theory can be made realistic again, although of course it loses a (wrong) unique prediction which does not arise naturally in the SM.

2.4 Spontaneous Symmetry Breaking of $\mathsf{SU(5)}$

In order for the theory to be realistic we need to spontaneously break the $\mathsf{SU(5)}$ gauge group into the SM. For a generic GUT we may need to break the group several times, but the $\mathsf{SU(5)}$ is a small group and it can be immediately broken into the SM. The field responsible for this process is the **24** scalar, denoted by Φ, which will break $\mathsf{SU(5)}$ into the SM group

$$\Phi : \mathsf{SU(5)} \rightarrow G_{SM} \ , \tag{2.79}$$

as was said and it will be clear throughout this section we choose an adjoint scalar field since it does not break the rank of the group, i.e. we preserve the Cartan sub-algebra of the group and so we will break the group into a maximal sub-group of the larger one. Recall that in the SM we use a fundamental representation scalar which breaks the rank by a factor of one, these results can be seen in [29]. The group will be broken when the scalar Φ acquires a non-vanishing vev

$$\Phi_0 = \langle \Phi \rangle \ , \tag{2.80}$$

which will break the group into a particular direction of the group. The direction, or the subgroup in which it will break, is constrained by the potential (2.62). Since we can only create group invariants from an adjoint using traces we can always apply a global $\mathsf{SU(5)}$ transformation so that Φ_0 will be diagonalized with real eigenvalues

$$\Phi^i_{0j} = \phi^i_0 \delta^i{}_j \ , \tag{2.81}$$

in fact the potential can be written as a one parameter matrix and there are not many possibilities for the breaking pattern as we shall see next.

Minimizing the Potential and Breaking Patterns

The minimization of the potential of an adjoint scalar is non trivial. A thorough study on the subject as well the final results can be seen in Appendix D and we refer here only to the conclusions. There are only three extrema for the potential of the form (2.62) and they can bee written as a diagonal one parameter matrices as

$$\langle \mathbf{24}_H \rangle = \begin{cases} v \frac{1}{\sqrt{15}} \text{ diag}(2,2,2,-3,-3) \ , \\ v_{41} \text{ diag}(1,1,1,1,-4) \ , \\ \text{diag}(0,0,0,0,0) \ , \end{cases} \tag{2.82}$$

where the first breaks into the SM, the second into $\mathsf{SU(4)} \times \mathsf{U(1)}$ and the last keeps $\mathsf{SU(5)}$ unbroken. We note that breaking into the SM is to let the group be broken into the λ_{24} direction, which means the breaking chooses the final hypercharge.

For a simplified version of the potential (2.62) with $c = 0$, which means an extra Z_2 symmetry for the scalar fields, the SM vev can be computed for the minimal potential value with the above structure, and one gets

$$v^2 = \frac{15\mu^2}{30a + 7b} \ . \tag{2.83}$$

For reference sake we note that for the general case $c \neq 0$, this means $\not{\mathsf{Z}}_2$, the value of v will be different and it reads

$$v^2 = 15 \left(\frac{c \pm \sqrt{c^2 + 4(30a+7b)\mu^2}}{60a + 14b} \right)^2 \ . \tag{2.84}$$

One thing we conclude is that in the $c \neq 0$ case we have a non unique minimized configuration for the potential which will make things more complicated. But for what is worth the minimal model we will use (2.83), it will enable us to construct rapidly a working model and study the global structure. We can also consider a t'Hoof type conjecture to hypothesize a small value for c and work with the Z_2 symmetry.

Gauge Bosons Masses

Due to the derivative term of the scalar field (2.61) the non vanishing vev will generate masses for some of the gauge bosons.

Recall the covariant derivative for an adjoint field (2.60), so that the interaction term for a adjoint scalar is then

$$\mathcal{L}_H^{\mathbf{24}} = \frac{1}{2}\text{Tr}\left\{(D_\mu \mathbf{24}_H)^\dagger (D^\mu \mathbf{24}_H)\right\} = \frac{1}{2}\text{Tr}\left\{\left(\partial_\mu \mathbf{24}_H + ig_5 \left[\underset{\sim}{A}_\mu, \mathbf{24}_H\right]\right)^\dagger \left(\partial^\mu \mathbf{24}_H + ig_5 \left[\underset{\sim}{A}^\mu, \mathbf{24}_H\right]\right)\right\} \ , \tag{2.85}$$

and when it acquires an vev it reduces to

$$\mathcal{L}^{\langle \mathbf{24}_H \rangle} = \frac{1}{2}\text{Tr}\left\{\left(ig_5 \left[\underset{\sim}{A}_\mu, \langle \mathbf{24}_H \rangle\right]\right)^\dagger \left(ig_5 \left[\underset{\sim}{A}^\mu, \langle \mathbf{24}_H \rangle\right]\right)\right\} \ , \tag{2.86}$$

which is now straightforward to compute. We get the mass terms

$$\mathcal{L}^{\langle \mathbf{24}_H \rangle} = \mathcal{L}_m^X = \frac{5}{6} g_5^2 v^2 \left(\overline{X_\mu^i} X^{i\mu} + \overline{Y_\mu^i} Y^{i\mu} \right) \ , \tag{2.87}$$

this means we have the mass spectrum

$$M_X^2 = M_Y^2 = \frac{5}{6} g_5^2 v^2 \ . \tag{2.88}$$

Note that only the non SM gauge bosons acquire mass, the remaining gauge symmetry is then the SM gauge group.

As we already did before we will sometimes refer to the breaking scale as the same mass scale of these bosons, $\Lambda_{GUT} \sim M_X$, just like we do with the SM, $\Lambda_{SM} \sim M_Z$, this is intuitive since $M_X \propto v$ apart from some numerical factors.

Adjoint Higgs Masses

Since we break into the direction of the hypercharge we must ask the question of what happens to the other fields in the adjoint. Just like we do with the boson matrix we have four kinds of fields in the adjoint: an $\mathsf{SU(3)}$ octet, an $\mathsf{SU(2)}$ triplet, a leptoquark configuration and a singlet. This means we can represent the scalar adjoint by

$$\mathbf{24}_H = \Sigma_O \oplus \Sigma_T \oplus \Sigma_X \oplus \Sigma_{X^c} \oplus \Sigma_S \ , \tag{2.89}$$

where of course we used the notation Σ_O for the $\mathsf{SU(3)}$ octet, Σ_T for the $\mathsf{SU(2)}$ triplet, Σ_X for the leptoquark and Σ_S for the singlet. The masses of these fields are read from the small oscillations near the vev of the scalar adjoint. Explicitly in a more symbolic way, that we have already introduced, we have

$$\langle \mathbf{24}_H \rangle + \mathbf{24}_H = \begin{pmatrix} \Sigma_O & \Sigma_{X^c} \\ \Sigma_X & \Sigma_T \end{pmatrix} + (v + \Sigma_S) \lambda_{24} \ . \tag{2.90}$$

We have organized explicitly the fields in order for the non vanishing vev part be separated from the vanishing vev part. It must be clear at this point that the SM is the unbroken group when the singlet is the only field to acquire an vev, that is what it means to break in the λ_{24} direction. Plugging this in the potential one can read the mass spectrum for the rest of the scalar fields of this representation, one gets

$$m^2(\Sigma_O) = \frac{1}{3} v^2 b \ , \ m^2(\Sigma_T) = \frac{4}{3} v^2 b \ , \ m^2(\Sigma_S) = 2\mu^2 \ , \ m^2(\Sigma_X) = \ m^2(\Sigma_{X^c}) = 0 \ . \tag{2.91}$$

Recall that we can obtain the SM if $b > 0$, check Appendix D, this is analogous to the SM statement that scalars with a non negative mass term do not acquire a non vanishing vev. Also, all the massive fields have masses proportional to v which makes them of the order of the GUT scale by a naturality argument[2]. The massless terms, just like in the SM, are would-be Goldstone bosons and can be rotated away in order to be assimilated as longitudinal degrees of freedom of the now massive X_μ and Y_μ. As the masses are large they will not interfere greatly in the running of the theory's parameters and so our previous analysis stands valid.

[2] By not letting the potential parameters differ too much between them.

Higgs Sector and the Doublet-Triplet splitting problem

We still have a scalar field in a fundamental representation denoted by H. Since the previous remaining fields do not acquire non vanishing vevs we can not use them to break the SM into the electromagnetism group and that is why we can not use only an adjoint scalar. This field is to acquire an vev at the SM scale, which is much lower than the GUT scale where the adjoint's singlet acquired the vev, and so we have to rewrite the total potential (2.65) into an effective potential where the adjoint scalar is in the vev state

$$V \to V_{eff}(\mathbf{5}_H) \ , \tag{2.92}$$

which becomes

$$V_{eff}(\mathbf{5}_H) = -\frac{\mu_5^2}{2}\mathbf{5}_H^\dagger \mathbf{5}_H + \frac{\lambda}{4}(\mathbf{5}_H^\dagger \mathbf{5}_H)^2 + \alpha \mathbf{5}_H^\dagger \mathbf{5}_H \ \mathrm{Tr}\left\{\langle \mathbf{24}_H \rangle^2\right\} + \beta \mathbf{5}_H^\dagger \langle \mathbf{24}_H \rangle^2 \mathbf{5}_H + \delta \mathbf{5}_H^\dagger \langle \mathbf{24}_H \rangle \mathbf{5}_H \ , \tag{2.93}$$

once again we will consider the Z_2 symmetry over the scalar fields by imposing $\delta = 0$. Rearranging the terms we get

$$V_{eff}(\mathbf{5}_H) = \mathbf{5}_H^\dagger \left(-\frac{\mu_5^2}{2}\mathbb{1}_5 + \alpha \ \mathrm{Tr}\left\{\langle \mathbf{24}_H \rangle^2\right\}\mathbb{1}_5 + \beta \langle \mathbf{24}_H \rangle^2\right)\mathbf{5}_H + \frac{\lambda}{4}(\mathbf{5}_H^\dagger \mathbf{5}_H)^2 \ . \tag{2.94}$$

We have now to separate, or split, the $\mathsf{SU(5)}$ quintuplet into the $\mathsf{SU(3)}$ triplet and the SM doublet since $\mathsf{SU(5)}$ is broken at this stage, we get

$$V_{eff}(\mathbf{5}_H) = H^\dagger H\left(-\frac{\mu_5^2}{2} + \frac{v^2}{15}(30\alpha + 9\beta)\right) + T^\dagger T\left(-\frac{\mu_5^2}{2} + \frac{v^2}{15}(30\alpha + 4\beta)\right) + \frac{\lambda}{4}(\mathbf{5}_H^\dagger \mathbf{5}_H)^2 \ , \tag{2.95}$$

We did not expand the overall quartic terms since they will not matter for the rest of the discussion. So, since the two scalars were once part of the same representation it is intuitive to study a breaking where both would eventually get an vev and then minimize the potential. For that let the vev take the form

$$T \to \langle T \rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} v_c \\ 0 \\ 0 \end{pmatrix} \ , H \to \langle H \rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_W \end{pmatrix} \ . \tag{2.96}$$

One constraint that arises immediately is that the minimum is achieved only for three types of configuration: the trivial unbroken SM with $v_c = v_W = 0$, a broken $\mathsf{SU(3)}$ with $v_c \neq 0$ and unbroken $\mathsf{SU(2)}$ due to $v_W = 0$ or finally the realistic broken SM with $v_c = 0$ and $v_W \neq 0$. One can conclude this by minimizing the potential or by recalling that a fundamental representation can only break into a direction by *killing* a diagonal generator, and in such reasoning we have that H is an $\mathsf{SU(5)}$ fundamental representation and that the potential in (2.94) stills holds a global $\mathsf{SU(5)}$ symmetry.

Now we want to conclude that the SM is the correct breaking path. Note that the only term sensitive for the direction of the breaking is

$$\frac{1}{30}\beta\left(4v_c^2 + 9v_W^2\right)v^2 \tag{2.97}$$

and that for $\beta < 0$ the SM is a global minimum of this potential. With this configuration the triplet will have mass

$$m_T^2 = -\frac{1}{6}\beta v^2 \ , \tag{2.98}$$

and again a non negative mass term forbids a scalar of acquiring a non vanishing vev.

This potential is slightly different from the one of the SM as it will be the vev and the SM masses. The vev that will break the remaining SM group is now

$$v_W^2 = \frac{2}{\lambda}\left[\mu_5^2 + v^2\left(-4\alpha + \frac{6}{5}|\beta|\right)\right] \ , \tag{2.99}$$

and the W_μ mass is now

$$M_W^2 = \frac{1}{4}g^2 v_W^2 = \frac{g^2}{2\lambda}\left[\mu_5^2 + \frac{6M_X^2}{5g_5^2}\left(-4\alpha + \frac{6}{5}|\beta|\right)\right] \ , \tag{2.100}$$

which is a problem because on the one hand we have $M_W \sim 10^2$ GeV and in the other hand $M_X > 10^{14}$ GeV, this means we have a fine tuning problem in order for this theory to be realistic. Fine tuning problems are not formal problems, but they make the theory to be unnatural by spanning the theory's parameters in a narrow range of necessary values for it to be valid. This is the so called Doublet-Triplet Splitting problem, other way of considering this is noting that the triplet mass (2.98) is expected to be heavy, due to its proportionality with v, while the new excitation of H_{SM} is expected to be of the order of the SM scale.

One could assume that the triplet would be light, but unfortunately there are serious experimental constraints due to proton decay. This is so because the triplet can mediate proton decay and, as we will see, experimental bounds on proton decay require it to be heavy.

2.5 Proton Decay and Baryon Number Violation in SU(5)

Consider the new leptoquark bosons, due to them we now have tree level diagrams where a leptonic line changes into a quark line. This means we have a tree level violation of barionic and leptonic numbers which are protected by an accidental symmetry in the SM. By considering an hadronic bound state such as a proton it will be then possible to make the proton decay. To make things worse this is not the only possible tree level diagram which will lead to a proton decay operator: the proton decay can be mediated by the colour triplet scalar through Yukawa couplings.

Proton decay is probably the most striking signal for GUT, unfortunately there is no experimental evidence for proton decay at the time of this writing. With that in mind the study of proton decay in a GUT framework is very important and so we will discuss it in depth in the remaining of this section.

Nowadays there are many experiments sensitive to proton decay events. In Table 2.1 we summarize some of the current bounds, we choose the most striking bounds through π channels and clean lepton final states channels, full tables can be found in [27].

The bounds have evolved greatly in the last years and new experiments are planned or in construction that will further either increase the bounds or find evidence.

Keep in mind that different decay channels can have contributions from different vertexes, so it is important to describe all the barionic and leptonic number violating vertexes and to understand the structure of a proton decay event.

As we said before the new gauge bosons can mediate proton decay, the underlying interactions will come from (2.55) and we will use the leptoquark gauge boson matrix as given in (2.56)

Table 2.1: Experimental Lower Bounds of Proton Decay

Channel	τ_{pd} (10^{30} years)
$p \to$ Invisible	0.21
$p \to \pi^0 e^+$	8200
$p \to \pi^0 \mu^+$	6600
$p \to \pi^+ \nu$	25
$p \to e^+ \gamma$	670
$p \to \mu^+ \gamma$	478

$$\mathcal{L}_{int} = \mathcal{L}_{SM} + \mathcal{L}_X \ , \tag{2.101}$$

after we study those interactions we will study the proton decay possibility via the Yukawa interactions mediated by the scalar colour triplet, whose Lagrangian terms can be found in (2.66).

The new interactions affecting the fermions due to the leptoquark bosons are then

$$\mathcal{L}_X = -g_5 \mathrm{Tr}\left\{\overline{\mathbf{10}_F}\gamma^\mu \underset{\sim}{A}{}^X_\mu \mathbf{10}_F\right\} + g_5 \overline{\overline{\mathbf{5}}_F}\gamma^\mu (\underset{\sim}{A}{}^X_\mu)^T \overline{\mathbf{5}}_F \ . \tag{2.102}$$

Now, recall that the gauge bosons X_μ and Y_μ are in fact in the same representation of $\mathsf{SU(5)}$ with quantum numbers $(X_\mu, Y_\mu) \sim (\mathbf{3}, \mathbf{2}, -5/6)$, and as we have already seen they in fact acquire the same mass. If one want to use our symbolic writing we will then consider them as the same field with two indexes: an $\mathsf{SU(3)}$ index and an $\mathsf{SU(2)}$ index. We will then work with

$$\underset{\sim}{A}{}^X_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & X^c_\mu \\ X_\mu & 0 \end{pmatrix} \ , \tag{2.103}$$

such that

$$X^c_\mu \to (X^c_\mu)^\alpha{}_a \ , \ X_\mu \to (X_\mu)^a{}_\alpha \ , \tag{2.104}$$

where a stands for $\mathsf{SU}_c\mathsf{(3)}$ index and α for $\mathsf{SU}_L\mathsf{(2)}$, this means that a dictionary for our previous notation is such we make the substitution $a = 1 \to X$ and $a = 2 \to Y$.

We already used this simplified by-blocks notation when computing the Yukawa structure. Recall our fermionic fields are then assigned the form

$$\overline{\mathbf{5}}_F \to \begin{pmatrix} d^c \\ \epsilon_2 L \end{pmatrix} \ , \ \mathbf{10}_F \to \begin{pmatrix} \epsilon_3 u^c & q \\ -q^T & \epsilon_2 e^c \end{pmatrix} \ . \tag{2.105}$$

Now note that for ordered indexes, i.e. by -blocks notation, the a and α span different ranges and so one can transpose different types of indexes guilt-free, this ultimately means we can do equivalence $(X_\mu)_\alpha{}^a \leftrightarrow (X_\mu)^a{}_\alpha$ which will enable us to simplify these terms into

$$\mathcal{L}_{XY} = \frac{g_5}{\sqrt{2}}\left\{((\overline{d^c})^\alpha \gamma^\mu \epsilon_{ab} L^b - \overline{e^c}\epsilon_{ba}\gamma^\mu q^{\alpha b} + \overline{q}_{\beta a}\gamma^\mu \epsilon^{\alpha\beta\gamma} u^c_\gamma)(X_\mu)^a{}_\alpha + \right.$$
$$\left. +((\overline{L})_b \epsilon^{ba}\gamma^\mu d^c_\alpha - \overline{q}_{b\alpha}\gamma^\mu \epsilon^{ab} e^c + \epsilon_{\gamma\beta\alpha}(\overline{u^c})^\gamma \gamma^\mu q^{a\beta})(X^c_\mu)^\alpha{}_a \right\} \ , \tag{2.106}$$

this result is written in a simple form where the first half terms are readily read as the hermitian conjugate of the last half.

Now for the Yukawa mediated interactions we recall the terms in (2.66). Remind that Y_5 and Y_{10} are matrices in family space despite the fact the family indices are implicit. We approach these terms in the same way, the Y_5 is trivial but for the Y_{10} term we need to manipulate the different indexes with care. Consider then (2.70) and fix $m = \alpha$ where $\alpha = 1, 2, 3$, this means that now we have changed the span in which the last index, m, runs into the $\mathsf{SU(3)}$ index range, this changes the computations a little and in the end we have

$$\frac{1}{8}\epsilon_5 \mathbf{10}_F Y_{10} \mathbf{10}_F H \to -\frac{1}{2}\epsilon_{ab}\epsilon_{\alpha\beta\gamma}(q^{\alpha a})Y_{10}(q^{\beta b})T^\gamma - \frac{1}{2}e^c Y_{10} u^c_\gamma T^\gamma - \frac{1}{2}u^c Y_{10} e^c_\gamma T^\gamma \ , \tag{2.107}$$

putting this together with the Y_5 terms in T and we get the Yukawa terms mediated by T

$$\mathcal{L}_T = (-\epsilon_2 L Y_5 q + d^c Y_5 \epsilon_3 u^c)T^* - \left(\frac{1}{2}\epsilon_2\epsilon_3 q Y_{10} q + e^c Y_{10} u^c\right) T + \text{ h.c. } . \tag{2.108}$$

Note that neither (2.106) nor (2.108) consist only of baryon or lepton number violating terms. But note that the new Feynman rules that break B and L preserve $\Delta(B-L)$, this means that we have a new symmetry which is accidental, just like B and L are conserved in the SM by chance and not by demand.

We want to study the low energy process in which proton decays. At so low energies the propagator of these mediators will eventually be a collapsed four-fermion effective interaction. We can then study all the proton decay phenomenology with effective operators. For that we will construct all effective operators that arise from these new interactions and in the end keep only the operators that can account for proton decay.

$d = 6$ Effective Operators for Proton Decay

The equations of motion for a generic field ϕ can be gotten from

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \ . \tag{2.109}$$

If the field is heavy, i.e. its mass is heavier than the scale where our problem unfolds, it can be integrated out. This means its derivative terms are very small and so the equations of motion can be obtained only through

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \ . \tag{2.110}$$

We will now do this for the leptoquark bosons fields and for the colour triplet scalar as follows.

Let us start with the new gauge bosons by adding the mass term to the interaction Lagrangian (2.106)

$$\mathcal{L}_{M_X} = -\frac{M_X^2}{2}(X^c_\mu)^\alpha{}_a (X^\mu)^a{}_\alpha \tag{2.111}$$

in such way that all the important information of the leptoquark fields as long as all the interactions of interest are contained in $\mathcal{L}_{XY} + \mathcal{L}_{M_X}$ in such way that we use it now as the Lagrangian to compute the effective operators for the leptoquark fields. We get

$$(X_\mu^c)^\alpha{}_a = \frac{2}{\sqrt{2}}\frac{g_5}{M_X^2}\left\{(\overline{d^c})^\alpha\gamma_\mu\epsilon_{ab}L^b - \overline{e^c}\epsilon_{ba}\gamma_\mu q^{\alpha b} + \overline{q}_{\beta a}\gamma_\mu\epsilon^{\alpha\beta\gamma}u_\gamma^c\right\} \ , \tag{2.112}$$

and for its hermitian conjugate

$$(X_\mu)^a{}_\alpha = \frac{2}{\sqrt{2}}\frac{g_5}{M_X^2}\left\{(\overline{L})_b\epsilon^{ba}\gamma_\mu d_\alpha^c - \overline{q}_{b\alpha}\gamma_\mu\epsilon^{ab}e^c + \epsilon_{\gamma\beta\alpha}(\overline{u^c})^\gamma\gamma_\mu q^{a\beta}\right\} \ , \tag{2.113}$$

we now have just to plug these back in the interaction Lagrangian (2.106) and, after preserving only the baryon and lepton number violating terms we get the following $d = 6$ operators [3]

$$\mathcal{L}_{d=6} = \frac{g_5^2}{M_X^2}\epsilon_{\alpha\beta\gamma}(\overline{u^c})^\alpha\gamma_\mu q^{a\beta}\left\{\overline{e^c}\epsilon_{ab}\gamma^\mu q^{\gamma b} + (\overline{d^c})^\gamma\gamma^\mu\epsilon_{ab}L^b\right\} + \text{ h.c.} \tag{2.114}$$

Lets use this in an example, consider the following proton decay channel

$$p \to \pi^0 e^+ \ , \tag{2.115}$$

which is symbolically represented in Figure 2.2.

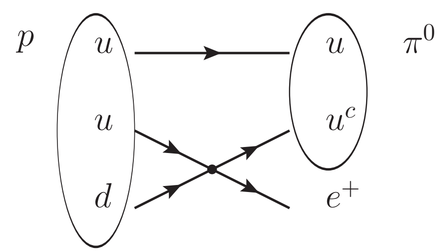


Figure 2.2: The $\pi^0 e^+$ channel for proton decay in Mininal SU(5) GUT.

The contributions to this process are

$$O^I = \frac{g_5^2}{M_X^2}\epsilon_{\alpha\beta\gamma}(\overline{u^c})^\alpha\gamma_\mu u^\beta\overline{e^c}\gamma^\mu d^\gamma \ , \tag{2.116}$$

$$O^{II} = \frac{g_5^2}{M_X^2}\epsilon_{\alpha\beta\gamma}(\overline{u^c})^\alpha\gamma_\mu u^\beta(\overline{d^c})^\gamma\gamma^\mu e \ . \tag{2.117}$$

Now (2.116) and (2.117) seem rather difficult to work, but we can get easily an order of magnitude estimation for the width [30]

$$\Gamma_{pd} \sim \alpha_5^2\frac{m_p^5}{M_X^4} \ . \tag{2.118}$$

We can use this now to set estimates on the leptoquark boson mass by using the bounds on Table 2.1, considering the proton decay lifetime to be

$$\tau(p \to \pi^0 e^+) \sim 10^{34} \text{ years} \ , \tag{2.119}$$

[3]Dimension refers to mass dimension due to field operators.

from (2.33) we consider the unified coupling of the order

$$\alpha_5 \sim 1/40 \ , \tag{2.120}$$

and for the proton mass we use [27]

$$m_p = 0.938 \text{ GeV} \ , \tag{2.121}$$

to finally get

$$M_X \sim 4 \times 10^{15} \text{ GeV} \ . \tag{2.122}$$

This result has a deep interest, recall we had already estimated the GUT scale, and therefore these bosons mass scale, in (2.32) has to be of the order $\sim 6 \times 10^{14}$ GeV only with the argument of unification. The unification by itself is wrong in this minimal setup, but it is highly coincidental that the scale of unification is compatible with the requirements from proton decay bounds with just an estimate.

We do now the same for the colour triplet. Consider its mass term

$$\mathcal{L}_{m_T} = -m_T^2 T^* T \ , \tag{2.123}$$

and add it to the interaction terms (2.108), then integrate out the field and keep only the terms that violate baryon and lepton number, we get

$$\mathcal{L}_{d=6}^T = \frac{1}{2m_T^2}(qY_{10}q)(LY_5q) - \frac{1}{m_T^2}(d^cY_5u^c)(e^cY_{10}u^c) + \text{ h.c. } . \tag{2.124}$$

These operators are of the generic form

$$\mathcal{L}_{d=6}^T \sim \frac{1}{m_T^2} Y_{10}Y_5 \ qqqe \ , \tag{2.125}$$

which means that m_T^2 is less constrained than the leptoquark gauge bosons and reason is simple: the Yukawas for the first generation, from which the quark is made of, are very small of the order $\lesssim 0.01$ which will loosen the stress on m_2^T by about four orders of magnitude, so we get

$$m_T^2 \geq 10^{12} \text{ GeV} \ . \tag{2.126}$$

We also note that in (2.124) it is implicit an antisymmetric structure in the colour gauge numbers since the mediator carries quark-like colour number and the only way to construct $qq + ql$ colour invariant is to antisymmetrize the colour. This will soften the decay rate through this channel since only the antisymmetric part of the Yukawas will contribute, i.e. the new channel is naturally constrained due to symmetries.

Some remarks on the operational technique concerning the effective Lagrangian. First we did a very summarized example, for a complete listing of the operations we refer again to [30]. Also, keep in mind we used a minimal notation with gauge eigenstates, for a complete and correct analysis one has to write these operators in a physical basis where the parameters of V_{CKM} play an important role. Also, in (2.118) a series of results from Chiral Lagrangian Technique are used in order to deal with the hadronic part of the operators, one can not forget that hadronic physics is a complicated subject and completely out of the scope of this text and so these results are presented here without further arguments.

2.6 Supersymmetric Minimal SU(5) GUT Model

As we will see, GUTs extended by Supersymmetry (SUSY) are naturally consistent and phenomenologically interesting, specially in a minimal configuration. As we will discuss some of the features of SU(5) supersymmetric GUTs (SUSY GUTs) we now briefly describe the minimal implementation of SUSY in the SM, in what is called the Minimal Supersymmetric Standard Model (MSSM), as a review of supersymmetric model building and after we will extend the minimal SU(5) with it and study the consequences. SUSY is an interesting subject by its own merit and we will see that although it works well with GUTs neither GUTs nor SUSY depend on each other success.

Recall that SUSY is a graded extension of the Lorentz algebra which then extends the symmetries of a Lagrangian. The formal aspects will not matter for the great part of this work, but there is a most striking phenomenological consequence which is the relation between fermions and bosons that arises from it: for every fermionic degree of freedom there is a corresponding bosonic degree of freedom.

The MSSM is the extension of the SM for a low energy realization of SUSY at a scale just above the SM: $M_{SUSY} \sim 1$ TeV. The MSSM at least doubles the fields of the SM through the supersymmetric partners, and this scale is to be interpreted as the scale where the superpartners appear. The value of this scale is rather important since it cures the already discussed divergence of the Higgs mass if it is not too high. This is one of the main reasons why we will consider the SUSY scale to be of the order of 1 TeV.

In order to obtain the MSSM we extend the SM symmetries and add only the fields necessary to satisfy SUSY algebra. This means we will need to add fermion fields for every boson fields and vice versa. The supersymmetric version of the SM Lagrangian will not be thoroughly described here, instead we turn our attention to the Yukawa and scalar potential sectors since they will be responsible for the mass spectrum and any correction to the symmetry breaking of the potential. The interactions will be deduced through an heuristic derivation using R-Parity[4], e.g. for a two fermion and one scalar term we derive all the Feynman rules through the SM particles and change at most two lines into their superpartners.

The Yukawa and the scalar parts of the Lagrangian are derived from D-Terms and from the superpotential, W, which for the MSSM is

$$W_Y = \epsilon_2 Q Y_u U H_u + \epsilon_2 Q Y_d D H_d + \epsilon_2 L Y_e R H_d \ , \tag{2.127}$$

where we note that the fields correspond to the superfields. As the superpotential can not depend on the field and in its conjugate we need two distinct Higgs representations H_u and H_d with the quantum numbers shown in Table 2.2. The Higgs part of the potential will then be

$$W_H = -\mu \epsilon_2 H_d H_u \ , \tag{2.128}$$

where the scalar part of the superfields reads

$$H_u = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \ , \ H_d = \begin{pmatrix} H^- \\ -H^0 \end{pmatrix} \ . \tag{2.129}$$

The Yukawa and scalar sector of the Lagrangian are obtained through the so called D and F terms by

$$\mathcal{L}_{Y,\text{scalar}} = \frac{1}{2} D^a D^a + F_i F_i^* \ , \tag{2.130}$$

[4]R-Parity is an imposed symmetry within SUSY, it forbids spartners to decay solely into SM particles and so it predicts a stable spartner which can be a candidate for Dark Matter.

Table 2.2: The Higgs Fields in the MSSM

Fields	$(\mathsf{SU(3)_c},\mathsf{SU(2)_L},\mathsf{U(1)_Y})$
H_d	$(\mathbf{1},\mathbf{2},\text{-}1)$
H_u	$(\mathbf{1},\mathbf{2},1)$

where

$$D^a = g(\phi)^*_i (T^a)^i{}_j (\phi)^j \ , \ F_i = \frac{\partial W}{\partial \phi^i} \ , \tag{2.131}$$

where ϕ stands for the scalar part of the supermultiplet.

When constructing the model one wants the SM superpartners to be heavy while having low mass Higgs fields. Superpartners are heavy since we know from experiment that SUSY is not realized at low energies and so it must be broken at some scale. The study of the mechanisms which break SUSY and the study of the MSSM particle spectrum are beyond the scope of this work and we will not refer them again.

Just like in the SM we can test whether the couplings unify in the MSSM according to $\mathsf{SU(5)}$, for that we need to run again the couplings considering now the contribution of the superpartners for energies above 1 TeV, the calculations are done in Appendix A and the new b_i coefficients are

$$b_1 = \frac{33}{5} \ , \ b_2 = 1 \ , \ b_3 = -3 \ , \tag{2.132}$$

and the running of the couplings can be seen in Figure 2.3. We see that, without any other assumption besides the SUSY scale we have a strong hint for unification and one can estimate the unification by $\mathcal{O}(\Lambda_{\mathrm{GUT}}) \sim 10^{16}$ GeV .

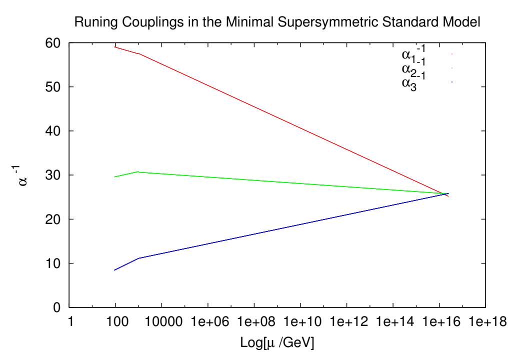


Figure 2.3: Running Couplings in the $\mathsf{SU(5)}$ Minimal Supersymmetric Standard Model.

In order to test the model we run the weak angle up to gut scale and to and compare it to $\sin^2\theta_W(\Lambda_{GUT}) = 3/8 = 0.375$, one gets

$$\sin^2\theta_W(M_X) \gtrsim 0.370 \ , \tag{2.133}$$

which differs from the correct value by an error $\sim 1\%$. Recall that the SM fails by about 10% in its estimate of the Z mass at 1-loop.

If we do as in the minimal model we can run the couplings by imposing unification in order to retrieve the theory's parameters, we get an unification scale of

$$\Lambda_{GUT} \simeq 2.244 \times 10^{16} \text{ GeV} , \tag{2.134}$$

with the value for the unified coupling being

$$\alpha_5^{-1} \simeq 24.268 . \tag{2.135}$$

The natural occurrence of unification through SUSY is a great motivation to study supersymmetric versions of GUTs. Of course this result can be merely coincidental, and if so it does not invalidate neither SUSY nor GUTs, but of course the naturally arising unification is an elegant result which combining with the other interesting features of each theory we get an even more interesting theory with SUSY GUTs. This is our motivation to discuss the supersymmetric versions of $\mathsf{SU(5)}$ based models throughout this text.

We will begin with the minimal supersymmetric $\mathsf{SU(5)}$ theory since, as the minimal $\mathsf{SU(5)}$, many of its features and problems will also occur with extensions based on this model.

The Yukawa contribution for the superpotential is

$$W_Y = \overline{\mathbf{5}}_F Y_5 \mathbf{10}_F H_{\overline{\mathbf{5}}} + \frac{1}{8}\epsilon_5 \mathbf{10}_F Y_{10} \mathbf{10}_F H_{\mathbf{5}} , \tag{2.136}$$

where the fields are superfields. The Higgs contribution to the superpotential is now

$$W_{\mathbf{24}_H,H} = \frac{\tilde{\mu}}{2}\,\mathrm{Tr}\left\{\mathbf{24}_H^2\right\} + \frac{\tilde{c}}{3}\,\mathrm{Tr}\left\{\mathbf{24}_H^3\right\} + \tilde{\mu}_5 H_{\overline{\mathbf{5}}} H_{\mathbf{5}} + \tilde{c}_1 H_{\overline{\mathbf{5}}} \mathbf{24}_H H_{\mathbf{5}} , \tag{2.137}$$

where the parameters are not the same as in the minimal model and hence we put a tilde in order to change notation. Note that we can not have in the superpotential the field and its complex conjugate and so we need two SM Higgs, which means we need two $\mathsf{SU(5)}$ Higgs, one in the fundamental representation and other in the anti-fundamental representation,

$$H_{\mathbf{5}} = \begin{pmatrix} T^i \\ H^i \end{pmatrix} = \begin{pmatrix} T^1 \\ T^2 \\ T^3 \\ H^+ \\ H^0 \end{pmatrix} , \; H_{\overline{\mathbf{5}}} = \begin{pmatrix} T_i^c \\ \epsilon_2 H^c \end{pmatrix} = \begin{pmatrix} T_1^c \\ T_2^c \\ T_3^c \\ H^- \\ -H^0 \end{pmatrix} . \tag{2.138}$$

One can then derive the potential from the supersymmetric action with the $\mathsf{SU(5)}$ gauge symmetry and one gets degenerated vacua between the minima $\mathsf{G_{SM}}$ and $\mathsf{SU(4)}\times\mathsf{U(1)}$. Supergravity restores the hierarchy of the vacuum as is shown in [30] and we will not worry about this issue and assume the $\mathsf{G_{SM}}$ to be the minimum of the potential.

Regarding the proton decay, remember we can read from the superpotential the Feynman rules and even effective operators. Just like we did in (2.66) where we retrieved proton decay operators through the colour triplet, we will now turn to proton decay contributions arising from the superpotential involving superparticles. The colour triplet superpotential contribution reads

$$W_T = (-\epsilon_2 L Y_5 q + d^c Y_5 \epsilon_3 u^c) T^c - \left(\frac{1}{2}\epsilon_2\epsilon_3 q Y_{10} q + e^c Y_{10} u^c\right) T , \tag{2.139}$$

and one can directly study proton decay channels due not only the colour triplet but also due to superpartners like the process in Figure 2.4.

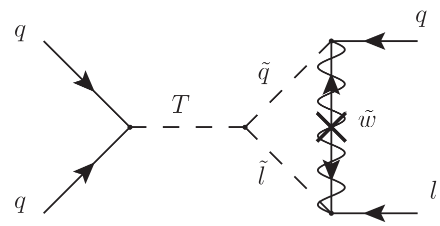


Figure 2.4: Example of a Proton Decay Chanel in the $\mathsf{SU(5)}$ Minimal Supersymmetric Standard Model.

As the superpartners and the colour triplet are clearly much heavier than the hadronic energy scale we will now derive the effective operators for proton decay. For that we will integrate out the colour triplet considering that its mass term is

$$W_{m_T} = -m_T T^c T \ . \tag{2.140}$$

We obtain two $d = 5$ effective operators

$$W_{d=5} = \frac{1}{2m_T}(qY_{10}q)(LY_5q) - \frac{1}{m_T}(d^cY_5u^c)(e^cY_{10}u^c) \ , \tag{2.141}$$

usually called $LLLL$ and $RRRR$ channels due to the fields involved. Note that the operators are $d = 5$ at Lagrangian level and this normally means a larger decay rate than a $d = 6$ process.

Applying the same reasoning used in the study of proton decay in the minimal model we have that the fifth dimension operators in the Lagrangian go with

$$\mathcal{L}_{d=5} \propto \frac{1}{m_T} Y_{10}Y_5 qq\tilde{q}\tilde{l} \ , \tag{2.142}$$

and so we collapsed the colour triplet propagator as it can be seen in Figure 2.5. It is now obvious that these operators are more problematic than the ones mediating proton decay through the colour triplet in the minimal model, where we had a decay width $\Gamma \propto m_T^{-2}$. On the other hand, just like in the non SUSY model, (2.142) has an implicit antisymmetric structure in the colour quantum numbers, this will lead in turn to a restriction in the Yukawas that contribute to the process and this will, in turn, loosen up a little the bounds from proton decay. Nevertheless we have that SUSY worsens the constraints on the colour triplet mass. As the gauge mediated proton decay is still present we then need to account for relevant contributions from different mediators.

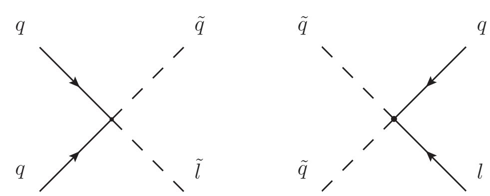
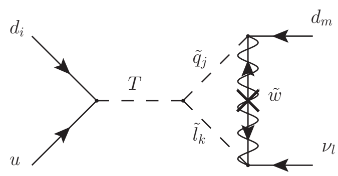


Figure 2.5: Undressed Effective $d = 5$ Operators for Proton Decay

As the superpartners are heavy we need to dress the process of Figure 2.5 with hadronic scale particles and in the end, after integrating out the superpartners, we will have an overall $d = 6$ operator, although it is still proportional to only one inverse power of the colour triplet mass.

The study carried out so far is heuristic and symbolic whose purpose is to understand the qualitative problems of this model. A full and consistent study of these decay channels for phenomenological purposes requires that one takes into consideration the parameters from CKM and PMNS matrices. Consider for example the decay channel $p \to K^+\overline{\nu}$, which is the dominant channel in supersymmetric $\mathsf{SU(5)}$ [31,32]



$$\propto (D^T Y_u U)_{i1} (N^T \tilde{E}^*)_{ik} (\tilde{E}^T Y_e^T \tilde{U})_{jk} (\tilde{U}^\dagger D)_{jm}$$

where the cross in the wino propagator means that the decay width goes with $\Gamma \propto m_{\tilde{w}}^2/(m_{\tilde{q}_j}^2)$, the indices are mass eigenstates indices and where the matrices that mix the gauge eigenstates and the mass egeinstates are such that

$$U^T Y_u U_c = Y_U^d \ , \ D^T Y_D D_c = Y_D^d \ , \ E_c^T Y_e E = Y_e^d \ , \tag{2.143}$$

with the notation X rotates x and these matrices are such that we get the CKM and PMNS matrices through

$$U^\dagger D = V_{\text{CKM}} \ , \ N^\dagger E = V_{\text{PMNS}} \ . \tag{2.144}$$

So, as one can see the study of proton decay is a phenomenological complex subject and we refer to [30,33,34] for listings and derivations of proton decay widths in supersymmetric models. The current estimates on proton decay through the main channel, $p \to K^+\overline{\nu}$, are very restrictive but the theory is not yet dead. In [35] a pre-Kamiokande phenomenological study was carried out assuming $\tau_{pd} > 10^{32}$ years and unification and the colour triplet mass was found to be bound

$$2.2 \times 10^{13} \text{ GeV} < m_T < 2.3 \times 10^{17} \text{ GeV} \ . \tag{2.145}$$

After the superKamiokande the proton lifetime due to the channel $p \to K^+\overline{\nu}$ was set to $\tau_{pd} > 6.7 \times 10^{32}$ years, alongside with more precise measurements of the couplings from LEP motivated Murayama et al [36] early claims of the death of the theory when unification constrained the triplet mass to be at the range

$$3.5 \times 10^{14} \text{ GeV} < m_T < 3.6 \times 10^{15} \text{ GeV} \tag{2.146}$$

while a realist proton decay decay width required

$$m_T > 7.6 \times 10^{16} \text{ GeV} \ . \tag{2.147}$$

In their work in [36] Murayama et al even choose the most favourable values for the hadronic parameters computed through lattice QCD, then the lowest bound due to proton decay was found to be $m_T > 5.7 \times 10^{16}$ GeV which is still not in agreement with unification constraints.

Later in the same year it was proposed [33] that one forfeits natural assumptions on the hierarchy of masses between the squarks by considering the possibility $m_{\tilde{q}_3} < m_{\tilde{q}_{1/2}}$. In previous works one usually sets the SM hierarchy throughout the squarks sector, and since the third family squark masses are experimentally constrained it would constrain the other families' masses. Eventually the proton decay processes involving the first family squarks would be bound by the third family squark masses. In fact this assumption is not difficult to argue, just recall that it is not well understood the physics that underlays the SM quark masses in the first place. By assuming unnatural masses for the squarks one can save the minimal supersymmetric version of the minimal $\mathsf{SU}(5)$.

2.7 Closing Remarks and Critique of the Minimal SU(5) GUT Model

Now that we have studied the minimal implementation of a GUT based on $\mathsf{SU}(5)$ we will discuss some of its features and problems just like we did in the SM.

- No unification

  The first great problem with the minimal model is that it fails do unify the SM's gauge coupling, this makes the theory useless as a GUT.

  We developed an indicative test for unification through the B-Test in (2.37), which simplifies the study on constraints of new particles masses through their quantum numbers in order to get unification. We saw that contributions that rise the fraction B_{23}/B_{12} are favourable to unification and so one searches for light particles with good quantum numbers while expects the others heavy.

  In the minimal $\mathsf{SU}(5)$ one has the additional fields presented in Table 2.3. As one can see apart from the $\mathsf{SU}(2)$ triplet, Σ_T and the new gauge bosons, X_μ, the new fields worsen the unification attempt. On the other hand, we need the new gauge bosons to be heavy due to proton decay bounds so we get only one field contributing favourably to unification. But naturality arguments will make Σ_T heavy of the order of GUT.

  Table 2.3: B-Test contributions from minimal $\mathsf{SU}(5)$

	X_μ	T	Σ_O	Σ_T
B_{23}	$-\frac{11}{3}r_{X_\mu}$	$-\frac{1}{6}r_T$	$-1r_{\Sigma_O}$	$\frac{2}{3}r_{\Sigma_T}$
B_{12}	$\frac{-22}{3}r_{X_\mu}$	$\frac{1}{15}r_T$	0	$-\frac{2}{3}r_{\Sigma_T}$

  Ignoring naturality arguments suppose one has Σ_O and X_μ of the order of Λ_{GUT}, as proton decay does not constrain much the colour triplet we fix its mass at a order of 10^{12} GeV, then Σ_T would need a mass of about 0.02 GeV in order to save unification, which is unrealistic and, of course, highly unnatural. Also this would be wrong if the cubic term in $\mathbf{24}_H$ in the Higgs potential is absent, since (2.91) $m(\Sigma_T) = 4m(\Sigma_O)$, as one can see by (2.84) the cubic term lifts this constraint and would allow this to happen.

  We have seen though that we could save the theory by expanding the particle content, namely through SUSY. One could have expanded the field content through other not SUSY extensions. The use of SUSY is solely based on the phenomenological interest because it is a theory testable at the near future colliders such as the LHC, and SUSY models and extensions are highly regarded as predictive. We could have added other particles of course but that addition should be motivated and for now we lack the motivation for adding new particles. As we shall see next, new representations of fields can be used to solve some problems of the minimal $\mathsf{SU(5)}$ and their inclusion might save the unification requirement.

In the next chapter we will see that we can make consistent realistic theories with an extended model with or without SUSY and with interesting predictions and consequences.

- No neutrino mass

  There is no right-handed neutrino, or one should say the $(\nu^c)_L$ field, and so one can not get tree level renormalizable Dirac mass terms for the neutrino. The problem is essentially the same as in the SM and one can imagine extensions where a generalized Weinberg operator would be present in the $\mathsf{SU(5)}$. But such operator would be of higher scale than the remainder of the $\mathsf{SU(5)}$ fields and so we would have very heavy and untestable seesaw mechanisms. Of course this problem is present also in the SM, where one can have seesaw mechanisms generated by very heavy fields, but now we have a new scale for physics because of the GUT and so we could expect some $\mathsf{SU(5)}$ field (representation) that would induce a seesaw mechanism through a SM Weinberg operator.

  This is in fact easily incorporated into minimal $\mathsf{SU(5)}$ model and these models, as we shall see in the next chapter, have TeV scale predictions. We have then a new motivation to introduce new fields which, in turn, might be able to save unification just as it was already discussed.

- Wrong charged lepton-down quarks masses

  A very problematic and sad result of the minimal $\mathsf{SU(5)}$ GUT is the wrong prediction of the charged leptons-down quarks Yukawas and masses relations. This wrong prediction needs to be eliminated somehow from the theory. Of course this will make the theory less predictive, but a wrong prediction is of no interest.

  In order to save the theory one needs to change its Yukawa structure. This can be made by adding new terms or by adding new scalar representations which couple with the SM matter fields and so give rise to new mass contributions. The first option goes through by adding non-renormalizable terms since all the renormalizable terms with the current particle content were already considered, this approach is preferred if one does not want to increase much the number of parameters although one gets a non-renormalizable theory. The second option is more intuitive if one wants to construct a complete model and one may get new predictions due the new fields. For example, it is easy to realize that the first option by itself will not save the unification problem, since we are not adding new fields to change the running of the couplings, as the second option due the new fields may have new observable particles at some energy range. The two approaches are discussed in the following chapter.

- No family structure

  We still do not have any family structure. This is somehow a generalized problem in GUTs since there is no obvious and natural way that some type of family symmetry arises from the gauge group or its representations. Also we still have the problem that matter fields are divided in separated representations, as it would be more elegant to have them all in the same representation.

  Even so, we reduced the number of necessary representations in order to incorporate all the matter fields of the SM. Recall that this led to a wrong prediction of the relation between the mass of the down quarks and the charged leptons and so we have somehow gain some family structure, or at least some constraints. This eventually points out for $\mathsf{SU(5)}$, and GUTs in general, to be a good framework to study further family symmetries and structure of the Yukawa couplings, see for example [37].

- Hierarchy problems

  Apart from the inclusion of SUSY we still have splitting masses when running them in the minimal $\mathsf{SU}(5)$. On the other hand we have splitting naturality problems in the Higgs representation. Splitting masses are always problematic unless other physics involved can cure it, but usually one has to fine-tune the theory's parameters in order to make it consistent and realistic.

- Hint on the unification scale is compatible with proton decay limits

  Of course not all is bad with minimal setup and we have general predictions and parameter constraints that will be useful when studying the extended model. One interesting aspect of the model discussed in this chapter is that when computing the unification scale through two different approaches, unification and proton decay, we get compatible values. This means that the general structure of the theory predicts naturally a narrow proton decay width and a high unification scale.

  While this is fortunate and exciting we still do not have any experimental evidence for proton decay. This would be a striking evidence for new physics, namely with baryon and lepton number violating physics such as a GUT naturally is.

- Charge quantized

  An impressive result from the minimal $\mathsf{SU}(5)$ model is the charged quantization of the quarks and fermion charges. The quantization arises naturally when one has a constrained gauge group where the diagonal generators that will be associated with the hypercharge have their diagonal entries constrained by the generator normalization conditions.

  Charge quantization is a trivial result in GUT scenarios and one of their most beautiful results and predictions.

- Low energies predictions

  While having a very high energy scale, GUTs are highly predictive at low energies. We studied the proton decay prediction, which is one of the most emphasized low energies consequence, but we also studied the low energy mass predictions as well the low energy gauge couplings predictions. For example recall from (2.15) that the weak angle is fixed at the GUT scale and so by running it into the SM scale we get a prediction for its value.

  The capability of making such predictions makes GUTs to be of interest and physically good theories.

Chapter 3

SU(5) Extensions

In this chapter we will address the problems enumerated in the previous one regarding the minimal SU(5) model. We will study an ensemble of realistic extensions, with correct quark-fermion mass relations and unification. We will follow loosely the classification done by Perez in [38] where we can split the extended models into two classes: the non-renormalizable and the renormalizable.

The classification is based on the approaches we use to cure the quark-fermion mass relations as it was pointed out in the last section of the previous chapter. We can cure the wrong relations by changing the Yukawa sector and one can do this in a renormalizable fashion. The non-renormalizable approach will lead to a incomplete theory, one can always argue there are other problems and inconsistencies with the theory and so there must be a larger one which eventually cures them and gives a consistent effective lower energy Yukawas; but if one wants a renormalizable theory we will need to change the Yukawa sector by extending the particle content. Keep in mind that one can not have a realistic non-renormalizable model with the particle content of the minimal SU(5) since unification fails, so a non-renormalizable model ought to have also an extended particle content. Either approach is valid and, as we will see, both are of interest with high predictive power.

3.1 Non-Renormalizable Models

The inclusion of non renormalizable terms was first proposed by Ellis and Gaillard [39] and consists of the addition of the following non-renormalizable Yukawa terms

$$
\begin{aligned}
\Delta\mathcal{L}_Y = & \overline{\mathbf{5}}_F Y_5^{(1)} \mathbf{10}_F \left(\frac{\Phi}{\Lambda}\mathbf{5}_H\right)^* + \overline{\mathbf{5}}_F Y_5^{(2)} \left(\frac{\Phi}{\Lambda}\mathbf{10}_F\right)\mathbf{5}_H^* + \\
& + \frac{1}{8}\epsilon_5 \mathbf{10}_F Y_{10}^{(1)} \mathbf{10}_F \left(\frac{\Phi}{\Lambda}\mathbf{5}_H\right) + \frac{1}{8}\epsilon_5 \mathbf{10}_F Y_{10}^{(2)} \left(\frac{\Phi}{\Lambda}\mathbf{10}_F\right)\mathbf{5}_H + \text{ h.c. },
\end{aligned}
\tag{3.1}
$$

where Λ is the cut-off scale where the effective operators cease to be valid, i.e. a scale where their internal structure becomes relevant. One can impose the scale to be the Planck scale, but this is quite arbitrary because we need not to know the higher energy theory and so we decided the consider this scale an arbitrary scale. Note however that we do not have a wide range of energies for possible physics, since the GUT and the Planck scale differ by about three orders of magnitude.

As one can see the non-renormalizable extension brings new Yukawa constants, which will change the final SM Yukawas. By collecting the SM Higgs terms from (3.1) one gets the following SM Yukawa matrices

$$Y_e = Y_5 - \sqrt{\frac{3}{5}}\frac{v}{\Lambda}Y_5^{(1)} - \sqrt{\frac{3}{5}}\frac{v}{\Lambda}Y_5^{(2)} \tag{3.2}$$

$$Y_d = Y_5^T - \sqrt{\frac{3}{5}}\frac{v}{\Lambda}Y_5^{(1)} + \frac{2}{\sqrt{15}}\frac{v}{\Lambda}Y_5^{(2)} \tag{3.3}$$

$$Y_u = -\frac{1}{2}(Y_{10} + Y_{10}^T) + \frac{3}{2\sqrt{15}}\frac{v}{\Lambda}(Y_{10}^{(1)} + Y_{10}^{(1)T}) - \frac{1}{4\sqrt{15}}\frac{v}{\Lambda}(2Y_{10}^{(2)} - Y_{10}^{(2)T}) \ , \tag{3.4}$$

and so we have enough parameter space freedom to fit experimental data. Obviously we have lost predictability, but we saved the theory from a wrong prediction.

Note that by changing the Yukawa sector through the inclusion of these terms instead of an extension of the Higgs sector we avoided eventual problems from having multiple scalar representations. For example, if we extend the SM's Higgs sector we will eventually be faced with the issue of the hierarchy between the vev, i.e. a splitting problem.

Although simple, these terms do not generate a realistic theory by themselves because they do not alter the running of the theory's parameters and so they do not contribute to the unification. This means that a non-renormalizable model must be extended also in the particle content.

The most interesting theories are those where the new fields are added with other motivations. We have already seen the minimal supersymmetric extension of the minimal SU(5) model, where unification is achieved naturally at the cost of restrictive experimental bounds on either the scalar colour triplet and the new gauge bosons. Non-renormalizable minimal SUSY SU(5) models have already been proposed, a striking and not obvious result presented by Emmanuel-Costa et al in [34] from a consistent SUSY model shows that the non-renormalizable terms can loosen the experimental constrains on proton decay.

There are of course other motivations to include extra fields. The more interesting ones which we will present in this chapter are those who incorporate representations that mediate seesaw mechanisms in order to have light massive left handed neutrinos. As we want these representations to contribute to the running of gauge couplings the seesaw mechanism will not be at higher scales than GUT and so we will have a seesaw mechanism similar to those that are proposed as SM extensions, i.e. the seesaw mechanisms we will get will be read at low energies from the $d = 5$ Weinberg operator (1.37). So we need a fermion singlet, a scalar SU(2) triplet or a fermion SU(2) triplet in order to have Type I, II or III seesaw mechanism, respectively. The seesaw mechanisms are not mutually exclusive and so the final neutrino mass can have contributions from different seesaw mechanisms.

The reasoning now is as follows: look for SU(5) representations which incorporate the above listed seesawable SM representations. Do not forget that a singlet fermion will not change the running of the couplings and so it can not save the unification problem, hence any model with Type I seesaw will need other fields to consolidate unification. As we will break SU(5) before the seesaw mechanism becomes an effective operator we will need to split the representation's fields, where the debris fields will have their masses constrained by unification. Finally, phenomenology can be studied in order to check if the model is experimentally testable.

Adjoint Fermion and Type I + III Seesaw

The most recent model was proposed by Bajc and Senjanovic [40, 41] where an adjoint (a SU(5) **24**) fermion representation is introduced. The reason is that it incorporates an SU(2) triplet and an overall singlet. We represent the new representation as

$$\mathbf{24}_F = \sum_{i=1}^{24} \frac{1}{\sqrt{2}} \Psi_i \lambda_i \ , \tag{3.5}$$

or symbolically through our usual by-blocks notation

$$\mathbf{24}_F = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_O & \Psi_{X^c} \\ \Psi_X & \Psi_T \end{pmatrix} + \frac{1}{\sqrt{2}} \Psi_S \lambda_{24} \ , \tag{3.6}$$

where the fields naming is self explanatory.

The seesaw mechanism will happen below the GUT scale, and the representation will be split by then due the interactions between $\mathbf{24}_F$ and $\mathbf{24}_H$. We will consider the most general terms and higher order contributions. The higher order contributions are considered for consistency, although these terms are not SM Yukawas we generally consider that higher energy physics might alter all the Yukawa-like interactions and not only the SM ones. As we will see these terms offer a more reliable model. We have then

$$\begin{aligned} \mathcal{L}_{\mathbf{24}} =& m_F \ \mathrm{Tr}\left\{\mathbf{24}_F^2\right\} + \lambda_F \ \mathrm{Tr}\left\{\mathbf{24}_F^2 \mathbf{24}_H\right\} + \\ &+ \frac{1}{\Lambda}\left(a_1 \mathrm{Tr}\left\{\mathbf{24}_F^2\right\} \mathrm{Tr}\left\{\mathbf{24}_H^2\right\} + a_2 (\mathrm{Tr}\left\{\mathbf{24}_F \mathbf{24}_H\right\})^2 + a_3 \mathrm{Tr}\left\{\mathbf{24}_F^2 \mathbf{24}_H^2\right\} + a_4 \mathrm{Tr}\left\{\mathbf{24}_F \mathbf{24}_H \mathbf{24}_F \mathbf{24}_H\right\}\right) \ , \end{aligned} \tag{3.7}$$

where λ_F and a_i are Yukawa-like couplings. m_F is an unconstrained mass term and we will not propose any Higgs-like mechanism to generate it. After the $\mathsf{SU(5)}$ breaking and the splitting of the representation we get the following mass spectrum for its embedded fields

$$m_S^F = m_F - \frac{1}{\sqrt{15}} v \lambda_F + \frac{v^2}{\Lambda}\left[2a_1 + 2a_2 + \frac{7}{15} a_3 + \frac{7}{15} a_4\right] \ , \tag{3.8}$$

$$m_T^F = m_F - \frac{3}{\sqrt{15}} v \lambda_F + \frac{v^2}{\Lambda}\left[2a_1 + \frac{3}{5} a_3 + \frac{3}{5} a_4\right] \ , \tag{3.9}$$

$$m_O^F = m_F + \frac{2}{\sqrt{15}} v \lambda_F + \frac{v^2}{\Lambda}\left[2a_1 + \frac{4}{15} a_3 + \frac{4}{15} a_4\right] \ , \tag{3.10}$$

$$m_X^F = m_F - \frac{1}{2\sqrt{15}} v \lambda_F + \frac{v^2}{\Lambda}\left[2a_1 + \frac{13}{30} a_3 - \frac{2}{5} a_4\right] \ . \tag{3.11}$$

We see that the non-renormalizable terms add enough parameters to split the masses in a wide range spectrum while avoiding naturality issues. As we will see we will need the masses to be somehow apart due to unification constrains. But if one suppresses the non-renormalizable terms one gets the relations

$$\lambda_F > 0 \Rightarrow m_T^F < m_S^F < m_X^F < m_O^F \ , \tag{3.12}$$

this serves only as an indication on the spectrum's hierarchy.

Now lets consider the new Yukawa contributions due to the new fermion representation. Again we will have contributions from non-renormalizable terms, and the new terms are

$$\mathcal{L}_{Y\mathbf{24}} = y_0^i \overline{\mathbf{5}}_F^i \mathbf{24}_F \mathbf{5}_H + \frac{1}{\Lambda} \overline{\mathbf{5}}_F^i \left(y_1^i \mathbf{24}_F \mathbf{24}_H + y_2^i \mathbf{24}_H \mathbf{24}_F + y_3^i \ \mathrm{Tr}\left\{\mathbf{24}_F \mathbf{24}_H\right\}\right) \mathbf{5}_H + \ \text{h.c.} \ , \tag{3.13}$$

we note the new Yukawa interactions are vectors and not matrices.

To study the seesaw mechanism we will isolate the neutrino terms which also couples to the SM Higgs doublet. We have then explicitly the terms

$$\mathcal{L}_{Y\nu} = \epsilon_2 L^i \left(-\frac{3}{\sqrt{30}} y_0^i \Psi_S + \frac{1}{\sqrt{2}} y_0^i \Psi_T \right) H + \frac{v}{\Lambda} \epsilon_2 L^i \left(\frac{3}{5\sqrt{2}} (y_1^i + y_2^i + y_3^i) \Psi_S - \sqrt{\frac{3}{10}} (y_1^i + y_2^i + y_3^i) \Psi_T \right) H + \text{ h.c.}\,, \tag{3.14}$$

notice that we will have a contribution from the singlet and the triplet and so we have a Type I+III seesaw mechanism. It is rather interesting to realize that the two types of seesaw happen naturally in this framework. One can rewrite in a more convenient and compact way

$$\mathcal{L}_{Y_L} = \epsilon_2 L^i \left(y_S^i \Psi_S + y_T^i \Psi_T \right) H + \text{ h.c.} \tag{3.15}$$

where the $y^i_{T/S}$ are linear combinations of the y^i_a, $a = 0, .., 3$.

The seesaw mechanism is now immediate, we just need to integrate out the heavy fields, getting then a SM Weinberg-type effective operator. For that consider the explicit form of the fermion triplet

$$\frac{1}{\sqrt{2}} \Psi_T = \begin{pmatrix} T^0/\sqrt{2} & T^+ \\ T^- & -T^0/\sqrt{2} \end{pmatrix} \,, \tag{3.16}$$

and the masses of the two fermions that couple with the neutrino

$$\mathcal{L}_{m_{T/S}} = -\frac{m_T^F}{2} T^0 T^0 - \frac{m_S^F}{2} \Psi_S \Psi_S + \text{ h.c.}\,. \tag{3.17}$$

It is by no coincidence that the neutral part of the triplet will be responsible for the seesaw mechanism. Also, we note that these fields have a Majorana mass term since we have not added their right-handed correspondents due to minimality arguments. Finally recall that a conjugation matrix is implicit in order to preserve Lorentz invariance.

As we want the low energy mass term of the neutrino, we consider the SM spontaneous symmetry breaking

$$\mathcal{L}_{m_\nu} = \frac{v+h}{\sqrt{2}} y_T^i \nu^i T^0 - \frac{m_T^F}{2} T^0 T^0 + \frac{v+h}{\sqrt{2}} y_S^i \nu^i \Psi_S - \frac{m_S^F}{2} \Psi_S \Psi_S + \text{ h.c.}\,, \tag{3.18}$$

and after integrating out the heavy fields we finally get the effective mass matrix for the neutrinos

$$m_\nu^{ij} = -\frac{v^2}{2} \left(\frac{y_T^i y_T^j}{m_T^F} + \frac{y_S^i y_S^j}{m_S^F} \right)\,. \tag{3.19}$$

Some remarks need to be made at this point. First we note that in order the theory to be realistic the Yukawas need to be small, even smaller than the SM Yukawas. On the other hand we have only two vector Yukawas, as the two of them can be simultaneously rotated in family space into a specific direction we will have only at most two massive neutrinos. At the time of the writing of this text we have only experimental evidence for two distinct non-vanishing neutrino masses.

As we want to impose unification, in order for the theory be realistic, we turn our attention to the change the new fields have in the gauge couplings running and for that we study how they influence the B-Test (2.37). The contributions to the B coefficients can be consulted in Table 3.1, where we used the tools in Appendix A to compute the group theoretical coefficients.

Table 3.1: B-Test contributions from $\mathbf{24}_F$

	Ψ_O	Ψ_X	Ψ_T
B_{23}	$-2r_{\Psi_O}$	$-\frac{2}{3}r_{\Psi_X}$	$\frac{4}{3}r_{\Psi_T}$
B_{12}	0	$\frac{2}{3}r_{\Psi_X}$	$-\frac{4}{3}r_{\Psi_T}$

So, in order to have unification the octet and the leptoquark must be heavy, at the same time the triplet must be light. One can deduce easily that, since we have two unification unfavourable fields, we want the triplet to be as near the SM scale as possible, in fact if one pushes the unfavourable fields all to the GUT scale and the favourable ones as near the SM scale as possible one gets unification with bosonic and fermionic triplets to have mass of the order of 1 TeV and the fermion octet can also be relatively light having its mass constrained due unification in the range that spans from 10^5 GeV to 10^8 GeV [40, 41].

We have then built a model with unification, natural Type I+III seesaw mechanism that generates small masses for left-handed neutrinos, correct quark-lepton mass relations at low energies and new particles detectable at near future colliders such as the LHC. These new particles decay mainly through W bosons, the fermion triplet decay modes are

$$T^0/T^{\pm} \to (W^{\pm}l^{\mp},\ Z\nu,\ h\nu)/(W^{\pm}\nu,\ Zl^{\pm},\ hl^{\pm}) \ , \tag{3.20}$$

also the decay width is proportional to the new Yukawa couplings [40]

$$\Gamma(\Psi_T) \sim |y_T|^2 (m_T^F)^2 \ , \tag{3.21}$$

and so we might not only have a collider testable seesaw mechanism but a seesaw mechanism where we can measure Yukawas responsible for neutrino masses. This is a very interesting and exciting result.

The model has been put to extensive phenomenological studies [42] due to the predictions that may be testable at the LHC. The conclusion is that the theory needs unavoidably light triplets in order for unification to happen. This in turn means that either one detects the weak triplets in LHC or some experimental proton decay evidence in the next generation experiments, or else the theory is unrealistic as it is.

$\mathbf{15}_H$ and Type II Seesaw

The other option to generate light neutrino masses via seesaw mechanism is to add a representation which one of its constituents is a scalar $\mathsf{SU(2)}$ triplet. Dorsner and Perez [43, 44] accomplished this by considering the $\mathsf{SU(5)}$ symmetric **15** dimensional representation, which incorporates a $\mathsf{SU(2)}$ triplet

$$\mathbf{15} = (\mathbf{6}, \mathbf{1}, -4/3) \oplus (\mathbf{3}, \mathbf{2}, 1/3) \oplus (\mathbf{1}, \mathbf{3}, 2) \ . \tag{3.22}$$

We will denote the scalar fields in this representation as

$$\mathbf{15}_H = \begin{pmatrix} \phi_{\mathbf{6}} & \phi_q \\ \phi_q^T & \Delta \end{pmatrix} \ , \tag{3.23}$$

where the triplet part is taken in the usual notation

$$\Delta = \vec{\Delta} \cdot \vec{\tau} \ . \tag{3.24}$$

This representation extends greatly the scalar potential of the theory, the new terms are

$$\begin{aligned}
\Delta V = & - \frac{\mu_{15}^2}{2} \mathrm{Tr}\left\{ \mathbf{15}_H^\dagger \mathbf{15}_H \right\} + \frac{a_{15}^2}{4} \mathrm{Tr}\left\{ \mathbf{15}_H^\dagger \mathbf{15}_H \right\}^2 + \frac{b_{15}^2}{4} \mathrm{Tr}\left\{ \mathbf{15}_H^\dagger \mathbf{15}_H \mathbf{15}_H^\dagger \mathbf{15}_H \right\} + \\
& + c_2 \mathrm{Tr}\left\{ \mathbf{15}_H^\dagger \mathbf{24}_H \mathbf{15}_H \right\} + c_3^* \mathbf{5}_H^\dagger \mathbf{15}_H \mathbf{5}_H^* + c_3 \mathbf{5}_H \mathbf{15}_H^\dagger \mathbf{5}_H + \\
& + b_1 \mathrm{Tr}\left\{ \mathbf{15}_H^\dagger \mathbf{15}_H \right\} \mathrm{Tr}\left\{ \mathbf{24}_H^2 \right\} + b_3 \mathbf{5}_H^\dagger \mathbf{5}_H \mathrm{Tr}\left\{ \mathbf{15}_H^\dagger \mathbf{15}_H \right\} + \\
& + b_5 \mathbf{5}_H^\dagger \mathbf{15}_H \mathbf{15}_H^\dagger \mathbf{5}_H + b_6 \mathrm{Tr}\left\{ \mathbf{5}_H \mathbf{15}_H^\dagger \mathbf{24}_H \mathbf{24}_H \right\} + b_7 \mathrm{Tr}\left\{ \mathbf{15}_H^\dagger \mathbf{24}_H \mathbf{15}_H \mathbf{24}_H \right\} \ ,
\end{aligned} \tag{3.25}$$

where the couplings naming was carried out with a mixture of faithfulness with our conventions and the one used in [43].

The $\mathbf{15}_H$ also couples to fermions through Yukawa interactions, the new contributions, including non-renormalizable terms, read

$$\Delta \mathcal{L}_Y = \overline{\mathbf{5}}_F Y_{15} \mathbf{15}_H \overline{\mathbf{5}}_F + \frac{1}{\Lambda} (\overline{\mathbf{5}}_F \mathbf{5}_H) Y_{15}^{(1)} (\overline{\mathbf{5}}_F \mathbf{5}_H) + \text{ h.c. } . \tag{3.26}$$

One gets the seesaw part of the Lagrangian as previously by breaking $\mathsf{SU(5)}$ and collecting the terms with neutrinos and the scalar triplet. One finds out that the seesaw sector is very similar to the usual Type II seesaw sector

$$\mathcal{L}_{seesaw} = -M_\Delta^2 \mathrm{Tr}\left\{ \Delta^\dagger \Delta \right\} + Y_{15} L \Delta L + c_3 H \Delta^\dagger H + \text{ h.c. } , \tag{3.27}$$

where M_Δ is the sum of all the contributions that arise from the $\mathsf{SU(5)}$ breakin in (3.25) to the quadratic term's mass parameter. Just like the usual Type II seesaw, when integrating out the neutral part of the triplet we conclude that the neutrino mass is then given by

$$m_\nu \simeq \frac{Y_{15} c_3}{M_\Delta^2} v_w^2 \ , \tag{3.28}$$

where now we have a matrix structure for the neutrino masses, note that family indexes were omitted for notation sake.

Regarding unification, by checking the B-Test contributions from the different scalars that come from the $\mathbf{15}_H$ in Table 3.2 we conclude that we need relatively light ϕ_q and Δ.

Table 3.2: B-Test contributions from $\mathbf{15}_H$

	ϕ_6	ϕ_q	Δ
B_{23}	$-\frac{5}{6} r_{\phi_6}$	$\frac{1}{6} r_{\phi_q}$	$\frac{2}{3} r_\Delta$
B_{12}	$\frac{8}{15} r_{\phi_6}$	$-\frac{7}{15} r_{\phi_q}$	$-\frac{1}{15} r_\Delta$

Unfortunately we have a new proton decay contributions due ϕ_q. The new contribution is implicit in

(3.26) where one can derive the new Feynman rule

$$d^c Y_{15} \phi_q L \ , \tag{3.29}$$

which is a B and L violating vertex which will lead to proton decay effective operators. As the mediation is carried by the scalar field ϕ_q in the Yukawa sector, its mass will be less constrained compared to the new vector bosons but still it will, in general, not be light in order to accommodate proton decay within experimental bounds. And so, once again, one has to play with unification constrains and proton decay bounds in order to make the theory realistic. A thorough study of the masses compatible with unification and proton decay was carried in [43].

Generally speaking, if one wants to extend the particle content minimally one will have to constrain the unification favourable fields masses in order to satisfy unification. By simplicity we will add as few new fields as possible, but on the other hand the lesser the number of favourable fields the lighter will have to be their masses. As such, it is expected minimal extensions to have new particles with masses ranging from the SM scale to the LHC scale and so new physics can be probed in the near future collider experiments. This makes these models particularly interesting as the LHC is running.

3.2 Renormalizable Models

We will now develop a renormalizable way to solve the wrong Yukawas predictions. In order to change the Yukawa sector (2.66) we need new scalar representations that couple with the SM fermion fields. We choose a representation that does not interfere with the spontaneous symmetry breaking pattern and that ultimately acquires a non vanishing vev alongside with the SM Higgs, i.e. it participates in the SM Higgs mechanism. The chosen representation will then need to have an $\mathsf{SU(3)} \times \mathsf{U(1)_Q}$ invariant vev structure. As we want to modify the final mass spectrum we expect this new $\mathsf{SU(5)}$ representation to have within itself a SM Higgs equivalent representation in order to mimic a double Higgs SM. Georgi and Jarlskog [45] proposed the addition of a $\mathbf{45}$ scalar which we will denote by $\mathbf{45}_H$. The study on the full scalar potential and its minimization can be seen in [46, 47].

Recall that the predictions in the minimal model for the Yukawas are

$$Y_e = Y_d^T \ , \ Y_u = Y_u^T \ , \tag{3.30}$$

and as we have only one SM Higgs field to contribute to the masses we will have then the mass matrices

$$M_e = Y_5 v_W \tag{3.31}$$

$$M_d = Y_5^T v_W \ . \tag{3.32}$$

As $\mathbf{45}_H$ will acquire a non vanishing vev there will be two distinct contributions to the mass matrices and so it is necessary to work with the mass matrices instead of the Yukawa matrices.

The new representation is derived from $\mathbf{10} \otimes \overline{\mathbf{5}} = \mathbf{5} \oplus \mathbf{45}$, and by computing the final representation using the usual notation where $\mathsf{SU(2)}$ and $\mathsf{SU(3)}$ indexes are separated we get the embedded SM representations

$$\begin{aligned}\mathbf{45} =&(\mathbf{8},\mathbf{2},1/2)\oplus(\overline{\mathbf{6}},\mathbf{1},-1/3)\oplus(\mathbf{3},\mathbf{3},-1/3)\oplus(\overline{\mathbf{3}},\mathbf{2},-7/6)\oplus\\ &\oplus(\mathbf{3},\mathbf{1},-1/3)\oplus(\overline{\mathbf{3}},\mathbf{1},4/3)\oplus(\mathbf{1},\mathbf{2},1/2)\ ,\end{aligned} \tag{3.33}$$

which we will identify by the fields

$$\mathbf{45}_H = \Phi_1\oplus\Phi_2\oplus\Phi_3\oplus\Phi_4\oplus\Phi_5\oplus\Phi_6\oplus H_2\ . \tag{3.34}$$

The new contributions to the Yukawa sector, excluding now the non-renormalizable terms since we want to construct a renormalizable model, are

$$\Delta\mathcal{L}_Y = \mathbf{10}_F Y_{45}\overline{\mathbf{5}}_F\mathbf{45}^*_H + \epsilon_5\frac{1}{8}\mathbf{10}_F Y'_{\mathbf{45}}\mathbf{10}_F\mathbf{45}_H + \text{ h.c. }. \tag{3.35}$$

As the $\mathbf{45}_H$ acquires a non vanishing vev these new Yukawas will generate new contributions to the masses of the matter fields.

The $\mathbf{45}_H$ has an antisymmetric nature in the contravariant indexes due the contribution from $\mathbf{10}$

$$(\mathbf{45}_H)^{ij}_k = -(\mathbf{45}_H)^{ji}_k\ ,\ (\mathbf{45}_H)^{ii}_k = 0\ , \tag{3.36}$$

and the vev structure is [46, 47]

$$\langle(\mathbf{45}_H)^{15}_1\rangle = \langle(\mathbf{45}_H)^{25}_2\rangle = \langle(\mathbf{45}_H)^{35}_3\rangle = v_{45}\ ,\ \sum_{i=1}^{3}\langle(\mathbf{45}_H)^{i5}_i\rangle = -\langle(\mathbf{45}_H)^{45}_4\rangle\ ,\ \langle(\mathbf{45}_H)^{55}_5\rangle = 0\ , \tag{3.37}$$

from which one can deduce immediately

$$\langle(\mathbf{45}_H)^{45}_4\rangle = -3v_{45}\ . \tag{3.38}$$

By plugging this in back into (3.35) we have the new mass matrices

$$M_e = Y_5 v_W + 2Y_{45}v_{45} \tag{3.39}$$

$$M_d = Y_5^T v_W - 6Y_{45}^T v_{45}\ . \tag{3.40}$$

It is clear now that we have enough parametric freedom to make the theory realistic. Unfortunately we increased the particle content of the theory through a scalar field, enlarging the number of free parameters. In fact the non-renormalizable alternative has less parameters than this one [48].

Inevitably this new representation will bring many new fields that will influence the running of SM gauge couplings, for a list of B-Test relevant factors check Table 3.3.

One sees immediately that there are more fields not contributing for a suitable B-Test than the ones that are favourable (of course that this statement is weighted by the fields masses) which are Σ_T, Φ_3 and H_2. The new SM Higgs is naturally expected to be light in order to generate the new mass contributions for the matter fields, and to counter the unfavourable contributions we will also want a light Φ_3 while the other fields will eventually be heavier in order not to spoil unification. As we did before we have a naturality

Table 3.3: B-Test contributions from $\mathbf{45}_H$

	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	H_2
B_{23}	$-\frac{2}{3}r_{\Phi_1}$	$-\frac{5}{6}r_{\Phi_2}$	$\frac{3}{2}r_{\Phi_3}$	$\frac{1}{6}r_{\Phi_4}$	$-\frac{1}{6}r_{\Phi_5}$	$-\frac{1}{6}r_{\Phi_6}$	$\frac{1}{6}r_{H_2}$
B_{12}	$-\frac{8}{15}r_{\Phi_1}$	$\frac{2}{15}r_{\Phi_2}$	$-\frac{9}{5}r_{\Phi_3}$	$\frac{17}{15}r_{\Phi_4}$	$\frac{1}{15}r_{\Phi_5}$	$\frac{16}{15}r_{\Phi_6}$	$-\frac{1}{15}r_{H_2}$

problem when splitting the masses of these fields throughout a wide range of viable values. Also, we expect the unification constraints on Φ_3 mass to make it light, unfortunately it contributes to proton decay through effective proton decay operators derived using the new Feynaman rules from (3.35)

$$qY_{45}\Phi_3 q \ , \ qY_{45}\Phi_3^* L \ . \tag{3.41}$$

If we set the Yukawas values through naturality one gets fairly heavy Φ_3 of about 10^{10} GeV. Interestingly the unification constraints imposes $10^9 < m_{\Phi_3}/(\text{GeV}) < 10^{12}$. Considering this by imposing the new proton decay mediator mass to be $m_{\Phi_3} \gtrsim 10^{10}$ GeV the unification constraint predicts the remaining spectrum to have the scalar octet mass of the order 10^5 GeV, beyond experimental reach but $m_{\Sigma_T} \simeq m_{\Phi_1} \simeq m_Z$[1], i.e. two LHC testable new particles. Note that also Φ_5 and Φ_6 contribute to proton decay, but since they ought to be heavy due to unification constraints we will not be faced with the problematic balance between the light mass requirement for unification and proton decay contribution like with Φ_6.

Summarizing, we can conclude then that we get a predictive and realistic unified model based on the $\mathsf{SU(5)}$ gauge group. At some extent this model betters the non-renormalizable theory where one needed to add fields to cure the unification problem. With this alternative approach we have new fields that cure the wrong Yukawa predictions and no extra fields seem necessary to maintain unification. Of course one expects looser unification constraints if one considers additional fields from the $\mathbf{24}_F$ or the $\mathbf{15}_H$. On the other hand we already saw that the theories with these fields eventually explain a small neutrino mass due to natural seesaw mechanisms. As one introduces the renormalizable approach for completeness sake one should also consider these extensions, since we will then have a neutrino mass theory. Having this in mind we will now discuss the extension of the renormalizable model by adding the fields that are responsible for the seesaw mechanisms. The discussion will be quick because most of the algebraic results were already presented throughout this chapter and many predictions are similar.

Adjoint Fermion and Type I + III Seesaw

We begin by the adjoint fermion field with Type-I+III seesaw mechanism. The model was first proposed by Perez [38, 49] and stands as a simple, predictive and interesting theory.

The new Yukawa contributions to the minimal model are now (3.35), the renormalizable terms of (3.13) and

$$\Delta\mathcal{L}_Y = h^i \overline{\mathbf{5}}_F^i \mathbf{24}_F \mathbf{45}_H + \text{ h.c. } . \tag{3.42}$$

[1]Recall that we still have to consider the contributions from Σ_T and from the other Σ fields that arise from the splitting of $\mathbf{24}_H$.

Computing the seesaw mechanism using the same approach as the one that led us to (3.19) we obtain a neutrino mass matrix

$$M_\nu^{ij} = \frac{a^i a^j}{m_T^F} + \frac{b^i b^j}{m_S^F} \ , \tag{3.43}$$

where now the structure is given not only by Yukawa couplings but mass contributions from different Yukawas and the two vev

$$a^i = \frac{1}{\sqrt{2}} y^i v_W - 3h^i v_{45} \ , \ b^i = \frac{\sqrt{15}}{2}\left(\frac{y^i v_W}{5\sqrt{2}} + h^i v_{45}\right) \ . \tag{3.44}$$

The new fermions masses can be read from (3.8)(3.9)(3.10)(3.11) apart from the non-renormalizable contributions and the new particle content B-Test contributions can be consulted in Tables 3.1 and 3.3.

We expect unification constraints to be less restrictive since we have more fields affecting the running of the couplings. On the other hand the renormalizable masses are more constrained with each other, since the non-renormalizable terms extend the freedom to fit the masses that respect proton decay bounds and unification constraints. Note that the fields favourable to unification whose masses are not constrained by each other are Σ_T, Ψ_T and Φ_3. This means that we lost some comfort zone in splitting the masses but we have more independent masses to fit unification demands. This theory is realistic in the sense it accommodates unification, small neutrino masses and proton decay within experimental bounds [38]. Again recall Φ_3 mediates proton decay, if one sets it as small as possible to aid unification one can have m_{Φ_1} and m_{Σ_O} as low as the SM scale and therefore testable at LHC, while the fermionic triplet stays beyond the reach of experiments with a mass of $m_{\Psi_T} \sim 10^{14}$ GeV. Other extreme scenario is to swap the fermionic triplet with the scalars. Of course there are viable values that leave the fields outside LHC reach but the unification constraints are usually easier to satisfy with LHC searchable fields.

$\mathbf{15}_H$ and Type II Seesaw

Finally we turn to the model with a new scalar field in a **15** representation. As we saw, this model induces a natural Type-II seesaw trough an SU(2) triplet embedded in the **15**. The renormalizable version of the model with an extra $\mathbf{15}_H$ differs only a little from the non-renormalizable version. The main problematic aspect is that we need to consolidate two distinct Yukawa mediated proton decay channels due the Φ_3 and ϕ_q interactions in the Yukawa sector.

As one can deduce easily no other new prediction will emerge than the ones we already discussed. The main phenomenological consequence of working with $\mathbf{15}_H$ in a renormalizable model is that we get many unrelated fields contributing for the running, namely we have many favourable contributions from Σ_T, Φ_3, Δ and ϕ_q. The mass spectrum of the enumerated fields is easily found as sparse since unification is easily accommodated with so many degrees of freedom. In fact the constraints are so loose and the parametric freedom is such that one finds that this model can easily be untestable either at colliders and in next generation proton decay experiments [50].

3.3 Comments on the SUSY Versions of the Models

What about the supersymmetric versions of these models? As we had seen in the previous chapter, extending the theory through SUSY was elegant in the sense that unification arose naturally and out-of-the-box. On the other hand, SUSY is an interesting theory by itself and has clear predictions at the TeV scale and so we get a theory with many phenomenological interesting predictions.

But as we have seen unification can be accomplished by adding new fields that are not the superpartners of the minimal $\mathsf{SU(5)}$ model. Fields that can either solve the wrong Yukawa relations between the down-quarks and charged leptons or can induce natural seesaw mechanisms which will generate light neutrino masses. Also, by avoiding SUSY one gains looser constrains due proton decay upon the masses of the scalars that may mediate proton decay.

SUSY versions of these models are still of interest and the phenomenological studies about them were made. For example Perez constructed the SUSY version of the renormalizable model with the adjoint fermionic [38] and concluded it was possible to consolidate proton decay by having some of the proton decay mediators above the GUT scale itself [51]. One can also speculate that since we do not have physics regulating the structure and the strength of the Yukawa couplings, some kind of symmetry could lead to *miraculous* cancellations that would prevent dangerously fast proton decay channels.

Chapter 4

Conclusions on **SU(5)** Models

In this remaining chapter we conclude this work with a discussion on what we have learned from SU(5) models, what are the current experimental bounds and the general features of these models.

We started by the SM problems that were not solved by either the minimal model or its extensions presented in the last chapter. Some of those problems are the non incorporation of Gravity, the negative vacuum energy density and non constrained Higgs potential parameters. Actually if one reviews the approach used when constructing the minimal model and the following extensions it becomes clear that we did not expect to solve these problems: gravity is not yet understood in the framework of QFT and attempts to gauge it have failed, hence it does not make sense to consider it at a minimal GUT setup; the negative vacuum energy density arises from the Higgs-like mechanisms, and since we are discarding any theory that would explain the positive sign that is experimentally measured we are left only with negative contributions to the vacuum energy density; finally any Higgs potential constructed solely through gauge symmetry principles is unconstrained unless we incorporate additional symmetries, these can be discrete or more complex ones such as SUSY, as we have seen. So we can conclude that these problems prevail in the GUT framework in great part for the same reasons they appear in the SM minimal framework.

Also we did not solve the family repetition problem of the SM. But we have reduced the number of representations for each family to two. This was an immediate result for a minimal setup of a GUT with a larger group unification, since the minimality in said framework will lead us to put the SM particles in larger representations without adding new fields and so we reduced the number of representations used. Unfortunately it was not possible to put all the matter fields in the same representation, as it is possible for example in SO(10) based GUTs were we can fit a full SM generation of matter fields into a **16**, including a $(\nu^c)_L$ that will be responsible for a natural Type-I seesaw mechanism. The reduction of the representations led us to wrong predictions in Yukawa relations which hints at GUTs to be an interesting framework to more extensive studies on family symmetries and Yukawa structure. So, after all we actually gained new insights on this matter.

To solve the erroneous prediction on the Yukawa relations we added either non-renormalizable terms or a new Higgs representation. The reasoning demanded either way since we needed to change the SM Higgs mechanism consequences on the fermion masses. This made the theory less predictive of course, but with such simple modifications we easily turned the theory into realistic models.

We also concluded that a minimal setup could not interpret the SM apparent couplings running into a unified regime under the SU(5) group. Although this is a problem in the sense that a GUT supposedly unifies the couplings, this is more a challenge than a real problem since unification was not far from the minimal setup, as it was argued in the finale of Chapter 1, and extensions that would eventually cure this could also account for a renormalizable framework and/or to create a neutrino mass theory. In this sense we can see the initial failure of unification as a further hint on new physics.

Recall that when adding the Higgs representation we concluded that new fields would have eventually a small mass in order to consolidate unification constrains, and at some extent we saw that some of the light masses would lay on the LHC range. But this was not the only collider signature we found: when adding new representations to account for seesaw mechanisms we had similar predictions for fields that would make the running couplings unify at the GUT scale. This means that if we want to create neutrino mass theories with unification constraint we will eventually have new particles detectable at the LHC. Recall for example that the non-renormalizable model with an adjoint fermionic was seen more easily accommodated within the experimental bounds if the $\mathsf{SU}(2)$ triplets are to be seen at the LHC.

The most striking experimental signature from GUTs is not through colliders though, but due to proton decay. In the minimal framework proton decay is governed by either the new gauge bosons or by the colour triplet that is embedded in the scalar $\mathsf{SU}(5)$ fundamental representation alongside with the SM Higgs. The scalar mediated proton decay was carried out through the Yukawas of the first family, having this in mind after we study the effective operators of proton decay we concluded that the colour triplet's mass is not so constrained due to proton decay bounds as the vector bosons' mass.

One incredibly result, or coincidence, we found when studying the minimal $\mathsf{SU}(5)$ model was the concordance between two distinct estimates for the GUT scale and therefore the new gauge bosons' mass: first by computing the unification scale with the SM matter fields, and then by estimating the new gauge bosons' masses in order proton decay width to be in agreement with experimental bounds. This result is impressive as one is working with a theory that spans its effective action throughout several orders of magnitude from a scale as high as 10^{16} GeV to the SM scale of about 100 GeV.

As proton decay was never experimentally observed, being the last bounds made by super-Kamiokande, it is important to study these models phenomenologically in order to understand what regions of parameter space are still valid. We did this for the minimal model and referred to other works after which we concluded that $\mathsf{SU}(5)$ based models are alive and well.

We studied extensions that contributed for proton decay. Namely SUSY, where we found that a new channel with $d = 5$ effective operators exist due the presence of scalar superpartners in Yukawa mediated proton decay. This channel is very dangerous as it goes with only one inverse power of the mass of the colour triplet, $\sim 1/m_T$, as in the non supersymmetric case the decays through the Yukawa sector go with $\sim 1/m_T^2$. Fortunately this decay operator as an overall antisymmetric colour structure that will eventually diminish the overall contribution from the Yukawa matrices. This feature is also present in the non-supersymmetric case but the loosen bounds on the mass makes it less relevant in that scenario.

With this work we can then conclude that $\mathsf{SU}(5)$ based GUT models are alive, realistic and highly predictive. Of course it is not without its flaws just like other theories. We still have to search for physical extensions to explain the Yukawa and family structures, natural ways to solve naturality problems such as the ones that arise from the splittings of different representations, and to constrain the potential's parameters. As GUTs are normally easy to incorporate with even larger theories such as the ones that try to treat gravity, it is expectable that eventually one can construct a more complete theory using GUTs.

Having said this we look forward to experimental input, either from the LHC or from the next generation of neutrino experiments that can also put bounds on proton decay, in order to understand better how a GUT can be realized in Nature.

Appendix A

Renormalization Group Equations and Results

In quantum field theory (QFT) one has to treat divergences in a systematic way so that one gets finite predictions for the observables and parameters. This treatment is applied through the renormalization group which accounts for the set of transformations between two different renormalization schemes that leave physics invariant.

The Renormalization Group Equations (RGE) for each parameter are calculated specifically, being the general result with second order contributions of the shape

$$\frac{d}{dt}X = \frac{1}{16\pi^2}\beta_X^{(1)} + \frac{1}{(16\pi^2)^2}\beta_X^{(2)} \ , \tag{A.1}$$

with X being the parameter, $\beta_X^{(i)}$ the beta function of the parameter at i^{th} order in perturbation theory (loosely speaking at i-loop corrections). The $\beta^{(i)}$ functions are calculated by computing all the contributing i^{th} order diagrams for that parameter, which is usually a complicated exercise. In supersymmetric field theories this simplifies due to non-renormalization theorems, specially higher order corrections. In this Appendix we will summarize the necessary results for the renormalization discussions held in the main text.

A.1 Running Couplings

The gauge coupling, g_i, RGE at 1-loop is

$$\frac{dg_i}{dt} = \frac{1}{16\pi^2}\beta_{g_i}^{(1)} \ , \tag{A.2}$$

where $t = \ln\mu$ and β is the so called beta function at 1-loop corrections which for a gauge coupling is

$$\beta_{g_i}^{(1)} = g_i^3 b_i \tag{A.3}$$

and b_i accounts for the contribution from all effective fields at the energy range of interest through group theoretical results due the gauge symmetry of the interaction. Using the fine structure notation $\alpha_i = g_i^2/(16\pi^2)$ we integrate the equation and get

$$\alpha_i^{-1}(\mu_2) = \alpha_i^{-1}(\mu_1) - \frac{b_i}{4\pi}\ln\left(\frac{\mu_2^2}{\mu_1^2}\right) \ , \tag{A.4}$$

where $\mu_2 > \mu_1$.

The gauge information is then solely contained in b_i coefficients, they can be calculated generally by [52,53]

$$b_i = \frac{1}{3} \sum_R s(R) t_i(R) \prod_{j \neq i} dim_j(R) \ , \tag{A.5}$$

where R is a field in some representation, $t_i(R)$ is the Dynkin of the representation in which the field is[1], the last term accounts for the dimensions that the field is concerning the other gauge groups[2], and finally $s(R)$ is given by

$$s(R) = \begin{cases} 1 & \text{for } R \text{ scalar,} \\ 2 & \text{for } R \text{ chiral fermion,} \\ -11 & \text{for } R \text{ gauge boson.} \end{cases} \tag{A.6}$$

In the SM the b_i coefficients are then easily calculated. Note we do not have a general structure for family physics and so we need to account for N_g number of families for every repeated fields (this excludes the Higgs multiplet at minimal framework). We have then the coefficient for $\mathsf{SU(3)}$ interaction

$$\begin{aligned} b_3 &= -\frac{11}{3} \times 3 + N_g \left[\frac{2}{3} \times \frac{1}{2} \times 2 + \frac{2}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} \right] \\ &= -7 \ , \end{aligned} \tag{A.7}$$

and for the $\mathsf{SU(2)}$

$$\begin{aligned} b_2 &= -\frac{11}{3} \times 2 + N_g \frac{1}{3} \left(2 \times \frac{1}{2} \times 3 + 2 \times \frac{1}{2} \right) + \frac{1}{3} \times \frac{1}{2} \\ &= -\frac{19}{6} \ , \end{aligned} \tag{A.8}$$

finally we calculate the b_1 coefficient, which in the framework of $\mathsf{SU(5)}$ is computed by the eigenvalues of the diagonal generator (recall (2.10)) and we have

$$\begin{aligned} b_1 &= \frac{3}{5} \left[N_g \left(\frac{2}{3} \left(\frac{1}{2} \right)^2 \times 2 + \frac{2}{3} \times (1)^2 + \frac{2}{3} \times \left(\frac{1}{6} \right)^2 \times 2 \times 3 + \frac{2}{3} \times \left(\frac{2}{3} \right)^2 \times 3 + \frac{2}{3} \times \left(\frac{1}{3} \right)^2 \times 3 \right) + \frac{1}{3} \times \left(\frac{1}{2} \right)^2 \times 2 \right] \\ &= \frac{41}{10} \ . \end{aligned} \tag{A.9}$$

We note further that in our group generators normalization convention the Dynkin index for the fundamental representation is 1/2 and for de adjoint of an $\mathsf{SU(n)}$ group is n. For other representations it is sometimes easier to compute it via another group invariant which is the quartic Casimir operator

$$C_i^2(R) = \sum_a (T_i^a)^2 \ , \tag{A.10}$$

where the generators T_i^a are represented in the same representation as the field R. Both invariants relate with each other by

[1]Note that for an abelian group this equals the squared eigenvalue of the field in respect to that group's generator.

[2]Ultimately it counts the multiplicity of the field due to other gauge symmetries.

$$C_i^2(R)^a{}_b = \frac{n_g}{dim(R)} t_i(R)\delta^a{}_b \ , \tag{A.11}$$

where n_g stands for the number of the generators of the group and $dim(R)$ is the dimension of the representation. Computing the Casimir operator is relatively easy if one has a grasp on the Young tableaux mechanics, see for example [54] but the reader must be advised that normalization conventions differ. Nowadays one can dismiss the hand made calculations with algebraic computational software, such as the group theoretical built-in functions in Susyno [55].

In the Minimal Supersymmetric Standard Model (MSSM) we have the so called supersymmetric partners at a scale just above the SM scale $\mu > M_{SUSY} \sim 1$ TeV. At that scale we get scalar partners for the SM matter fields and new Dirac fields partners of the bosonic degrees of freedom of the SM. Also we need to account for at least two Higgs doublets due to superpotential constraints. Keeping this in mind we can recompute the above b_i coefficients just by considering the extra contributions from superpartner fields $\tilde{R}$ by considering

$$s(R) \to s(R) + s(\tilde{R}) = \begin{cases} 1+2=3 & \text{for } R \text{ scalar,} \\ 2+1=3 & \text{for } R \text{ chiral fermion,} \\ -11+2=-9 & \text{for } R \text{ for gauge boson.} \end{cases} \ , \tag{A.12}$$

the supersymmetric coefficients, computed with the $\mathsf{SU(5)}$ normalization for the hypercharge, are then

$$b_1 = \frac{33}{5} \ , \ b_2 = 1 \ , \ b_3 = -3 \ . \tag{A.13}$$

A.2 Yukawa and Masses Renormalization

The masses and Yukawa renormalizations are very similar. To realize that recall that a mass term is an Yukawa multiplied by an vev constant. The rigorous way to check how Yukawa relations at high energies influence the low energy masses would be to run down the Yukawa until the spontaneous symmetry breaking scale and then run the mass as a different parameter. In this work, however, we intend only to show that the Yukawa relations predictions in the minimal $\mathsf{SU(5)}$ fail and so we will treat the mass as parameter to be run all through the GUT scale into the SM scale.

The integrated 1-loop mass running equation is

$$m_f(t) = m_f(0) \exp\left[\int_{g(0)}^{g(t)} 16\pi^2 \frac{\gamma_{m_f}^{(i)}(x)}{\beta_i(x)} dx\right] \ , \tag{A.14}$$

where there is an implicit sum in i which stand for the gauge groups, $\gamma_m^{(i)}$ are called the mass anomalous dimensions and β_i are the already introduced first order beta functions, $\beta_i = g_i^3 b_i$.

The values for $\gamma_m^{(i)}$ can be found in [56] which we rewrite here with our notation and conventions for completion

$$\gamma_{m_f}^{(1)} = -\frac{3}{8\pi^2}\frac{3}{5}\left(\frac{Y}{2}\right)_{f_L}\left(\frac{Y}{2}\right)_{f_R} g_1^2 \ , \ \gamma_{m_f}^{(2)} = -\frac{9}{32\pi^2} g_2^2 \ , \ \gamma_{m_f}^{(3)} = \begin{cases} -\frac{1}{2\pi^2} g_3^2 & \text{for } f \text{ quarks,} \\ 0 & \text{for } f \text{ letpons} \end{cases} \ , \tag{A.15}$$

note again that the they are calculated in the $\mathsf{SU(5)}$ framework.

The factors in the exponential are then

$$\frac{\gamma_{m_f}^{(3)}}{\beta_3} = \begin{cases} -8/(g_3 b_3) = c_3/g_3 & \text{for } f \text{ quarks,} \\ 0 & \text{for } f \text{ letpons} \end{cases} , \tag{A.16}$$

$$\frac{\gamma_{m_f}^{(2)}}{\beta_2} = -\frac{9}{2b_2 g_2} = \frac{c_2}{g_2} , \; \frac{\gamma_{m_f}^{(1)}}{\beta_1} = -\frac{6}{g_1 b_1} \left(\frac{\lambda_{24}}{2}\right)_{f_L} \left(\frac{\lambda_{24}}{2}\right)_{f_R} = \frac{c_1(f)}{g_1} . \tag{A.17}$$

We see that they are all proportional to the inverse of the integrand so the integral is immediately evaluated into logarithms

$$m_f(t) = m_f(0) \exp\left[c_1(f) \log\left(\frac{g_1(t)}{g_1(0)}\right) + c_2 \log\left(\frac{g_2(t)}{g_2(0)}\right) + c_3 \log\left(\frac{g_3(t)}{g_3(0)}\right)\right] . \tag{A.18}$$

As we are interested in study the relation between the masses of charge leptons and down quarks we get

$$\frac{m_{d_i}(t)}{m_{e_i}(t)} = \frac{m_{d_i}(0)}{m_{e_i}(0)} \exp\left[(c_1(d_i) - c_1(e_i)) \log\left(\frac{g_1(t)}{g_1(0)}\right) + c_3 \log\left(\frac{g_3(t)}{g_3(0)}\right)\right] , \tag{A.19}$$

where i stands now for family index and it is not summed, is just a label such as d_i is a down quark of the i^{th} family and e_i is the charge lepton of the same i^{th} family.

We note also that the difference of the two c_1 factors is independent of the generation

$$c_1(d_i) - c_1(e_i) = \frac{2}{b_1} . \tag{A.20}$$

Appendix B

The SM L/R structure and Charge Conjugation Matrix

In the SM the gauge eigenstates are historically chosen with the $\mathsf{SU(2)_L}$ playing the main role. Mainly because not all particles are in non trivial representations, i.e. some are singlets, under this symmetry which is eventually broken through the Higgs mechanism. The same is done with the $\mathsf{SU(3)_c}$ representations by differentiating colour singlets from the fundamental represented matter fields, i.e. leptons and quarks respectively. But $\mathsf{SU(2)_L}$ has a particular structure: it only acts on left-handed particles.

Handedness is an abuse of terminology, since what we are talking about are the chirality eigenstates which are the eigenstates of the projector operator

$$P_{L/R} = \frac{1 \mp \gamma_5}{2} \ , \tag{B.1}$$

one can easily check that the above operators form a complete set of projection operators. This operator however is equivalent to the helicity operator

$$\frac{\vec{\sigma} \cdot \vec{p}}{p} \ , \tag{B.2}$$

in the ultrarelativistic limit, or one might say in the massless limit. This is easily understood as chirality is a Lorentz invariant quantum number but obviously helicity is frame dependent: there is a boost in which the projection of the momentum upon the spin will flip sign. But is due to this equivalence that chirality eigenstates are called Left (L) and Right (R) handed states, since the sign of the projection upon γ_5 will be identified with the sign of the helicity operator which in turn we call through the resemblance of positive and negative projections.

But since L and R interact differently within the SM gauge symmetries we will have to be careful to construct group and Lorentz invariant terms, for example a fermion mass term is to be written as

$$\overline{\psi}\psi = \psi^\dagger \gamma^0 \psi \ , \tag{B.3}$$

now by sandwiching $\mathbb{1} = (P_L + P_R)(P_L + P_R)$ in between the spinors and recalling $\{\gamma_5, \gamma^\mu\} = 0$, if we denote the chirallity eigenstates $P_{L/R}\psi = \psi_{L/R}$ we get

$$\overline{\psi}\psi = \overline{\psi}_R \psi_L + \overline{\psi}_L \psi_R \ . \tag{B.4}$$

But how can we construct group invariants if R fields are $\mathsf{SU(2)_L}$ singlets? The problem is solved by considering the fields to be massless at first and that they become massive with their mass being the Yukawa couplings multiplied by the non vanishing vev of the Higgs doublet. The symmetry is then broken and we call this the Higgs mechanism that spontaneously breaks the symmetry. We note that the Lorentz invariance is assured by the L and R fields and the γ^0 matrix while the group invariance comes from the addition of a scalar doublet. This is in fact the only way to create a renormalizable, Lorentz and group invariant which generates masses for the fermions in the context of the SM.

We can, however, change notation in order to be easier to read the quantum numbers to write down invariants. We will do this by considering the charge conjugation matrix and the charge conjugated fields. We will follow the notation used in [57] and present explicit calculations for clarity sake.

We start by stating that there is a matrix C such that

$$C^\dagger = C^T = C^{-1} = -C \ , \ C\gamma_\mu C^{-1} = -\gamma_\mu^T \ , \tag{B.5}$$

its explicit form is of no interest for us and depends on the representation for the gamma matrices. Now if we define the charge conjugated field as

$$\psi^c = C\overline{\psi}^T \ , \tag{B.6}$$

definition which also respects

$$(\psi^c)^c = \psi \ , \ \overline{\psi^c} = \psi^T C \ , \tag{B.7}$$

one can then write the Dirac equations for charged fermion's spinor and its conjugate as

$$(i\not{D} - m)\psi = 0 \ , \tag{B.8}$$

$$(i\not{D}^* - m)\psi^c = 0 \ , \tag{B.9}$$

where the second equation clearly governs the fermion with the conjugated charges of the first.

So we clearly have constructed a notation where it is easier to treat the field and its charge conjugated. We will now apply this to the L and R parts of the spinnors since we already saw that chirality eigenstates play a huge role in the structure of the Lagrangian. It is easy to prove that

$$(\psi_L)^c = (\psi^c)_R \ , \ (\psi_R)^c = (\psi^c)_L \ , \tag{B.10}$$

which will lead us to conclude that

$$\overline{\psi}_R = (\psi^c)_L^T C \ , \tag{B.11}$$

and so we will then be able to write down invariants without using the right-handed field notation. For example we can now write the RL part of the mass term as

$$\overline{\psi}_R \psi_L = (\psi^c)_L^T C \psi_L = \psi_L^T C (\psi^c)_L \ , \tag{B.12}$$

and all the mass term is now

$$\overline{\psi}\psi = \psi_L^T C(\psi^c)_L + \overline{\psi}_L C\overline{(\psi^c)_L}^T \ , \tag{B.13}$$

where the second term is to be read as the hermitian conjugate of the first term.

This will lead to an abusive notation where one omits the T of transpose and the charge conjugation matrix C.

The SM right handed fields will then be rewritten according the prescription

$$f_R \to (f^c)_L \sim f^c \ , \tag{B.14}$$

where the quantum numbers are the same as $(f_R)^c$.

Appendix C

Group Theory and Representations of SU(5)

We briefly summarize the important group and representation theory results for the SU(5). We do explicit calculations whenever we deemed necessary for clarity and notation sake.

G_{SM} is a maximal subgroup of SU(5), loosely speaking this means they share the same Cartan subalgebra which is the subgroup of the algebra of the diagonal generators. More rigorously this means that the generators of SU(3) and SU(2) can be extended to five dimensional matrices through a direct sum and, as they still respect the algebra commutation relations, can be interpreted as SU(5) generators.

In turn this defines a base in which we write explicitly the SU(5) generators with a by-blocks fashion where the SU(3) and SU(2) parts are separated.

C.1 The SU(5) Gell-Mann Matrices

We begin by enumerating the generalized Gell-Mann matrices for the SU(5). Recall that the generalized Gell-Mann matrices are, apart from a normalization factor, the generator matrices in the fundamental representation of an SU(n) group. They are hermitian traceless matrices which form, together with the n-dimensional identity matrix, the basis of the vectorial space $\mathcal{M}_{n\times n}(\mathbb{C})$.

We will extensively use the fact that G_{SM} is a maximal subgroup of SU(5) and so we will immediately identify some of the generators through the known forms of the other generators.

SU(3) Generators

We choose to reserve the first three indexes for the colour indexes. So the embedding of the SU(3) is then

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 & & \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 & & \\ i & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 & & \\ 0 & -1 & 0 & & \\ 0 & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i & & \\ 0 & 0 & 0 & & \\ i & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 1 & 0 & & \\ & & & & \\ & & & & \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & -i & & \\ 0 & i & 0 & & \\ & & & & \\ & & & & \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & -2 & & \\ & & & & \\ & & & & \end{pmatrix} .$$

Mixed Quantum Numbers Generators

The mixed quantum numbers generators, i.e. the generators with non vanishing $\mathsf{SU(3)}$ and $\mathsf{SU(2)}$ quantum numbers, are not present in the SM as they are obtained when one unifies the algebras of the $\mathsf{G_{SM}}$ subgroups. They can be computed through the commutator relations between the generators of $\mathsf{SU(3)}$ and the generators of $\mathsf{SU(2)}$. Alternatively one can compute them using the basis argument.

$$\lambda^9 = \begin{pmatrix} & & & 1 & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix} \quad \lambda^{10} = \begin{pmatrix} & & & -i & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ i & 0 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix} \quad \lambda^{11} = \begin{pmatrix} & & & 0 & 0 \\ & & & 1 & 0 \\ & & & 0 & 0 \\ 0 & 1 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix}$$

$$\lambda^{12} = \begin{pmatrix} & & & 0 & 0 \\ & & & -i & 0 \\ & & & 0 & 0 \\ 0 & i & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix} \quad \lambda^{13} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & 1 & 0 \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \end{pmatrix} \quad \lambda^{14} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & -i & 0 \\ 0 & 0 & i & & \\ 0 & 0 & 0 & & \end{pmatrix}$$

$$\lambda^{15} = \begin{pmatrix} & & & 0 & 1 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \end{pmatrix} \quad \lambda^{16} = \begin{pmatrix} & & & 0 & -i \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ i & 0 & 0 & & \end{pmatrix} \quad \lambda^{17} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 1 \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ 0 & 1 & 0 & & \end{pmatrix}$$

$$\lambda^{18} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & -i \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ 0 & i & 0 & & \end{pmatrix} \quad \lambda^{19} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & 0 & 1 \\ 0 & 0 & 0 & & \\ 0 & 0 & 1 & & \end{pmatrix} \quad \lambda^{20} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & 0 & -i \\ 0 & 0 & 0 & & \\ 0 & 0 & i & & \end{pmatrix} .$$

$\mathsf{SU(2)}$ Generators

The last two indexes are for $\mathsf{SU(2)}$ indexes, and so we use the same approach as we did for the $\mathsf{SU(3)}$.

$$\lambda^{21} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} \quad \lambda^{22} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & 0 & -i \\ & & & i & 0 \end{pmatrix} \quad \lambda^{23} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & 1 & 0 \\ & & & 0 & -1 \end{pmatrix} .$$

Diagonal Generator

Finally we have a diagonal generator that is not identified with any presented in the $\mathsf{SU(3)}$ and $\mathsf{SU(2)}$ Cartan subalgebras. We will identify it eventually with the SM hypercharge apart from a normalization factor that does not appear in the SM. It can be computed solely through demanding a traceless nature and that its eigenvalues are the same for the two separated parts of the other SM subgroups.

$$\lambda^{24} = \frac{1}{\sqrt{15}} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

One can check that all the Gell-Mann matrices, and the respective generators, respect the following normalization constraints, where we imposed our convention where the Dynkin index is equal to 1/2 for the fundamental representation

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab} \to \mathrm{Tr}\left(\lambda^a \lambda^b\right) = 2\delta^{ab} \ . \tag{C.1}$$

C.2 Representations, Transformations and Electric Charges

The group generators are important if one wants to build fields in the adjoint representation but, as we learn from the SM, the matter fields are usually in other representations. It is then important to study different representations in our basis of interest, compute their transformation rules and check the electric charges.

Representations

We will follow the Young tableaux notation for the SM subgroups representations, while we will leave the $\mathsf{SU(5)}$ representations indicated by a bold number which stands for its dimensionality.

Our conventions are such that the fundamental representation corresponds to a superscript in tensorial notation and the antifundamental (conjugated fundamental) to a subscript in tensorial notation[1], this means

$$\square \to \psi^i \ , \ \overline{\square} \to \psi_i \ . \tag{C.2}$$

In our basis of interest, where the SM's non-abelian subgroups are not overlapped, the fundamental representation of $\mathsf{SU(5)}$ is obtained by the direct sum of the $\mathsf{SU(3)}$ and $\mathsf{SU(2)}$ fundamental representations

$$\mathbf{5} = \mathbf{3} \oplus \mathbf{2} = \left(\square, \mathbf{1}, -2/3\right) \oplus \left(\mathbf{1}, \square, 1\right) \ , \tag{C.3}$$

where the quantum numbers are ordered as $(\mathsf{SU(3)}, (\mathsf{SU2}), \mathsf{U(1)})$. The hypercharge was assigned with respect to the SM field with the other quantum number configuration, so the fermion fundamental representations is constituted by the fields

[1] We will avoid identifying the position of the indexes with the contravariant and covariant terminology as one usually does in relativity.

$$\mathbf{5}^i = \begin{pmatrix} d_R^1 \\ d_R^2 \\ d_R^3 \\ e_L^c \\ -\nu_L^c \end{pmatrix} \quad , \tag{C.4}$$

where we note that the lepton part is the conjugate of the SM's lepton $\mathsf{SU(2)}$ fundamental representation, this means that the hypercharge has an opposite sign and that the $\mathsf{SU(2)}$ conjugated configuration is obtained by[2]

$$\epsilon_2 l \ . \tag{C.5}$$

As the hypercharges signs are opposite regarding the diagonal generator we will make use of the antifundamental representation of $\mathsf{SU(5)}$,

$$\overline{\mathbf{5}} = \overline{\mathbf{3}} \oplus \mathbf{2} = \left(\overline{\square}, \mathbf{1}, 2/3\right) \oplus \left(\mathbf{1}, \square, -1\right) \ . \tag{C.6}$$

By doing this we will transform a superscript into a subscript

$$\mathbf{5} \to \psi^i \ , \ \overline{\mathbf{5}} \to \psi_i \ , \tag{C.7}$$

and this holds also for the $\mathsf{SU(3)}$ and $\mathsf{SU(2)}$ indexes, the final form of the fermion antifundamental is then

$$\overline{\mathbf{5}}_i = \begin{pmatrix} d_i^c \\ (\epsilon_2)_{ia} l^a \end{pmatrix} = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ -\nu \end{pmatrix} . \tag{C.8}$$

From the fundamental and anifundamental representation we can derive other representations, either reducible or irreducible but only the irreducible are of interest. We start by the adjoint, which is the representation in which the gauge bosons are, which is obtained by non invariant part of the direct product of the fundamental with its conjugated

$$\mathbf{5} \otimes \overline{\mathbf{5}} = A^i{}_j \oplus \mathbf{1} \ . \tag{C.9}$$

We can do the computation explicitly using (C.3) and (C.6)

$$\mathbf{5} \otimes \overline{\mathbf{5}} = \left((\square, \mathbf{1}, -2/3) \oplus (\mathbf{1}, \square, 1) \right) \otimes \left((\overline{\square}, \mathbf{1}, 2/3) \oplus (\mathbf{1}, \square, -1) \right) \ , \tag{C.10}$$

and recalling that the $\mathsf{SU(3)}$ antifundamental representation is obtained through $\epsilon_3 \psi \psi$, i.e.

$$\overline{\square} = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \ , \tag{C.11}$$

we use the Young tableaux technique and finally get the composition in SM quantum numbers

[2]Recall that $\mathsf{SU(2)}$ is a real group, i.e. $\overline{\square} = \square$, and so a group invariant can be obtained with two fundamental representations using a two dimensional Levi-Civita symbol by $(\epsilon_2 \psi)_i \psi^i = \mathbf{1}$.

$$
\mathbf{5}\otimes\overline{\mathbf{5}} = \left(\yng(2,1),\mathbf{1},0\right)\oplus\left(\mathbf{1},\yng(2),0\right)\oplus\left(\square,\square,5/3\right)\oplus\left(\overline{\square},\square,-5/3\right)\oplus(\mathbf{1},\mathbf{1},0)\oplus(\mathbf{1},\mathbf{1},0)\ . \tag{C.12}
$$

We identify easily the quantum numbers of the gluons, the weak vector bosons and the hypercharge singlet vector boson. The third and the forth are new vector gauge bosons and their charge conjugated partner. The extra singlet is the emerging singlet configuration that arise from the direct product.

By performing the direct product of two fundamental representations we get an antisymmetric and a symmetric representation, for $\mathsf{SU(5)}$ this means

$$
\mathbf{5}\otimes\mathbf{5} = \mathbf{10}^{ij} + \mathbf{15}^{ij}\ , \tag{C.13}
$$

the explicit computation is then

$$
\begin{aligned}
\mathbf{5}\otimes\mathbf{5} &= \left(\left(\square,\mathbf{1},-2/3\right)\oplus\left(\mathbf{1},\square,1\right)\right)\otimes\left(\left(\square,\mathbf{1},-2/3\right)\oplus\left(\mathbf{1},\square,1\right)\right)\\
&= \left\{\left(\overline{\square},\mathbf{1},-4/3\right)\oplus\left(\square,\square,1/3\right)\oplus(\mathbf{1},\mathbf{1},2)\right\}\oplus\\
&\quad\oplus\left\{\left(\yng(2),\mathbf{1},-4/3\right)\oplus\left(\mathbf{1},\yng(2),2\right)\oplus\left(\square,\square,1/3\right)\right\}\ .
\end{aligned} \tag{C.14}
$$

The antisymmetric part is really interesting. As one can see with

$$
\mathbf{10}^{ij} = \left(\overline{\square},\ \mathbf{1},-4/3\right)\oplus\left(\square,\square,1/3\right)\oplus(\mathbf{1},\mathbf{1},2)\ , \tag{C.15}
$$

we have the quantum numbers of the remaining SM matter fields.

We gathered the symmetric configurations together to form the symmetric representation, but we had to gather them with a quark-like configuration that must be put symmetrically alongside the others

$$
\mathbf{15}^{ij} = \left(\yng(2),\mathbf{1},-4/3\right)\oplus\left(\mathbf{1},\yng(2),2\right)\oplus\left(\square,\square,1/3\right)\ , \tag{C.16}
$$

this representation is not used in minimal context because there is not either a **6** dimensional $\mathsf{SU(3)}$ nor a **3** dimensional $\mathsf{SU(2)}$ fermions in the SM. Recall, however, that a scalar **3** dimensional $\mathsf{SU(2)}$ is needed to perform Type-II seesaw mechanism.

One can write an explicit matrix form for the fermion **10** we must keep in mind we need to antisymmetrize the indexes, one has

$$
\mathbf{10}^{ij} = \begin{pmatrix} (\epsilon_3)^{ijk}u^c_k & q \\ -q^T & (\epsilon_2)^{ij}e^c \end{pmatrix} = \begin{pmatrix} 0 & u^c_3 & -u^c_2 & u_1 & d_1 \\ -u^c_3 & 0 & u^c_1 & u_2 & d_2 \\ u^c_2 & -u^c_1 & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{pmatrix}_L\ . \tag{C.17}
$$

Finally we present the **45** since we need it to build renormalizable realistic models as we do in Chapter 3, it can be computed through

$$
\mathbf{10}\otimes\overline{\mathbf{5}} = \mathbf{45}\oplus\mathbf{5}\ , \tag{C.18}
$$

and the composition in SM fields will have the quantum numbers

$$
\begin{aligned}
\mathbf{45} = {} & \left(\begin{array}{@{}l@{}l@{}}\square&\square\\ \square&\end{array}, \square, 1/2\right) \oplus \left(\overline{\square\square}, \mathbf{1}, -1/3\right) \oplus \left(\square, \square\square, -1/3\right) \oplus \left(\overline{\square}, \square, -7/6\right) \oplus \\
& \oplus \left(\square, \mathbf{1}, -1/3\right) \oplus \left(\overline{\square}, \mathbf{1}, 4/3\right) \oplus \left(\mathbf{1}, \square, 1/2\right) \ ,
\end{aligned} \tag{C.19}
$$

where we note the important feature of incorporating a SM-like configuration.

Transformation Rules and Notation

We now derive explicitly the transformation rules for the basic representations and then generalize them to an arbitrary field in tensor notation.

A field with a single superscript transforms trivial as

$$
\psi'^i = U^i{}_j \psi^j \ , \tag{C.20}
$$

as the transformation does not conjugate indexes, this can be seen easily by realizing that the transformation will have the index structure of the generators since we can apply the exponential map[3]

$$
U^i{}_j = \exp\left(-i\alpha^a \lambda^a/\sqrt{2}\right)^i{}_j \ . \tag{C.21}
$$

A double (upper) index field is treated in tensor notation as a direct product of two fundamental fields

$$
\Psi^{ij} = \psi^i \otimes \phi^j \ . \tag{C.22}
$$

For example, the antisymmetric representation is then

$$
\Psi^{ij}_{AS} = (\psi^i \otimes \phi^j)_{AS} = (\psi^i \wedge \phi^j) = (\psi^i \phi^j - \psi^j \phi^i) \ , \tag{C.23}
$$

which transforms

$$
\begin{aligned}
\Psi'^{ij}_{AS} = & (\psi'^i \phi'^j - \psi'^j \phi'^i) \\
= & (U^i{}_k \psi^k U^j{}_l \phi^l - U^j{}_l \phi^l U^i{}_k \psi^k) \\
= & U^i{}_k U^j{}_l \Psi^{kl}_{AS} \\
= & U^i{}_k \Psi^{kl}_{AS} (U^T)_l{}^j \ ,
\end{aligned} \tag{C.24}
$$

where the last result is in a suitable form to perform calculations using matrix formalism which is what we eventually do.

Similarly one can conclude for a symmetric representation with two indexes

$$
\Psi'^{ij}_S = U^i{}_k U_l{}^j \Psi^{kl}_S = U^i{}_k \Psi^{kl}_S (U^T)_l{}^j \ , \tag{C.25}
$$

and as one can decompose a tensor in its symmetric and antisymmetric part a two (upper) index representation transforms as

[3]Note that SU(5) is a simple classic Lie group and so one can use matrix notation by treating it as a Lie matrix group [58].

$$\Psi'^{ij} = U^i{}_k U_l{}^j \Psi^{kl} = U^i{}_k \Psi^{kl} (U^T)_l{}^j \ , \tag{C.26}$$

For the subscript indexes, we recall that a index changes with conjugation and so we have

$$\Psi'_i = (U^i{}_j \Psi^j)^\dagger = \Psi_j (U^\dagger)^j{}_i \ . \tag{C.27}$$

Since each index transforms separately and we can then see that a field with the two types of index

$$\Psi^j{}_i = \psi^j \otimes \phi_i \ , \tag{C.28}$$

will then transform as

$$\Psi'^i{}_j = U^i{}_k \Psi^k{}_l (U^\dagger)^l{}_j \ . \tag{C.29}$$

The generalization is now obvious, superscript indexes transform as

$$X^{i_1 i_2 \cdots} \to U^{i_1}{}_{j_1} U^{i_2}{}_{j_2} \ldots X^{j_1 j_2 \cdots} \ , \tag{C.30}$$

while the subscript indexes

$$X_{i_1 i_2 \ldots} \to X_{j_1 j_2 \ldots} (U^\dagger)^{j_1}{}_{i_1} (U^\dagger)^{j_2}{}_{i_2} \cdots \ , \tag{C.31}$$

which leads us to the general rule

$$X^{i_1 i_2 \cdots}{}_{j_1 j_2 \ldots} \to U^{i_1}{}_{k_1} U^{i_2}{}_{k_2} \ldots X^{k_1 k_2 \cdots}{}_{l_1 l_2 \ldots} (U^\dagger)^{l_1}{}_{j_1} (U^\dagger)^{l_2}{}_{j_2} \cdots \ . \tag{C.32}$$

Electric Charge

One particularly important generator is the electric charge generator, Q. The electric charge generator is the generator of the remaining abelian group, the electromagnetism, after the SM spontaneous symmetry breaking, in $\mathsf{SU(5)}$ it is identified as

$$Q = \frac{1}{2} \left(\lambda_{23} - \sqrt{\frac{5}{3}} \lambda_{24} \right) \ . \tag{C.33}$$

Since electromagnetism is an abelian group the quantum numbers will be read from the eigenvalues of the generator. On the other hand because it is diagonalized the transformations associated with the group can be written as

$$U^i{}_j = \exp\left(-i\alpha Q/\sqrt{2}\right)^i{}_j = \exp\left(-i\alpha Q^i/\sqrt{2}\right) \delta^i{}_j \ , \tag{C.34}$$

where no summation is assumed in i and Q^i is then the eigenvalue of the field in which the transformation is applied, namely to its i entry

$$Q\psi^i = Q^i \psi^i \ , \tag{C.35}$$

and we can also construct a two dimensional array for a double index field

$$Q\Psi^{ij} = Q^{ij}\Psi^{ij} \ . \tag{C.36}$$

To read the electric charge of a field with two superscript indexes we consider the transformation of the electromagnetism group made upon it by

$$U^i{}_k U^j{}_l = \exp\left(-i/\sqrt{2}\theta Q^i\right)\delta^i{}_k \exp\left(-i/\sqrt{2}\theta Q^j\right)\delta^j{}_l \ , \tag{C.37}$$

when applied to the field we will then have

$$U^i{}_k U^j{}_l \Psi^{kl} = \exp\left(-i\alpha/\sqrt{2}(Q^i + Q^j)\right)\delta^i{}_k \delta^j{}_l \Psi^{kl} = \exp\left(-i\alpha/\sqrt{2}Q^{ij}\right)\Psi^{ij} \ , \tag{C.38}$$

where we identified

$$Q^{ij} = Q^i + Q^j \ . \tag{C.39}$$

For a subscript index, i.e. a conjugated index, we use the fact that by being diagonal the transformation reads

$$U^\dagger = U^* \ , \tag{C.40}$$

and so we have

$$Q_i\psi_i = -Q^i\psi^i \ , \tag{C.41}$$

this leads us to the array of electric charges of a field with two subscript indexes

$$Q_{ij} = Q_i + Q_j = -Q^i - Q^j \ , \tag{C.42}$$

and for a field with the two types of index

$$Q^i{}_j = Q^i + Q_j = Q^i - Q^j \ . \tag{C.43}$$

The explicit forms of these arrays of electric charges are then

$$Q^i = \begin{pmatrix} -1/3 \\ -1/3 \\ -1/3 \\ 1 \\ 0 \end{pmatrix} \ , \tag{C.44}$$

$$Q^{ij} = \begin{pmatrix} -2/3 & -2/3 & -2/3 & 2/3 & -1/3 \\ -2/3 & -2/3 & -2/3 & 2/3 & -1/3 \\ -2/3 & -2/3 & -2/3 & 2/3 & -1/3 \\ 2/3 & 2/3 & 2/3 & -2 & 1 \\ -1/3 & -1/3 & -1/3 & 1 & 0 \end{pmatrix} \ , \tag{C.45}$$

and finally

$$Q^i{}_j = \begin{pmatrix} 0 & 0 & 0 & -4/3 & -1/3 \\ 0 & 0 & 0 & -4/3 & -1/3 \\ 0 & 0 & 0 & -4/3 & -1/3 \\ 4/3 & 4/3 & 4/3 & 0 & 1 \\ 1/3 & 1/3 & 1/3 & -1 & 0 \end{pmatrix} . \tag{C.46}$$

Note however that when reading the electric charge of an antisymmetric representation using (C.45) the diagonal entries will not have any meaning.

Appendix D

Extrema in an Adjoint Higgs Potential

Recall the most general renormalizable potential for an adjoint $\mathsf{SU(n)}$ represented scalar field Φ is

$$V(\Phi) = -\frac{\mu^2}{2}\ \mathrm{Tr}\left\{\Phi^2\right\} + \frac{a}{4}\ \mathrm{Tr}\left\{\Phi^2\right\}^2 + \frac{b}{4}\ \mathrm{Tr}\left\{\Phi^4\right\} + \frac{c}{3}\ \mathrm{Tr}\left\{\Phi^3\right\}\ . \tag{D.1}$$

We can always diagonalize Φ by a unitary global $\mathsf{SU(n)}$ transformation, we then get a diagonal vev

$$\Phi^i_{0j} = a_i \delta^i{}_j, \quad i,j = 1,...,n\ , \tag{D.2}$$

where the traceless nature of the adjoint matrices requires that

$$\sum_{i=1}^{n} a_i = 0\ . \tag{D.3}$$

The potential now reads

$$V(\Phi_0) = -\frac{\mu^2}{2}\sum_{i=1}^{n} a_i^2 + \frac{a}{4}\left(\sum_{i=1}^{n} a_i^2\right)^2 + \frac{b}{4}\sum_{i=1}^{n} a_i^4 + \frac{c}{3}\sum_{i=1}^{n} a_i^3\ . \tag{D.4}$$

Under these conditions holds the following lemma.

Lemma The potential admits extrema only if no more than two a_i are different.

The above lemma can be found in [59, 60] although the same conclusions were obtained earlier by Li in [29].

The breaking patterns of the $\mathsf{SU(5)}$ group is an important topic and so we briefly discuss the proof of this lemma.

Proof and discussion of the Lemma

We begin by studing a general fourth order polynomial function. We will consider for now the case where there is no cubic term.

Let $f(x_i)$ be a function with $n \geq 4$ real variables x_i given by

$$f(x_i) = \sum_{i=1}^{n} x_i^4\ . \tag{D.5}$$

Subject to the constraints[1]

$$\sum_{i=1}^{n} x_i^2 = \rho' \ , \quad \sum_{i=1}^{n} x_i = \sigma' \ . \tag{D.6}$$

We are looking for a necessary condition, so we begin by summoning a XIX century result concerning n^{th} order polynomials: it is a necessary condition for $f(x_i)$ to have extrema that it is extremal to four of its variables, being those arbitrarily chosen while the others are kept fixed.

We have then to consider the minimization problem for

$$f(x_i) = \sum_{i=1}^{4} x_i^4 \ . \tag{D.7}$$

Constrained to

$$\sum_{i=1}^{4} x_i^2 = \rho \ , \quad \sum_{i=1}^{4} x_i = \sigma \ . \tag{D.8}$$

Putting all together

$$f(x_i) = \sum_{i=1}^{4} x_i^4 + \lambda(\sum_{i=1}^{4} x_i^2 - \rho) + \mu(\sum_{i=1}^{4} x_i - \sigma) \ . \tag{D.9}$$

Being λ and μ Lagrange multipliers and we have chosen the four variables to be x_i with $i = 1, ..., 4$ without loss of generality.

Our stationarity conditions regarding the variables are

$$\frac{\partial f(x_i)}{\partial x_j} = 4x_i^3 + 2\lambda x_i + \mu = 0 \ . \tag{D.10}$$

We have also the equations for the Lagrange multipliers, but we will not need them as we shall see. These cubic equations are equal to all variables x_i, we conclude then there is only at most three different values the variables x_i can have. Noting that there are four of them, hence at least two of them must be equal.

For the case where the solutions are realized with three different possible values, these values summed vanish. This is easily shown, we consider now the stationarity equations

$$\begin{aligned} 4x_1^3 + 2\lambda x_1 + \mu = 0 \ , \\ 4x_3^3 + 2\lambda x_3 + \mu = 0 \ , \\ 4x_4^3 + 2\lambda x_4 + \mu = 0 \ , \end{aligned} \tag{D.11}$$

where, without loss of generality, we assumed $x_1 = x_2$ to be the variables with the same value. From these equations if we assume $x_1 \neq x_3 \neq x_4$ we conclude

$$(x_1 + x_3 + x_4) = 0 \ , \tag{D.12}$$

and so we conclude that when the three different values are obtained for a solution those values summed

[1]The first constraint is just a group invariant and so fixed. The second constraint is the general condition from which we will impose the traceless nature of the adjoint representation's matrices by setting $\sigma = 0$.

vanish.

Otherwise it is easy to see we will get two different types of solutions, one with $x_1 = x_2 \neq x_3 = x_4$ and the other with $x_1 \neq x_2 = x_3 = x_4$. We have then only to consider three types of solutions. Each case has its own consequences to the constraints that we discussed already, let us shortly resume:

- (i) $x_1 = x_2 \neq x_3 \neq x_4$

$$x_1 + x_2 + x_3 = 0 \ , \quad \sum_{i=1}^{4} x_i^2 = 2x_1^2 + x_3^2 + x_4^2 = \rho \ , \quad \sum_{i=1}^{4} x_i = 2x_1 + x_3 + x_4 = \sigma \ . \tag{D.13}$$

- (ii) $x_1 = x_2 \neq x_3 = x_4$

$$\sum_{i=1}^{4} x_i^2 = 2x_1^2 + 2x_3^2 = \rho \ , \quad \sum_{i=1}^{4} x_i = 2x_1 + 2x_3 = \sigma \ . \tag{D.14}$$

- (iii) $x_1 = x_2 = x_3 \neq x_4$

$$\sum_{i=1}^{4} x_i^2 = 3x_1^2 + x_4^2 = \rho \ , \quad \sum_{i=1}^{4} x_i = 3x_1 + x_4 = \sigma \ . \tag{D.15}$$

Since we have worked with necessary conditions, we can now solve these equations to obtain the values of x_i for which f respects the stationary conditions:

- (i) $x_1 = x_2 \neq x_3 \neq x_4$

$$x_1 = \sigma \ , \quad x_3 = \frac{1}{2}\left(-\sigma - \sqrt{2\rho - 5\sigma^2}\right) \ , \quad x_4 = \frac{1}{2}\left(-\sigma + \sqrt{2\rho - 5\sigma^2}\right) \ . \tag{D.16}$$

$$f(2,1,1) = \frac{1}{2}\left(\rho^2 - 2\rho\sigma^2 + 3\sigma^4\right) \ . \tag{D.17}$$

- (ii) $x_1 = x_2 \neq x_3 = x_4$

$$x_1 = \frac{1}{4}\left(\sigma - \sqrt{4\rho - \sigma^2}\right) \ , \quad x_3 = \frac{1}{4}\left(\sigma + \sqrt{4\rho - \sigma^2}\right) \ . \tag{D.18}$$

$$f(2,2) = \frac{1}{16}\left(4\rho^2 + 4\rho\sigma^2 - \sigma^4\right) \ . \tag{D.19}$$

- (iii) $x_1 = x_2 = x_3 \neq x_4$

$$x_1 = \frac{1}{12}\left(3\sigma \pm \sqrt{3}\sqrt{4\rho - \sigma^2}\right) \ , \quad x_4 = \frac{1}{4}\left(\sigma \mp \sqrt{3}\sqrt{4\rho - \sigma^2}\right) \ . \tag{D.20}$$

$$f_{\pm}(3,1) = \frac{1}{12}\left(7\rho^2 + \rho\sigma^2 - \frac{1}{4}\sigma^4 \pm \frac{1}{6}|\sigma|\left(12\rho - 3\sigma^2\right)^{3/2}\right) \ . \tag{D.21}$$

Note that in (D.21) we named the two possible values of f according to the possible sign σ can have.

We shall study the first case to rule it out, and so the lemma becomes proved since the other solutions have at most two different values for the variables of f. For that consider the difference that f has for the different solutions

$$\begin{aligned} f(2,1,1)-f(2,2) &= \frac{1}{16}\left(2\rho-5\sigma^2\right)^2 \ , \\ f_+(3,1)-f_-(3,1) &= \frac{|\sigma|\left(4\rho-\sigma^2\right)^{3/2}}{4\sqrt{3}} \ , \\ f(2,1,1)-f_+(3,1) &= -\frac{1}{12}\left(\rho^2+13\rho\sigma^2+\frac{37}{2}\sigma^4+\sqrt{3}\left(4\rho-\sigma^2\right)^{3/2}|\sigma|\right) \ . \end{aligned} \tag{D.22}$$

The first two equations are always positive while the third equation is always negative. We then conclude that

$$f(2,2) < f(2,1,1) < f_+(3,1) \ , \tag{D.23}$$

and so $f(2,1,1)$ is not an extremum. It is easily checked that the sign of $f(2,2)-f_-(3,1)$ is not unambiguously determined, and so we do not have a necessary condition for a minimum but we now know that the maximum is obtained for $f_+(3,1)$, i.e., a necessary condition for the maximum is that three of the variables get the same value.

Generalization of this result for the n variables case is straightforward: We have that for n variables our stationarity conditions are n equations with the exact same form as of (D.10), and remember that our necessary condition is such that we must find extrema for four variables keeping the others $n-4$ fixed; with these considerations we note that we will always choose four arbitrary variables and work with the same three possible types of solution. We then conclude that the extrema are always obtained when no more than two variables are different and that the absolute maximum is always obtained when $n-1$ of the variables have the same value.

We will now focus on the case where there is a cubic term. We will not introduce it as a constraint but rather as a new term of the function. Let then be

$$f(x_i) = \frac{1}{4}\sum_{i=1}^{4} x_i^4 + a\frac{1}{3}\sum_{i=1}^{4} x_i^3 \ . \tag{D.24}$$

At this point we have to put explicit parameters. The numerical factors are chosen so they cancel out other numerical factors that come from the derivatives. The function f is constrained by the same previous constraints. Introducing the constraints with Lagrange Multipliers the function now reads

$$f(x_i) = \frac{1}{4}\sum_{i=1}^{4} x_i^4 + a\frac{1}{3}\sum_{i=1}^{4} x_i^3 + b\frac{1}{2}\left(\sum_{i=1}^{4} x_i^2 - \rho\right) + c\left(\sum_{i=1}^{4} x_i - \sigma\right) \ . \tag{D.25}$$

The stationarity conditions now hold

$$\frac{\partial f(x_i)}{\partial x_j} = x_j^3 + a x_j^2 + b x_j + c = 0 \ . \tag{D.26}$$

The previous argument is still valid: we have the same cubic equation for all the variables, only a set of at most three values for all the variables will satisfy the stationarity conditions. As we did before, we first compute the implicit constraint that arises from the stationarity conditions when we study the solution with the three different values. We then choose $x_1 = x_2$ and get

$$
\begin{aligned}
x_1^3 + ax_1^2 + bx_1 + c = 0 \ , \\
x_3^3 + ax_3^2 + bx_3 + c = 0 \ , \\
x_4^3 + ax_4^2 + bx_4 + c = 0 \ .
\end{aligned}
\tag{D.27}
$$

After manipulating the equations we conclude

$$
x_1 + x_3 + x_4 = -a \ . \tag{D.28}
$$

We now compute the solutions as we did previously and the values f has for each of them:

- (i) $x_1 = x_2 \neq x_3 \neq x_4$

$$
x_1 = a + \sigma \ , \quad x_3 = -\frac{\sigma}{2} - a - \frac{1}{2}\sqrt{\frac{y_1^2 - y^2}{2}}, \quad x_4 = -\frac{\sigma}{2} - a + \frac{1}{2}\sqrt{\frac{y_1^2 - y^2}{2}} \ , \tag{D.29}
$$

$$
f(2,1,1) = \frac{1}{768}\left(3y^4 - 6y_1^2\left(y^2 - 2y\sigma + 3\sigma^2\right) + 4y\sigma^3 + 6y_1^4 - 9\sigma^4\right) \ , \tag{D.30}
$$

for

$$
y_1^2 > y^2 \ . \tag{D.31}
$$

- (ii) $x_1 = x_2 \neq x_3 = x_4$

$$
x_1 = \frac{1}{4}\left(\sigma - y_1\right) \ , \quad x_3 = \frac{1}{4}\left(\sigma + y_1\right) \ , \tag{D.32}
$$

$$
f(2,2) = \frac{1}{768}\left(\sigma^3(16a + 3\sigma) + 6\sigma y_1^2(8a + 3\sigma) + 3y_1^4\right) \ . \tag{D.33}
$$

- (iii) $x_1 = x_2 = x_3 \neq x_4$

$$
x_1 = \frac{1}{4}\left(\sigma \pm \frac{y_1}{\sqrt{3}}\right) \ , \quad x_4 = \frac{1}{4}\left(\sigma \mp \sqrt{3}y_1\right) \ , \tag{D.34}
$$

$$
f_\pm(3,1) = \frac{1}{2304}\left(3\sigma^3(16a + 3\sigma) \mp 8\sqrt{3}y_1^3(4a + 3\sigma) + 18\sigma y_1^2(8a + 3\sigma) + 21y_1^4\right) \ . \tag{D.35}
$$

Where we have rewritten some terms so the results would be easier to read by using

$$
y_1 = (4\rho - \sigma)^{1/2} \ , \ y = 4a + 3\sigma \ . \tag{D.36}
$$

It is important to check that these solutions return the same we had before when $a \to 0$. The differences between distinct values of f for the different stationary points are now

$$
\begin{aligned}
f(2,1,1) - f(2,2) &= \frac{1}{256}\left(y^2 - y_1^2\right)^2 \ , \\
f_+(3,1) - f_-(3,1) &= -\frac{y y_1^3}{48\sqrt{3}} \ , \\
f_\pm(3,1) - f(2,1,1) &= -\frac{1}{2304}\left(9y^4 \pm 8\sqrt{3}y_1^3 y - 18y_1^2y^2 - 3y_1^4\right) \\
&= \frac{1}{2304}(3y \mp \sqrt{3}y_1)\left[8yy_1^2 - (3y \pm \sqrt{3}y_1)(y_1^2 + y^2)\right] = \Delta^\pm \ , \\
f(2,2) - f_\pm(3,1) &= y_1^3\frac{3}{576}\left(\pm\frac{2}{\sqrt{3}}y - y_1\right) \ .
\end{aligned}
\tag{D.37}
$$

One can derive a lot of conditions from the above results, many of them are too constrained that it becomes difficult to deduce important conclusions. Instead we focus only on the following direct condition

$$
f(2,2) < f(2,1,1) \ . \tag{D.38}
$$

This important result states that a stationary point where the variables get the three different values is not a minimum of the function.

We want to show $f(2,1,1)$ is not extremum at all, for that we need to show it is not a maximum. Consider that it could be an extremum, since it could only be a maximum we would then have

$$
f_\pm(3,1) < f(2,1,1) \ , \tag{D.39}
$$

i.e.

$$
\Delta^\pm < 0 \ . \tag{D.40}
$$

We get the following conditions

$$
\begin{aligned}
9y^4 + 8\sqrt{3}y_1^3y - 18y_1^2y^2 - 3y_1^4 > 0 \ , \\
9y^4 - 8\sqrt{3}y_1^3y - 18y_1^2y^2 - 3y_1^4 > 0 \ .
\end{aligned}
\tag{D.41}
$$

Using (D.31) and (D.36) and reminding the above conditions ought to be simultaneously respected we conclude that $f(2,1,1)$ is impossible to be a maximum and therefore an extremum.

It is interesting to realize that the inclusion of cubic term in the function does not modify the extrema's structures.

Computing the vev and Breaking Patterns of SU(5)

We now discuss how to apply these results to the SU(5) GUT, in particular how do we retrieve all the breaking patterns and the vev.

It was already discussed that the diagonal vev has at most three different eigenvalues, we represent all the possibilities as

$$\mathrm{diag}(\alpha, \alpha, \alpha, \beta, -3\alpha - \beta) \tag{D.42}$$

$$\mathrm{diag}(\alpha, \alpha, \beta, \beta, -2\alpha - 2\beta) \tag{D.43}$$

$$\mathrm{diag}(\alpha, \alpha, \alpha, \alpha, -4\alpha) \tag{D.44}$$

$$\mathrm{diag}(\alpha, \alpha, \alpha, -3/2\alpha, -3/2\alpha) \tag{D.45}$$

$$\mathrm{diag}(0, 0, 0, 0, 0)\ . \tag{D.46}$$

All of the presented possibilities account already for the traceless nature of the `vev`. The last eigenvalue is bounded to the first two and it may be either different from the others or equal to one of them. We can determine all the possibilities straightforwardly: pick the last four eigenvalues and impose upon them the conditions we have encountered, those conditions will determine the true structure of the solutions. For completion and as an example we develop this reasoning for some possibilities.

We again begin by the vanishing cubic term case. The procedure is straightforward after we start by a possible diagonal for the `vev` we then pick the last four eigenvalues and apply to them the derived conditions. Start for example with (D.42) and now pick the set of eigenvalues $\{\alpha, \alpha, \beta, -3\alpha - \beta\}$. We know $\alpha \neq \beta$ but this set of four eigenvalues can still be $f(2,1,1)$, $f(2,2)$ or $f_{\pm}(3,1)$.

- $f(2,1,1)$

  In this case it holds $\alpha + \beta - 3\alpha - \beta = 0$, i.e.

$$\alpha = 0\ . \tag{D.47}$$

  So the `vev` actually depends only on one parameter v_{311} and can be written as

$$\Phi_0(311) = v_{311}\ \mathrm{diag}\ (0, 0, 0, 1, -1)\ . \tag{D.48}$$

  This solution will not respect the conditions for extrema for the potential, this is obvious since we used the results for a $f(2,1,1)$ solution which does not return an extremum. Nevertheless it is a stationary point of the potential.

- $f(2,2)$

  In this case we have $\beta = -3\alpha - \beta$, and so we conclude

$$\beta = -3/2\alpha\ . \tag{D.49}$$

  The `vev` is also an one parameter matrix and it is given by

$$\Phi_0 = v\ \mathrm{diag}\ (1, 1, 1, -2/3, -2/3)\ . \tag{D.50}$$

- $f_{\pm}(3,1)$

  As $\alpha \neq \beta$ we only get

$$\alpha = -3\alpha - \beta \ , \tag{D.51}$$

which leads to $\beta = -4\alpha$ and once again we get an one parameter matrix whose structure is given by

$$\Phi_0 = v_{41} \text{ diag } (1,1,1,1,-4) \ . \tag{D.52}$$

With the same procedure we can evaluate (D.43) which will return a new type of solution apart from other solutions already returned from (D.42). The cases (D.44) and (D.45) are trivial, each of them corresponds to a one parameter diagonal form of the vev, which were already obtained by the previous cases. Finally (D.46) corresponds to the vanishing vev, i.e. to the SU(5) unbroken by this adjoint scalar field.

Putting together all the possible vev we conclude that all the possible vev correspond to the following one parameter diagonal forms:

$$\Phi_0 = v\frac{1}{\sqrt{15}} \text{ diag } (2,2,2,-3,-3) \tag{D.53}$$

$$\Phi_{41} = v_{41} \text{ diag } (1,1,1,1,-4) \tag{D.54}$$

$$\Phi_{221} = v_{221} \text{ diag } (1,1,-1,-1,0) \tag{D.55}$$

$$\Phi_{311} = v_{311} \text{ diag } (0,0,0,1,-1) \tag{D.56}$$

$$\Phi_{trivial} = \text{ diag } (0,0,0,0,0) \tag{D.57}$$

We notice we have redefined (D.53) in order it to be explicitly proportional to the hypercharge generator.

As all of the possible vev are one parameter matrices, we can now compute the value of that parameter in the minimum of the potential. We note that the first two cases, (D.53) and (D.54) break the SU(5) group into the SM group and $\mathsf{SU(4)} \times \mathsf{U(1)}$ respectively, while the others will not be extrema of the potential.

We get

$$v^2 = \frac{15\mu^2}{30a+7b} \tag{D.58}$$

$$v_{41}^2 = \frac{\mu^2}{20a+13b} \tag{D.59}$$

$$v_{221}^2 = \frac{\mu^2}{2a+b} \tag{D.60}$$

$$v_{311}^2 = \frac{\mu^2}{4a+b} \ . \tag{D.61}$$

All these possibilities return a negative value for the potential, i.e. they all stand for a potential lower than the one we would get if there would be a vanishing vev of the type (D.57). It is then easily seen that when all the solutions are valid the hierarchy between the different vacua is

$$V(\Phi_0) < V(\Phi_{221}) \ , V(\Phi_{311}) \ , V(\Phi_{41}) \text{ for } b > 0 \tag{D.62}$$

$$V(\Phi_{41}) < V(\Phi_{221}) \ , V(\Phi_{311}) \ , (\Phi_0) \text{ for } b < 0 \ . \tag{D.63}$$

And so $\mathsf{SU(5)}$ breaks into the SM if $b > 0$ and into $\mathsf{SU(4)} \times \mathsf{U(1)}$ for $b < 0$. One easily checks that all (non vanishing) vacua are degenerate for $b = 0$.

The presence of the cubic term does not alter the type of solutions, at least the extrema. This is clearly seen: the cubic term will only modify the $f(2,1,1)$ type stationary points, which are not extrema, by imposing $x_1 + x_3 + x_4 = -a$.

On the other hand the cubic term lifts the degeneracy for the possible values of v and v_{41} by eliminating a Z_2 symmetry over the adjoint field, we now have

$$v(\pm) = \sqrt{15}\frac{c \pm \sqrt{c^2 + 4(30a + 7b)\mu^2}}{60a + 14b} \ , \tag{D.64}$$

$$v_{41}(\pm) = \frac{1}{2}\frac{3c \pm \sqrt{9c^2 + 4\mu^2(20a + 13b)}}{20a + 13b} \ . \tag{D.65}$$

It is not obvious what is the hierarchy between the vacua[2]. We can still treat this result: by demanding boundness from below for either solutions we separate the parameter space into two different regions

$$b > 0 \wedge a/b > -7/30 \ , \tag{D.66}$$

$$b < 0 \wedge a/b > -13/20 \ . \tag{D.67}$$

Actually these are the same regions one gets without the cubic term. The difference resides in the fact that without the cubic term these regions are also the distinct regions for the SM minimum and $\mathsf{SU(4)} \times \mathsf{U(1)}$ minimum.

With this conditions one finds out immediately

$$V(v(+)) > V(v(-)) \text{ for } c < 0 \ , \ V(v(+)) < V(v(-)) \text{ for } c > 0 \ , \tag{D.68}$$

and similarly

$$V(v_{41}(+)) > V(v_{41}(-)) \text{ for } c < 0 \ , \ V(v_{41}(+)) < V(v_{41}(-)) \text{ for } c > 0 \ . \tag{D.69}$$

When computing the difference of the potential $V(v) - V(v_{41})$ in both cases one gets a simple result: the $V(v_{41})$ is minimal for (D.67) while $V(v)$ is minimal for (D.66) independently of c's sign, which is the exact same result that without the cubic term. This conclusion is neither obvious nor immediate and so it is interesting.

[2]We excluded from this discussion the other stationary points, we hope that at this point no doubts are left that they are not extrema.

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