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INSTITUTO SUPERIOR TÉCNICO

Hydrodynamic behavior of a Degenerate Microscopic Dynamics with Slow Reservoirs

Renato Ricardo de Paula

Supervisor: Doctor Ana Patrícia Carvalho Gonçalves

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To my family.

To my friend Marcela Nakamura.

Resumo

Esta tese trata da análise do limite hidrodinâmico de um sistema de partículas em interação (SPI) com taxas degeneradas, nomeadamente, o modelo em meios porosos (MMP) evoluindo no espaço discreto $\{0, ..., n\}$, onde $n > 1$ e os sítios 0 e n representam reservatórios lentos. O nome lento significa que os reservatórios possuem um fator de escala $n^{-\theta}$, e quanto maior o valor de $\theta \geq 0$, mais lenta é a dinâmica do processo na fronteira. Mais especificamente, as partículas podem ser inseridas no sistema no sítio 1 (resp. $n-1$) com taxa $\kappa\alpha n^{-\theta}$ (resp. $\kappa\beta n^{-\theta}$), e podem ser removidas do sistema através do sítio 1 (resp. $n-1$) com taxa $\kappa(1-\alpha)n^{-\theta}$ (resp. $\kappa(1-\beta)n^{-\theta}$), onde $\alpha,\ \beta \in (0,1)$, $\theta \geq 0$, e $\kappa > 0$. A ideia de adicionar esses reservatórios é ver se essas perturbações têm impacto sobre o comportamento macroscópico do sistema. Normalmente, essas perturbações, sendo locais, não destroem a natureza da equação macroscópica, mas em vez disso, trazem condições de fronteira adicionais.

A nossa estratégia para caracterizar o comportamento do MMP com reservatórios lentos baseia-se no Método de Entropia de Guo, Papanicolau e Varadhan [27]. Este procedimento limite estabelece que a densidade espacial das partículas do MMP com reservatórios lentos converge para a única solução fraca da equação macroscópica correspondente, chamada equação hidrodinâmica, e representada nesta tese pela equação em meios porosos (EMP). No entanto, este método não pode ser aplicado de forma direta, pois há configurações que não evoluem de acordo com a dinâmica do processo (as chamadas configurações bloqueadas). Para evitar esse problema, perturbamos ligeiramente a dinâmica de forma que o comportamento macroscópico do sistema continue seguindo a equação em meios porosos, mas com condições de fronteira que dependem da intensidade dos reservatórios. Mais especificamente, obtemos três tipos diferentes de condições de fronteira: Dirichlet, Robin e Neumann. Como consequência do limite hidrodinâmico, obtemos a lei de Fick. Obtemos também estimativas de energia suficientemente fortes que permitem obter informação detalhada sobre o comportamento na fronteira das soluções fracas da equação em meios porosos.

Palavras-chave: Limite hidrodinâmico, Sistema de partículas em interação, Modelo em meios porosos, Equação em meios porosos, Condições de fronteira.

Abstract

This thesis is concerned with the analysis of the hydrodynamic limit of an interacting particle system (IPS) with degenerate rates, namely, the porous medium model (PMM) evolving in a discrete space $\{0,\ldots,n\}$, where $n>1$ and the sites 0 and n stand for slow reservoirs. The name slow means that the reservoirs are scaled by a factor $n^{-\theta}$, and the higher the value of $\theta\geq 0$, the slower the boundary dynamics. More specifically, particles can be inserted into the system at the site 1 (resp. $n-1$) with rate $\kappa\alpha n^{-\theta}$ (resp. $\kappa\beta n^{-\theta}$), and can be removed from the system through the site 1 (resp. $n-1$) with rate $\kappa(1-\alpha)n^{-\theta}$ (resp. $\kappa(1-\beta)n^{-\theta}$), where $\alpha,\ \beta\in(0,1)$, $\theta\geq 0$, and $\kappa>0$. The idea of adding these reservoirs is to see whether these perturbations have an impact over the macroscopic behavior of the system. Usually, these perturbations, being local, do not destroy the macroscopic equation's nature, but instead, they bring up additional boundary conditions.

Our strategy to characterize the behavior of the PMM with slow reservoirs relies on the Entropy Method of Guo, Papanicolau, and Varadhan [27]. This limit procedure states that the spatial density of particles of the PMM with slow reservoirs converges to the unique weak solution of the corresponding macroscopic equation, the so-called hydrodynamic equation, represented in this thesis by the porous medium equation (PME). However, this method cannot be straightforwardly applied, since there are configurations that do not evolve under the dynamics (the so-called blocked configurations). In order to avoid this problem, we slightly perturbed the dynamics in such a way that the macroscopic behavior of the system keeps following the porous medium equation, but with boundary conditions that depend on the reservoirs' strength. More specifically, we obtain three different types of boundary conditions (Dirichlet, Robin, and Neumann) depending on the intensity of the rate at the reservoir's dynamics. As a consequence of the hydrodynamic limit, we derived Fick's law of diffusion. We also derived sufficiently strong energy estimates which allow obtaining detailed information about the boundary behavior of the weak solutions of the porous medium equation.

Keywords: Hydrodynamic limit, Interacting particle system, Porous medium model, Porous medium equation, Boundary conditions.

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Chapter 1

Introduction

Throughout our civilization's history to the present day, we, human beings, have been trying to understand better the world that we live in. While the pursuit is never ending, this curious journey has shown us how different and extraordinary is the world through different scales. For example, our ancestors would not have imagined how different astronomy could be after the telescope (1609), and the biology after the microscope (1674) have been invented. This thesis focuses on the derivation of macroscopic patterns that arise by studying microscopic random models. Let us take the glass formation as an example. Despite glass being a very present material in our lives and having produced them on a large scale, understanding how its formation process happens has been a subject of great interest for scientists and industries worldwide. The idea behind making glass is to take a hot liquid mixture of silica, sodium carbonate, calcium carbonate and cool it rapidly to avoid crystal structure formation. There are several video examples available on the internet in which we can see this process. Now, suppose that we augment our sensory perception with a microscope to see the process again. Thus, in the beginning, we would observe the molecules moving randomly without any specific purpose. Then, while cooling it, we would observe the same chaotic motion of the particles, but now they can not move easily, so they move slower and slower until they ultimately get stuck. The question here is: how do these particles interact with each other in such a way as to form the glass state as we know it? In principle, the analysis could be done using the standard mathematical approach of Classical/Quantum Mechanics, which consists of two main ingredients: the system's complete state at a given time and an equation of motion. However, since the number of particles in the system is huge (of the order of Avogadro's number), we can not use the standard approach to precisely describe the microscopic state of the system, i.e., the position and velocity of each particle throughout time. The idea then to overcome this problem comes from Statistical Mechanics, whose concepts add some uncertainty to the system's microscopic state, i.e., instead of considering a deterministic motion of the particles, consider it as being stochastic. Thus, our goal is to show how we can derive a macroscopic pattern from a random movement of a collection of particles like the one described above.

The process that describes this collection of particles' evolution is the so-called *interacting particle system (IPS)*. The framework of IPS was introduced independently in the $1970s$ by Spitzer [40] and

Dobrushin [12, 13]. We will focus on a family of IPSs widely used to study certain aspects of glassy behavior, called *kinetically constrained lattice gases (KCLG)*. Therefore, this thesis aims presenting a rigorous mathematical proof of how we can derive a certain *partial differential equation (PDE)* - the *porous medium equation (PME)* - from a member of this family - the *porous medium model (PMM)* - whose ends are attached to slow reservoirs.

In recent years, there has been an intensive research activity around the derivation of PDEs with boundary conditions from IPSs [25, 31]. This derivation is known as hydrodynamic limit, and consists in proving, rigorously, that the conserved quantity of a random microscopic dynamics is described by the solution of some PDE. Therefore, this PDE coins the name hydrodynamic equation. A vast literature has been produced on the hydrodynamic limit with many important techniques, e.g., the method of v-functions, see [8] for a review, the Entropy Method [27], the Relative Entropy Method [43], non-gradient techniques, and others. This thesis will focus on the Entropy Method of Guo, Papanicolau, and Varadhan introduced in [27]. Note that the hydrodynamic limit consists (probabilistically speaking), in a Law of Large Numbers for the empirical measure associated with the conserved quantities of the system (the density of particles in this thesis). More recently, there has been quite a lot of attention devoted to analyzing microscopic systems with local perturbations. One of the puzzling questions is to see whether these perturbations have an impact over the macroscopic behavior of the system. Usually, these perturbations, being local, do not destroy the PDE's nature, but instead, they bring up additional boundary conditions to the PDE [2, 6, 21, 23]. In case of microscopic systems with independent particles, we usually have linear hydrodynamic equations, otherwise, we have nonlinear hydrodynamic equations. See for instance [25] and references therein.

In light of these questions, we present the derivation of the porous medium equation with boundary conditions from a microscopic dynamics, which is placed in contact with reservoirs. As a consequence, we also derive Fick's law of diffusion. Up to our knowledge, this is the first derivation of a nonlinear degenerate parabolic PDE with boundary conditions that can be obtained as the hydrodynamic limit of an underlying microscopic random dynamics. More specifically, we obtain three different types of boundary conditions (Dirichlet, Robin, and Neumann) depending on the intensity of the rate at the reservoir's dynamics. We remark, however, that the first microscopic derivation of the PME was obtained in [14] and [17], in which the authors considered a model with continuous occupational variables. The first microscopic derivation considering discrete occupational variables was obtained in [26]. There, the authors considered a random microscopic dynamics - porous medium model - evolving in the discrete d-dimensional torus $\mathbb{T}_n^d$ without the presence of reservoirs. Therefore, the PME did not have any type of boundary conditions. This motivated us to work with discrete occupational variables in order to derive the PME, that is, to consider as the random microscopic dynamics, an ad-hoc version of the PMM analyzed there. With the aim to derive boundary conditions in the PME, we combined the microscopic dynamics of [26] with the boundary dynamics of [2]. In the latter article, the dynamics was given by the *symmetric simple exclusion process (SSEP)* in contact with reservoirs that has a parameter that regulates its strength (called *slow reservoirs*). Thus, the authors obtained the heat equation with different boundary conditions, namely Dirichlet, Robin, and Neumann.

Now let us precisely describe the *PMM with slow reservoirs*. First, in order to do that we fix a scaling parameter $n \in \mathbb{N}$, and a parameter $m \in \mathbb{N}$ whose role is to regulate the constraints strength. Then, we fix the discrete space where particles will be moving around, that is, the space $\Sigma_n := \{1, \ldots, n-1\}$, that we call *bulk*. We call an element $x \in \Sigma_n$ by site, which can be empty or occupied, and we note that we can not have more than one particle per site (*exclusion rule*). We denote the configuration of particles by the function $\eta : \Sigma_n \to \{0,1\}$ since according to IPS framework, they are usually denoted by a Greek letter. For $x \in \Sigma_n$, we write down $\eta(x)$ for the occupation variable, that represents the number of particles at site x in the configuration η. Since $\eta(x)$ takes values in $\{0,1\}$, then $\eta(x) = 0$ (resp. $\eta(x) = 1$) stands for an empty (resp. occupied) site. Therefore, the state space of our process is $\{0,1\}^{\Sigma_n}$ and η is an element of this set. The PMM is a continuous-time Markov process where particles jump to nearest-neighbor sites under the exclusion rule, however, within some constraints. For instance, let $\eta \in \{0,1\}^{\Sigma_n}$ and suppose that a particle at site x wants to perform a jump to the site $x+1$. This jump happens with the rate $c^m_{x,x+1}(\eta)$, as long as at least one of the set of points below is full of particles

$$\{x-(m-1), \ldots, x-1\}, \{x-(m-2), \ldots, x-1, x+2\}, \ldots, \{x+2, x+3, \ldots, x+m\},$$

see Figure 1.1.

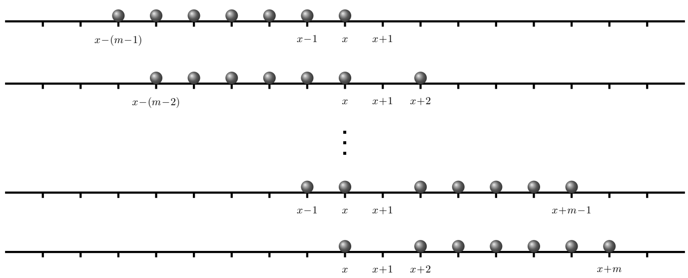


Figure 1.1: Sets of points for which the particle jump from site x to site $x+1$ in the PMM dynamics.

The jump mechanism from $x+1$ to x is the same as described above. Now, replace each set of points above by occupation variables' products, i.e., $\{x-(m-1), \ldots, x-1\}$ by $\eta(x-(m-1)) \cdots \eta(x-1)$, $\{x-(m-2), \ldots, x-1, x+2\}$ by $\eta(x-(m-2)) \cdots \eta(x-1)\eta(x+2)$ and so on. The jump rate $c^m_{x,x+1}(\eta)$ is defined as the sum of these particles' products. For example, for $m = 2$, the jump rate is given by $c^2_{x,x+1}(\eta) = \eta(x-1) + \eta(x+2)$ and a particle at site x performs the jump if, and only if, $c^2_{x,x+1}(\eta) > 0$. In the general case the exact form for the jump rate is given by

$$c^m_{x,x+1}(\eta) = \sum_{k=1}^{m} \prod_{\substack{j=-(m-k) \\ j \neq 0,1}}^{k} \eta(x+j).$$

The choice of m regulates the strength of the kinetic constraint, therefore the higher its value, the slower the movement of the particles. For example, when $m = 1$ the jump rate is $c^1_{x,x+1}(\eta) = 1$ and we recover the SSEP. However, for $m > 1$, for certain configurations $c^m_{x,x+1}(\eta) = 0$ might happen, i.e., if particles do not have a minimal number of neighboring sites (in our case $m-1$), they will not be able to move. As we mentioned above, in the case $m = 2$ the jump rate is given by the sum of the number of particles at sites $x-1$ and $x+2$. Due to the constraint of the model's rates, and since one can have configurations in which the distance between two successive particles is larger than two, the model exhibits the so-called *blocked configurations*, i.e., configurations that do not evolve under the dynamics. See Figure 1.2 below.

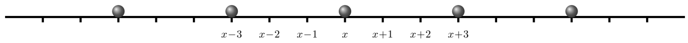


Figure 1.2: Example of blocked configuration for PMM (with $m = 2$).

This jump rate is chosen in such a way that the local constraints slow down the low density dynamics so that at the macroscopic level, we obtain a diffusion coefficient that degenerates with a low density of particles. We can see this relation by a heuristic argument used to derive the macroscopic equation, which in turn rules the space-time evolution of the density of particles of the PMM - the porous medium equation. We present this heuristic argument for the PMM with slow reservoirs in Chapter 3.

Now, let us add a technicality needed to finish the description of the bulk dynamics. Since the PMM presents blocked configurations, it is a reducible Markov process. However, as we will see later, to prove the hydrodynamic limit using the Entropy Method [27], we must have an irreducible Markov process, i.e., we have to avoid blocked configurations. Thus, to accomplish this, we will superpose the PMM dynamics on the SSEP dynamics on the bulk so that the macroscopic hydrodynamic behavior of this perturbed dynamics still evolves according to the PME. This means that when scaling the time diffusively, we tune the SSEP dynamics so that its impact is not seen at the macroscopic level.

At the boundary, we use the same dynamics introduced in [2], that is, a Glauber dynamics at sites 1 and $n-1$, which play the role of *reservoirs*. These reservoirs will also be scaled by a parameter which can be taken to infinity, and the highest its value, the slowest its impact. More specifically, the reservoirs dynamics can be described as follows. Particles can be inserted into the system at the site 1 (resp. $n-1$) with rate $\kappa\alpha n^{-\theta}$ (resp. $\kappa\beta n^{-\theta}$), and can be removed from the system through the site 1 (resp. $n-1$) with rate $\kappa(1-\alpha)n^{-\theta}$ (resp. $\kappa(1-\beta)n^{-\theta}$), where $\alpha, \beta \in (0,1)$, $\theta \geq 0$, and $\kappa > 0$. These reservoirs break down the conservation of the number of particles since there is a mass transfer between the reservoirs and the bulk. The factor $n^{-\theta}$ is the one that scales the boundary dynamics, and the higher the value of θ, the slower the boundary dynamics. However, we might ask what the role of the factor κ is. Actually, parameter κ is very important to study convergence results at the macroscopic level. Although κ is always present at the microscopic level, at the macroscopic level it only appears in the case $\theta = 1$. We will see below that in this case, our macroscopic equation is the porous medium equation with Robin boundary conditions depending on κ. Thus, the role of κ is to study the convergence of weak solutions of the PME with Robin boundary conditions as in [9] and [22]. This convergence depends on obtaining sufficiently strong energy estimates which is one of the main results of this thesis, and it is presented in Chapter 7. The argument we employ to their proof can be described as follows. We consider a proper L^2 space that gives adequate weights at the boundary points 0 and 1, which are explicitly given by (7.1). From this space, we define an energy functional which we prove to be finite on functions ξ such that their m-th power, namely ξ^m, belongs to $L^2(0,T;\mathcal{H}^1)$, where $\mathcal{H}^1$ is the usual Sobolev space. We also prove that the weak derivative of ξ^m exists, lives on the introduced L^2 space, and also satisfy the Robin boundary conditions, as given in (2.17), almost everywhere in time. Given this result, we then prove that the solution ρ^κ of the PME with Robin boundary conditions has finite energy with respect to that energy functional, so that all the aforementioned results come for free for ρ^κ. This is the content of Theorem 7.0.4.

The solution of the hydrodynamic equation is called *hydrodynamic profile*. Our hydrodynamic profiles are weak solutions of the PME with different boundary conditions depending on the range of the parameter θ. For $0 \leq \theta < 1$, we obtain the PME with Dirichlet boundary conditions, which is given by,

$$\begin{cases} \partial_t \rho_t(u) = \Delta\left(\rho_t(u)\right)^m, \quad (t,u) \in (0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, \quad t \in (0,T]. \end{cases} \tag{1.1}$$

For $\theta = 1$, the boundary dynamics is slowed enough so that the boundary conditions of Dirichlet type are replaced by a type of Robin boundary conditions given by

$$\begin{cases} \partial_t \rho_t(u) = \Delta\left(\rho_t(u)\right)^m, \quad (t,u) \in (0,T] \times (0,1), \\ \partial_u(\rho_t(0))^m = \kappa(\rho_t(0) - \alpha), \quad t \in (0,T], \\ \partial_u(\rho_t(1))^m = \kappa(\beta - \rho_t(1))\,, \quad t \in (0,T], \end{cases} \tag{1.2}$$

where $\kappa \in (0,\infty)$. Finally, for $\theta > 1$, the boundary is sufficiently slowed so that the Robin boundary conditions are replaced by Neumann boundary conditions,

$$\begin{cases} \partial_t \rho_t(u) = \Delta\left(\rho_t(u)\right)^m, \quad (t,u) \in (0,T] \times (0,1), \\ \partial_u(\rho_t(0))^m = \partial_u(\rho_t(1))^m = 0, \quad t \in (0,T]; \end{cases} \tag{1.3}$$

which dictate that, macroscopically, there is no flux of particles from the boundary reservoirs. A consequence of the degeneracy of these equations is that (depending on the initial condition) we do not have classical solutions of the problems above, see Chapter 5 of [42]. The solutions that we will obtain from the particle system are not classical solutions, and for that reason we just need to require the initial condition $\rho_0 : [0,1] \to [0,1]$ to be a measurable function. Therefore we need to introduce an appropriate concept of generalized solution of the equation. There are different ways of defining generalized solutions of partial differential equations. The weak formulation is obtained by multiplying the equation by suitable test functions, integrating by parts all the terms and using the boundary conditions. One also needs to ask from the solution a regularity that allows the weak formulation to make sense. In this case, we say that the solution is a weak solution. The notions of weak solutions of the equations above are given in Definitions 4 and 5, respectively. We stress that in the regime $\theta < 0$, that is, when the boundary dynamics is fast, the macroscopic behavior of the system will be the same as in the case $0 \leq \theta < 1$, i.e., given by PME with Dirichlet boundary conditions. The difference between these regimes is on their notion of weak solution. On one hand, in the case $0 \leq \theta < 1$, we consider test functions that vanish at the boundary. On the other hand, in the case $\theta < 0$, we have to consider more restricted test functions in the notion of weak solution, i.e., functions with compact support. Despite the similarity in the notion of weak solutions between these cases, we stick to $\theta \geq 0$ because the proof of uniqueness in the case $\theta < 0$ requires more effort and we leave it for a future work.

In order to better understand the hydrodynamic behavior of our model, we start by observing that the PME, $\partial_t \rho = \Delta(\rho^m)$, $m > 1$, is a nonlinear evolution equation of parabolic type. This equation has received a lot of attention in the last decades due to the mathematical difficulties of building a theory for nonlinear versions of the heat equation. One can rewrite the equation in divergence form as

$$\partial_t \rho = \nabla(D(\rho)\nabla\rho), \tag{1.4}$$

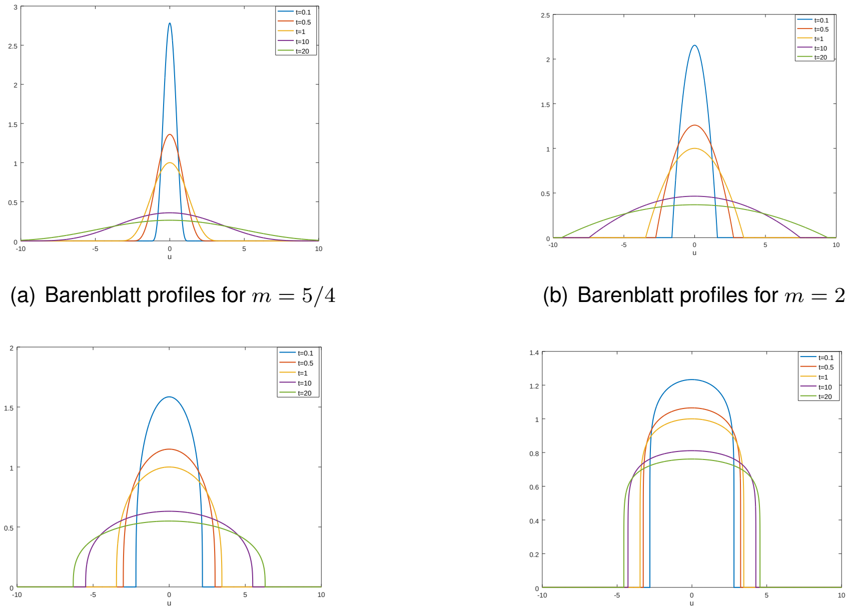
where $\rho = \rho(t,u)$ is a scalar function and $D(\rho) = m\rho^{m-1}$ is the diffusion coefficient. The space variable u takes values in some bounded or unbounded domain of $\mathbb{R}^d$ and the variable t satisfies $t \geq 0$. As mentioned above, the PME is also a degenerate parabolic equation, since the diffusion coefficient vanishes when ρ goes to zero, i.e., the equation is parabolic at the points $\rho \neq 0$, but it changes its character at the level $\rho = 0$. Thus, the regularity results for its solutions are weaker than the solutions of classical parabolic equations, and the techniques for studying the PME are much more refined. Matters as existence and uniqueness of classical and weak solutions are also affected by the degeneracy of this equation.

In this thesis, ρ represents a density (for example a gas density), and we look for solutions $\rho \geq 0$. In this way, a relevant problem for the PME consists in studying how the initial profile ρ_0, confined in a small region at time $t = 0$, evolves in time. From the physical point of view, one of the main differences between the PME and the heat equation is the so-called finite speed of propagation, that is, the solutions of the PME can be compactly supported at each fixed time. This property implies the appearance of a free boundary that separates the regions where there is gas ($\rho > 0$) from the empty region ($\rho = 0$). For example, across this boundary, the solution loses regularity. Therefore one of the main issues in the mathematical investigation of free boundaries consist in finding the evolution in time of the free boundary of regions with gas concentration, i.e., in the closure of $\{(t,u) : \rho(t,u) > 0\}$. Nevertheless, a way to obtain some information or insight about the free boundary, e.g, how it behaves when we vary the power m, is to look at the fundamental solution of the PME, the so-called *Barenblatt solution (or ZKB solution)*. This solution was obtained by Zel'dovich and Kompaneets [3] and Barenblatt [32]. These profiles are similar to the Gaussian profiles for the linear case. In the one-dimensional case, the Barenblatt solution has the following explicit form

$$\rho(t,u) = \begin{cases} t^{-\frac{1}{m+1}} \left(C - A \dfrac{u^2}{t^{\frac{2}{m+1}}} \right)^{\frac{1}{m-1}}, & \text{if } u^2 \leq \frac{C}{A} t^{\frac{2}{m+1}}, \\ 0, & \text{if } u^2 > \frac{C}{A} t^{\frac{2}{m+1}}, \end{cases}$$

where $C > 0$ is an arbitrary constant and $A = \frac{m-1}{2m(m+1)}$. Note that for every $t > 0$, the points $u = \pm\sqrt{\frac{C}{A}} t^{\frac{1}{m+1}}$ represent the region where there is gas, so that the free boundary is located at this distance

u. Considering it as a time function $G(t)$, we have therefore that the speed of propagation is $G'(t) = \pm\frac{1}{m+1}\sqrt{\frac{C}{A}}t^{-\frac{m}{m+1}}$. To see how the Barenblatt profiles evolve in time for different values of m see Figure 1.3 below.



(a) Barenblatt profiles for $m = 5/4$

(b) Barenblatt profiles for $m = 2$

(c) Barenblatt profiles for $m = 4$

(d) Barenblatt profiles for $m = 10$

Figure 1.3: Barenblatt solution for different values of m considering $C = 1$.

Note that when the parameter m increases, the solution varies more slowlier inside its support and tends to be like a step function in the interface of the support.

To summarize, the main contributions of this thesis are:

- *to derive for the first time the hydrodynamic limit for the PMM with slow reservoirs, and as a consequence, the Fick's law*;
- *to derive sufficiently strong energy estimates which are the keystone in the proof of convergence results for weak solutions of parabolic PDEs, as presented in [9, 22]*.

Here is the outline of this thesis. Chapter 2 aims stating the hydrodynamic limit. In Section 2.1, we introduce some notation and define the porous medium model with slow reservoirs. Section 2.2 is devoted to presenting the notion of weak solution of the hydrodynamic equations. In Section 2.3, we define the empirical measure associated to the process and we state the hydrodynamic limit and the Fick's law. In Chapter 3, we present a heuristic argument to derive the PME from the PMM. Chapter 4 shows tightness for the sequence of probability measures of interest and in Chapter 5, we characterize the limit points of that sequence. In Chapter 6, we present the proofs of all the replacement lemmas that are needed along with the proof's arguments. Chapter 7 deals with energy estimates, and Chapter 8 with the Fick's law. Finally, in Chapter 9, we prove the uniqueness of weak solutions of the hydrodynamic equations, and we devote Chapter 10 to the discussion of some topics to investigate in the near future. We finish the thesis in Appendix A by presenting some results used throughout the text.

Chapter 2

Statement of Results

From the big picture, this thesis is focused on the study of diffusion. Generally speaking, diffusion is a spread out of a substance (e.g., molecules, atoms, energy, etc.) from a region with high concentration to a region with low concentration. The diffusion is represented here in two scales: the macroscopic one - by means of a diffusion equation, the porous medium equation with boundary conditions; and the microscopic one - by means of a microscopic system in contact with reservoirs, the porous medium model with slow reservoirs. This chapter aims establishing a connection between these scales by a limit procedure that we will explain below. Thus, a very common question that arises when using such an approach is: from which of these worlds should we start our study? In fact, the starting point depends on the limit procedure technique that we are using and the problem at hand. In this thesis, in particular, we want to characterize the behavior of the PMM in contact with reservoirs. As we mentioned above, we use the Entropy Method introduced by Guo, Papanicolau, and Varadhan in [27] to accomplish this. Informally, the aforementioned limit procedure - known in the literature as hydrodynamic limit - states that the spatial density of particles converges to the unique weak solution of a partial differential equation, the so-called *hydrodynamic equation*.

The Entropy Method was one of the main contributions to the hydrodynamic limit theory, mainly due to its robustness and vast applicability, e.g., to prove laws of large numbers, equilibrium central limit theorems, and large deviations principles. However, the method at the hydrodynamic limit level requires the following assumptions regarding the hydrodynamic equation and the microscopic system:

1) uniqueness of weak solutions of the corresponding hydrodynamic equation;

2) irreducibility of the microscopic system over hyperplanes of configurations with k particles $\Omega_n^k := \left\{\eta \in \{0,1\}^{\Sigma_n} : \sum_{x\in\Sigma_n} \eta(x) = k\right\}$.

With these two conditions, the method proves the existence of such weak solutions. Let us now say some words about them. The hydrodynamic equations obtained through the hydrodynamic limit described above are the PME with Dirichlet, Robin, and Neumann boundary conditions, as given in (1.1), (1.2), and (1.3). We prove the uniqueness of their weak solutions in Chapter 9 and we are done with item $1)$ above. However, there are blocked configurations in the PMM dynamics so that we do not have irreducibility, and item $2)$ is not satisfied. Then, as mentioned above, we can overcome this problem by adding jumps of the SSEP dynamics in a time scale less than the diffusive one so that in the limit, it does not affect the hydrodynamic equation. For more details about the method, we refer the reader to [24, 25, 31, 41].

2.1 The Porous Medium Model

Let $n \in \mathbb{N}$ be a scaling parameter and fix $m \in \mathbb{N}$. We denote by $\Sigma_n = \{1, \ldots, n-1\}$ the discrete set of points of size $n-1$ that we call *bulk*. We call an element of the set Σ_n a *site*. The function $\eta : \Sigma_n \to \{0,1\}$ that evaluates each site and returns the value 0 (resp. 1) for an empty (resp. occupied) site is called a *configuration of particles*. The PMM is a continuous time Markov process that we denote by $\{\eta_t\}_{t\geq 0}$, which has state space $\Omega_n := \{0,1\}^{\Sigma_n}$ and infinitesimal generator L_n^m that we will define below. The dynamics of the model at the bulk can be described by associating $2^m - 1$ Poisson processes at each bond of the form $(x, x+1)$, with $x \in \{1, \ldots, n-2\}$ and with a parameter depending on the constraints of the model. Recall from the previous chapter that in the case $m = 2$, the jump rate of the process is given by $c^2_{x,x+1}(\eta) = \eta(x-1) + \eta(x+2)$. Thus, we associate three Poisson processes to each bond $(x, x+1)$ in the following way: $N^{x-1}_{x,x+1}(t)$ and $N^{x+2}_{x,x+1}(t)$ with parameter 1, and $N^{x-1,x+2}_{x,x+1}(t)$ with parameter 2. Now, note that in the general case, $c^m_{x,x+1}(\eta)$ is composed by a sum of m terms. The idea to associate the Poisson processes to each bond $(x, x+1)$ in this case, is to look at all possible combinations of positive sums of terms of $c^m_{x,x+1}(\eta)$, and for each one of them, we associate a Poisson clock with a parameter equal to the value of the positive sum and depending on the sites involved in this sum. Thus, for each $k = \{1, \ldots, m\}$ we will have $\binom{m}{k}$ Poisson processes with parameter k associated to the bond $(x, x+1)$, i.e., a total of $\sum_{k=1}^m \binom{m}{k} = 2^m - 1$. To define the dynamics at the boundary we first need to artificially add the sites 0 and n to the bulk. Hereafter, at the left boundary (resp. right boundary) we add Poisson clocks at the bonds $(0,1)$ (resp. $(n-1,n)$) and $(1,0)$ (resp. $(n, n-1)$) in the following way: $N_{0,1}(t)$ (resp. $N_{n,n-1}(t)$) with parameter $\kappa\alpha n^{-\theta}$ (resp. $\kappa\beta n^{-\theta}$) and $N_{1,0}(t)$ (resp. $N_{n-1,n}(t)$) with parameter $\kappa(1-\alpha)n^{-\theta}$ (resp. $\kappa(1-\beta)n^{-\theta}$), for some arbitrary $\theta \geq 0$, $\kappa > 0$ and $\alpha, \beta \in [0,1]$. We stress that all of these Poisson processes are independent.

Let us now define the infinitesimal generator of the process. Let $a \in (1,2)$. For $x, y \in \Sigma_n$ we denote the exchange and flip configurations by

$$\eta^{x,y}(z) = \begin{cases} \eta(z), \ z \neq x, y, \\ \eta(y), \ z = x, \\ \eta(x), \ z = y, \end{cases} \qquad \text{and} \quad \eta^x(z) = \begin{cases} \eta(z), \ z \neq x, \\ 1 - \eta(x), \ z = x. \end{cases}$$

With these notations, we define the infinitesimal generator of the process acting on functions $f : \Omega_n \to \mathbb{R}$ as

$$(L_n^m f)(\eta) = (L_P^m f)(\eta) + n^{a-2}(L_S f)(\eta) + (L_B f)(\eta), \tag{2.1}$$

where

$$(L_P^m f)(\eta) = \sum_{x=1}^{n-2} c_{x,x+1}^m(\eta)\{a_{x,x+1}(\eta) + a_{x+1,x}(\eta)\}[f(\eta^{x,x+1}) - f(\eta)], \tag{2.2}$$

is the generator of the PMM,

$$(L_S f)(\eta) = \sum_{x=1}^{n-2} \{a_{x,x+1}(\eta) + a_{x+1,x}(\eta)\}[f(\eta^{x,x+1}) - f(\eta)], \tag{2.3}$$

is the generator of the SSEP, and

$$(L_B f)(\eta) = \tfrac{\kappa}{n^\theta} I_1^\alpha(\eta)[f(\eta^1) - f(\eta)] + \tfrac{\kappa}{n^\theta} I_{n-1}^\beta(\eta)[f(\eta^{n-1}) - f(\eta)], \tag{2.4}$$

is the generator of the Glauber dynamics acting on sites 1 and $n-1$. Let η be a configuration of particles in Ω_n. For $x, y \in \{1, \ldots, n-2\}$, we define the exchange rates at the bulk as

$$c_{x,x+1}^m(\eta) = \sum_{k=1}^{m} \prod_{\substack{j=-(m-k) \\ j \neq 0,1}}^{k} \eta(x+j), \tag{2.5}$$

$$a_{x,y}(\eta) = \eta(x)(1 - \eta(y)), \tag{2.6}$$

and at the boundary as

$$I_z^b(\eta) = b(1 - \eta(z)) + (1 - b)\eta(z), \tag{2.7}$$

for $z \in \{1, n-1\}$ and $b \in \{\alpha, \beta\}$.

Remark 2.1.1. *Note that when $m = 1$ the rate* (2.5) *is equal to 1 and* (2.2) *is exactly the generator of the SSEP.*

Remark 2.1.2. *Above we used the convention*

$$\begin{aligned} \eta(x) &= \alpha, \quad \text{for} \quad x \leq 0, \\ \eta(x) &= \beta, \quad \text{for} \quad x \geq n. \end{aligned} \tag{2.8}$$

For example, if $m = 3$, we have that $c_{1,2}^3(\eta) = \eta(-1)\eta(0) + \eta(0)\eta(3) + \eta(3)\eta(4)$, which is equal to $\alpha^2 + \alpha\eta(3) + \eta(3)\eta(4)$.

Remark 2.1.3. *We take the constant $a \in (1, 2)$ for two reasons. First, we take the constant a less than 2 since we want to speed up the SSEP in a time scale less than diffusive, i.e., $n^a < n^2$. Second, the constant a must be bigger than 1 in order to control some terms that appear in the proof of the replacement lemmas, e.g., in* (6.9) *and* (6.15)*. This is technical and will be explained ahead.*

Remark 2.1.4. *The restriction imposed on the parameters $\alpha, \beta \in (0, 1)$ comes from the estimate in Lemma A.0.2, in which we have to take α and β different from 0 and 1 in order to control the entropy in the case $\theta < 1$. However, if $\theta \geq 1$ we do not need to impose this restriction due to the choice of profile in*

Lemma 4.2.1.

Remark 2.1.5. *We note that one could consider more general rates for the Glauber dynamics acting at the sites 1 and $n-1$, as in [10]. This could be done performing the following changes. Let $\alpha, \beta, \gamma, \delta \in (0,1)$, replace $1-\alpha$ (resp. $1-\beta$) by γ (resp. δ) in the reservoir's rates and consider the convention*

$$\eta(x) = \rho^{-}, \quad \text{for} \quad x \leq 0,$$
$$\eta(x) = \rho^{+}, \quad \text{for} \quad x \geq n,$$

with the densities $\rho^{-} = \frac{\alpha}{\alpha+\gamma}$ and $\rho^{+} = \frac{\beta}{\beta+\delta}$. The result would be the same as the one presented here but with a heavier notation. This is the reason we decided not to consider this case here. For more details about the action of this Glauber dynamics in the case $m = 2$, we refer the reader to [10].

The dynamics of the model in the case $m = 2$ is represented in Figure 2.1, where black balls stand for particles and blue balls stand for reservoirs. The clocks associated with the bonds represent all possible jumps that happen if a clock rings at that bond.

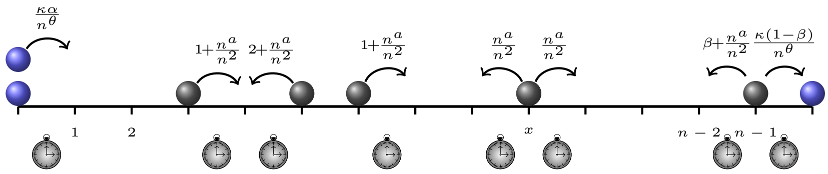


Figure 2.1: The porous medium model with slow reservoirs (with $m = 2$).

Recall that $m \in \mathbb{N}$ and $\alpha, \beta \in (0,1)$. Let $\eta \in \Omega_n$. We denote by $j^m_{x,x+1}(\eta)$ the instantaneous current of particles over the bond $(x, x+1)$. In other words, it is the rate at which the particle jumps from site x to $x+1$, minus the rate at which the particle jumps from site $x+1$ to x. Thus, the instantaneous current associated to the bond $(x, x+1)$ is given by

$$\begin{cases} j^m_{0,1}(\eta) = \frac{\kappa}{n^\theta}(\alpha - \eta(1)), \\ j^m_{x,x+1}(\eta) = \tau_x h^m(\eta) - \tau_{x+1} h^m(\eta), \quad \text{for} \;\; x \in \{1, \ldots, n-2\}, \\ j^m_{n-1,n}(\eta) = \frac{\kappa}{n^\theta}(\eta(n-1) - \beta), \end{cases} \tag{2.9}$$

where,

$$\tau_x h^m(\eta) = \sum_{k=1}^{m} \prod_{j=-(m-k)}^{k-1} \eta(x+j) - \sum_{k=1}^{m-1} \prod_{\substack{j=-(m-k) \\ j\neq 0}}^{k} \eta(x+j) + n^{a-2}\eta(x). \tag{2.10}$$

Observe that from the convention in (2.8), for $x = 1$ (resp. $n-1$), we have

$$\begin{aligned} \tau_1 h^m(\eta) &= \sum_{k=0}^{m-1} \alpha^k \prod_{j=1}^{m-k} \eta(j) - \sum_{k=1}^{m-1} \alpha^k \prod_{j=2}^{m+1-k} \eta(j) + n^{a-2}\eta(1), \\ \tau_{n-1} h^m(\eta) &= \sum_{k=0}^{m-1} \beta^k \prod_{j=1}^{m-k} \eta(n-j) - \sum_{k=1}^{m-1} \beta^k \prod_{j=2}^{m+1-k} \eta(n-j) + n^{a-2}\eta(n-1). \end{aligned} \tag{2.11}$$

Remark 2.1.6. *The identities above share a term of the form $n^{a-2}\eta(x)$. These terms come from the SSEP dynamics accelerated in a time scale less than diffusive, and since they vanish when $n \to \infty$ from here on, we ignore them and we look only at the remaining terms.*

Remark 2.1.7. *Note that due to the constraints of the PMM, it has configurations that do not evolve in time (blocked configurations) and we say that its dynamics is degenerate. This is directly related with the degeneracy of the diffusion coefficient of the PME, which is given by $D(\rho) = m\rho^{m-1}$, for $m \in \mathbb{N}$. This relation can be seen by analyzing the exchange rates of the process* (2.5) *and the diffusion coefficient of* (1.4)*, see, for example, the table below:*

m	$D(\rho)$	$c^m_{x,x+1}(\eta)$
1	1	1
2	2ρ	$\eta(x-1)+\eta(x+2)$
3	$3\rho^2$	$\eta(x-2)\eta(x-1)+\eta(x-1)\eta(x+2)+\eta(x+2)\eta(x+3)$

Table 2.1: Diffusion coefficient $vs.$ Exchange rates.

Remark 2.1.8. *The PMM conserves the total number of particles, and it is possible to write the action of the generator in $\eta(x)$ as the discrete gradient of the instantaneous current*

$$L^m_P\eta(x) = j^m_{x-1,x}(\eta) - j^m_{x,x+1}(\eta), \quad \textit{for } x \in \{1, \dots, n-2\},$$

where $j^m_{x,x+1}(\eta) = \tau_x h^m(\eta) - \tau_{x+1}h^m(\eta)$. Since it is also possible to write the instantaneous current in the bulk as the discrete gradient of a local function, i.e, as the difference of a local function and its translation, we can write the previous identity as

$$L^m_P\eta(x) = \tau_{x-1}h^m(\eta) + \tau_{x+1}h^m(\eta) - 2\tau_x h^m(\eta),$$

where $\tau_x h^m$ is given in (2.10)*. Every system with this previous condition is a gradient system. This is a significant property of the PMM since the analysis of the hydrodynamic limit for non-gradient systems is much more involved. See, for example, [31] Chapter 7. Moreover, the PMM superposed with the SSEP and the Glauber dynamics does not conserve the total number of particles. Actually, these reservoirs break down the conservation of particles since there is a mass transfer between the reservoirs (which have different rates) and the bulk. Since the process is superposed with the SSEP dynamics, it is an irreducible Markov process with a finite state space. Therefore only one invariant measure exists. In the equilibrium state, that is, when $\alpha = \beta$, the Bernoulli product measure, with a constant parameter $\rho = \alpha = \beta$, is the invariant measure of the process. Nevertheless, when $\alpha \neq \beta$, this measure is no longer invariant and we have no information on the system's invariant measure. We observe that the matrix product ansatz method of Derrida [11] can not be straightforwardly applied to this model due to the bulk dynamics' complicated action.*

Remark 2.1.9. *As we mentioned above, the PMM is a KCLG. This class of models was introduced in the 1980s in the physics literature [1] to study glassy dynamics. The KCLG models are usually classified as cooperative or non-cooperative. In this classification, a model is non-cooperative when its dynamical* *constraints are defined in such a way that it is possible to construct a finite group of particles (the mobile cluster), which can be moved to any position of the discrete space where particles evolve, by using strictly positive exchange rates. Any exchange is allowed when the mobile cluster is brought to the vicinity of the jumping particle, see [7, 38] for a review on the subject. All models which are not non-cooperative are said to be cooperative. Although we will not consider these models here, we stress that the Kob-Anderson (KA) model is a cooperative model extensively analyzed by physicists. In [39] Shapira proved the hydrodynamic limit for this model and also proved that cooperative models are all non-gradient.*

Remark 2.1.10. *The non-cooperativity of the model and the fact that we can perturb its dynamics with the SSEP dynamics, are crucial properties of the model that will be extensively used in our arguments. More precisely, when proving the hydrodynamic limit, in order to recognize the solution as a weak solution to the PME, we will have to derive some replacement lemmas, which are stated and proved in Chapter 6. In their proofs we will have to analyze the irreducibility of the model in the sense that we will have to send a particle from a site x to some site y at a distance depending on the size of the bulk. In spite of having available the SSEP dynamics, one could think that this could be accomplished easily. Nevertheless, the problem can not be overcomed just by using the SSEP jumps since they will be scaled in a time scale less than the diffusive one and for this reason, particles can not travel to sites at the distance we want. To push the argument further, we could try to use the PMM jumps, but to do that we need the jumping particle to have particles in its vicinity and many times that does not happen. The trick is then to fix a finite size window around the jumping particle, create a mobile cluster in that window and once the mobile cluster is created we can just use the PMM jumps to move the particles. After sending the particle to where we want we destroy the mobile cluster and we put the particles back to their initial position. We remark that the jumps used to create and destroy the mobile cluster on the finite size window are SSEP jumps, in all the rest of the path, we use PMM jumps.*

Remark 2.1.11. *From now on let $\{\eta_{tn^2}\}_{t\geq 0}$ denote the Markov process speeded up in the diffusive time scale tn^2 and with infinitesimal generator $n^2 L_n^m$.*

2.2 The Porous Medium Equation

The analysis of diffusion equations started with linear parabolic equations, particularly the well-studied heat equation $\partial_t \rho = \Delta \rho$. The equation was proposed by J. Fourier [20] as a model for heat propagation, and throughout history, it served as inspiration for the construction of the theory of nonlinear parabolic equations. The interest in studying nonlinear versions of the heat equation arises due to the development of the theory for linear parabolic equations. The idea was to use the existing theoretical framework for linear equations to compare matters of existence, uniqueness, regularity, maximum principle, applications, fundamental solutions, etc. However, due to the mathematical difficulties faced when studying such equations, a new theory needed to be created. In this way, in 2007, Juan Luiz Vázquez published the seminal book [42] focused on the porous medium equation to present part of the framework for nonlinear parabolic equations. There are many physical applications of the PME, most of them used to describe processes involving diffusion or heat transfer. In [44], the equation was used to study the heat radiation in plasmas, and in [28, 29], the authors used the PME to describe migratory diffusion of biological populations. The name PME was motivated by the work [37], in which the equation (with $m = 2$) was used to model the density of a gas flowing through a porous medium.

Let us now describe the different partial differential equations and the respective notion of weak solutions that we will consider along this thesis. We start by introducing some notations and definitions needed to state the hydrodynamic limit in Theorem 2.3.1. Fix an interval $I \subset \mathbb{R}$, $T > 0$ and $n, p \in \mathbb{N} \cup \{0\}$. We denote by:

- $C^{n,p}([0,T] \times I)$, the set of all real-valued functions defined on $[0,T] \times I$ that are n times differentiable on the first variable and p times differentiable on the second variable (with continuous derivatives);

- $C^n([0,1])$ (resp. $C_c^n((0,1))$), the set of all n times continuously differentiable real-valued functions defined on $[0,1]$ (resp. and with compact support in $(0,1)$);

- $C_c^{n,p}([0,T] \times (0,1))$, the set of all real-valued functions $G \in C^{n,p}([0,T] \times (0,1))$ with compact support in $[0,T] \times (0,1)$;

- $C_0^{n,p}([0,T] \times [0,1])$, the set of all real-valued functions $G \in C^{n,p}([0,T] \times [0,1])$ such that $G_s(0) = G_s(1) = 0$, for all $s \in [0,T]$. When $n = \infty$ or $p = \infty$ it means that the function is infinitely differentiable in the corresponding variable;

- $\langle \cdot, \cdot \rangle$, the inner product in $L^2([0,1])$ with corresponding norm $\| \cdot \|_2$.

Definition 1. *Let $G, H \in L^2([0,T] \times [0,1])$. We denote the inner product in $L^2([0,T] \times [0,1])$ by*

$$\langle\langle G, H \rangle\rangle := \int_0^T \langle G_s, H_s \rangle \, ds. \tag{2.12}$$

Definition 2 (Sobolev space)**.** *Let $\mathcal{H}^1$ be the set of all locally summable functions $\varphi : [0,1] \to \mathbb{R}$ such that there exists a function $\partial_u \varphi \in L^2([0,1])$ satisfying*

$$\langle \varphi, \partial_u g \rangle = -\langle \partial_u \varphi, g \rangle,$$

for all $g \in C_c^\infty\big((0,1)\big)$. For $\varphi \in \mathcal{H}^1$, we define the norm

$$\|\varphi\|_{\mathcal{H}^1}^2 := \|\varphi\|_2^2 + \|\partial_u \varphi\|_2^2.$$

Recall that the function $\varphi \in \mathcal{H}^1$ can be extended to $[0,1]$ by setting $\varphi(0) := \varphi(0^+)$ and $\varphi(1) := \varphi(1^-)$.

Definition 3. *Let $L^2(0,T;\mathcal{H}^1)$ be the set of all measurable functions $\zeta : [0,T] \to \mathcal{H}^1$ such that*

$$\|\zeta\|_{L^2(0,T;\mathcal{H}^1)}^2 := \int_0^T \|\zeta_t\|_{\mathcal{H}^1}^2 \, dt < \infty. \tag{2.13}$$

Remark 2.2.1. *Note that using the notation in* (2.12) *we can rewrite* (2.13) *as*

$$\|\zeta\|^2_{L^2(0,T;\mathcal{H}^1)} = \langle\!\langle \zeta, \zeta \rangle\!\rangle + \langle\!\langle \partial_u \zeta, \partial_u \zeta \rangle\!\rangle.$$

For more details about the definitions and notations outlined above, we refer the reader to [15]. Along the text we fix the parameters

$$\alpha, \beta \in (0,1) \text{ and } m \in \mathbb{N}. \tag{2.14}$$

Definition 4 (PME with Dirichlet boundary conditions)**.** *Let $T > 0$ and $g : [0,1] \to [0,1]$ a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the porous medium equation with Dirichlet boundary conditions*

$$\begin{cases} \partial_t \rho_t(u) = \Delta\left(\rho_t(u)\right)^m, \quad (t,u) \in (0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, \quad t \in (0,T], \\ \rho_0(u) = g(u), \quad u \in [0,1], \end{cases} \tag{2.15}$$

if the following conditions hold:

1. $\rho^m \in L^2(0,T;\mathcal{H}^1)$;

2. ρ *satisfies the integral equation:*

$$\begin{aligned} F_{Dir}(G,t,\rho,g,m) = \langle \rho_t, G_t \rangle - \langle g, G_0 \rangle - \int_0^t \langle \rho_s, (\partial_s G_s + (\rho_s)^{m-1} \Delta G_s) \rangle \, ds \\ + \int_0^t \left\{ \beta^m \partial_u G_s(1) - \alpha^m \partial_u G_s(0) \right\} ds = 0, \end{aligned} \tag{2.16}$$

   for all $t \in [0,T]$ and all functions $G \in C_0^{1,2}([0,T] \times [0,1])$;

3. *for almost every $t \in (0,T]$, $\rho_t(0) = \alpha$ and $\rho_t(1) = \beta$.*

Definition 5 (PME with a type of Robin boundary conditions)**.** *Let $T > 0$, $\kappa > 0$ and $g : [0,1] \to [0,1]$ a measurable function. We say that $\rho^\kappa : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the porous medium equation with Robin boundary conditions*

$$\begin{cases} \partial_t \rho_t^\kappa(u) = \Delta\left({\rho_t}^\kappa(u)\right)^m, \quad (t,u) \in (0,T] \times (0,1), \\ \partial_u (\rho_t^\kappa(0))^m = \kappa(\rho_t^\kappa(0) - \alpha), \quad t \in (0,T], \\ \partial_u (\rho_t^\kappa(1))^m = \kappa(\beta - \rho_t^\kappa(1)), \quad t \in (0,T], \\ \rho_0^\kappa(u) = g(u), \quad u \in [0,1], \end{cases} \tag{2.17}$$

if the following conditions hold:

1. $(\rho^\kappa)^m \in L^2(0,T;\mathcal{H}^1)$;

2. ρ^κ satisfies the integral equation:

$$\begin{aligned} F_{Rob}(G,t,\rho^\kappa,g,m) = \langle \rho^\kappa_t, G_t\rangle - \langle g, G_0\rangle - \int_0^t \langle \rho^\kappa_s, (\partial_s G_s + (\rho^\kappa_s)^{m-1}\Delta G_s)\rangle\, ds \\ + \int_0^t \big\{(\rho^\kappa_s(1))^m \partial_u G_s(1) - (\rho^\kappa_s(0))^m \partial_u G_s(0)\big\}\, ds \\ - \kappa \int_0^t \big\{G_s(0)(\alpha - \rho^\kappa_s(0)) + G_s(1)(\beta - \rho^\kappa_s(1))\big\}\, ds = 0, \end{aligned} \tag{2.18}$$

for all $t \in [0,T]$ and all functions $G \in C^{1,2}([0,T]\times[0,1])$.

Remark 2.2.2. *For $\kappa = 0$, we obtain above Neumann boundary conditions.*

Remark 2.2.3. *Observe that for $m = 1$ the equations above become the heat equation with different boundary conditions.*

We observe that the weak solutions of (2.15), (2.17), and (2.17) with $\kappa = 0$, in the sense given above are unique. We present a proof of this fact in Chapter 9. For a deeper discussion of the porous medium equation, we refer the reader to [42].

2.3 Hydrodynamic Limit

Let us begin this subsection by introducing the empirical measure associated to the process $\{\eta_{tn^2}\}_{t\geq 0}$. For $\eta \in \Omega_n$, this measure gives weight $1/n$ to each particle in our dynamics

$$\pi^n(\eta, du) := \frac{1}{n}\sum_{x\in\Sigma_n} \eta(x)\delta_{x/n}(du), \tag{2.19}$$

where δ_u is a Dirac mass on $u \in [0,1]$. In order to analyze the temporal evolution of the empirical measure, we define the process of empirical measures as $\pi^n_t(\eta, du) := \pi^n(\eta_{tn^2}, du)$. For a test function $G : [0,1] \to \mathbb{R}$, we denote the integral of G with respect to the empirical measure π^n_t, by $\langle \pi^n_t, G\rangle$, which is equal to

$$\langle \pi^n_t, G\rangle = \frac{1}{n}\sum_{x\in\Sigma_n} G\left(\tfrac{x}{n}\right)\eta_{tn^2}(x).$$

Note that the notation $\langle\cdot,\cdot\rangle$ above is not related with the inner product in $L^2([0,1])$. Fix $T > 0$ and $\theta \geq 0$. Let μ_n be a probability measure in Ω_n. We denote by $\mathcal{D}([0,T],\Omega_n)$ the Skorokhod space, that is, the space of right continuous with left limits functions *(càdlàg)* defined in $[0,T]$ and taking values in Ω_n. We denote by $\mathbb{P}_{\mu_n}$ the probability measure on the space $\mathcal{D}([0,T],\Omega_n)$ induced by the accelerated Markov process $\{\eta_{tn^2}\}_{t\geq 0}$ and the initial measure μ_n. The corresponding expectation is denoted by $\mathbb{E}_{\mu_n}$. Let $\mathcal{M}_+$ be the space of positive measures on $[0,1]$ with total mass bounded by 1 and equipped with the weak topology. We denote by $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ the sequence of probability measures on $\mathcal{D}([0,T],\mathcal{M}_+)$, induced by the Markov process $\{\pi^n_t\}_{t\geq 0}$ and by the initial distribution μ_n. The corresponding expectation is denoted by $\mathbb{E}_n$.

In order to state the hydrodynamic limit for $\{\eta_{tn^2}\}_{t\geq 0}$, we need to impose some conditions on the initial distribution of the process. Given a measurable function $g : [0,1] \to [0,1]$, we say that a sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ on Ω_n is *associated with* $g(\cdot)$, if for any continuous function $G:[0,1]\to\mathbb{R}$ and any $\delta>0$

$$\lim_{n\to\infty}\mu_n\left(\eta\in\Omega_n:\left|\langle\pi^n,G\rangle-\int_0^1 G(u)g(u)\,du\right|>\delta\right)=0. \tag{2.20}$$

The aim of the hydrodynamic limit is to show that the empirical measure $\pi^n_\cdot$ converges in probability, with respect to $\mathbb{P}_{\mu_n}$, when $n\to\infty$, to a deterministic trajectory of measures $\pi_\cdot$, such that for each t, $\pi_t(du)$ is absolutely continuous with respect to the Lebesgue measure, that is, $\pi_t(du)=\rho(t,u)\,du$ and $\rho(t,u)$ is the weak solution of the corresponding partial differential equation with certain boundary conditions and with initial condition g.

Theorem 2.3.1. *Let $g:[0,1]\to[0,1]$ be a measurable function and $\{\mu_n\}_{n\in\mathbb{N}}$ a sequence of probability measures on Ω_n associated with $g(\cdot)$. Then, for any $t\in[0,T]$ and any $\delta>0$,*

$$\lim_{n\to\infty}\mathbb{P}_{\mu_n}\left(\eta_\cdot\in\mathcal{D}([0,T],\Omega_n):\left|\langle\pi^n_t,G\rangle-\int_0^1 G(u)\rho_t(u)\,du\right|>\delta\right)=0,$$

where

- $\rho_t(\cdot)$ *is a weak solution of* (2.15)*, for* $0\le\theta<1$*;*
- $\rho_t(\cdot)$ *is a weak solution of (2.17), for* $\theta=1$*;*
- $\rho_t(\cdot)$ *is a weak solution of (2.17) (with* $\kappa=0$*), for* $\theta>1$*.*

Remark 2.3.2. *We note that if we consider the regime $\theta<0$ (when the reservoirs are fast) $\rho_t(\cdot)$ would be a weak solution of the PME with Dirichlet boundary conditions, with the same notion of weak solution as above but considering test functions $G\in C_c^{1,2}([0,T]\times[0,1])$. Since in this case $\partial_u G_s(0)=\partial_u G_s(0)=0$ the last term in* (2.16) *vanishes and we would not be able to distinguish this notion of weak solution from the others. Thus, due to the similarity with the regime $0\le\theta<1$, we will not consider the fast regime here. However, whenever necessary, we will make comments on it throughout the text.*

The theorem above is a corollary of the next result:

Proposition 2.3.3. *The sequence of probability measures $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ converges weakly to $\mathbb{Q}$, when $n\to\infty$, where $\mathbb{Q}$ is a Delta of Dirac measure on top of the trajectory of measures that are absolutely continuous with respect to the Lebesgue measure, i.e., $\pi_t(du)=\rho_t(u)du$, and the density $\rho_t(u)$ is the unique weak solution of the corresponding hydrodynamic equation.*

Here follows an outline of the next chapters. In Chapter 3 we present a heuristic argument to derive the hydrodynamic equations from the PMM for each range of θ by using Dynkin's formula. Then, the following chapters aim proving Theorem 2.3.1 by the Entropy Method. The proof relies on first showing tightness of the sequence of probability measures $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$, i.e., that the sequence has limit points, which we prove in Chapter 4. In that chapter we also define the Dirichlet forms, the carré du champ operator and present some estimates for the Dirichlet forms necessary to prove the replacement lemmas and the energy estimates. Then, the next step is to characterize the limit points by showing that they are concentrated on trajectories of measures that are absolutely continuous with respect to the Lebesgue measure. We do not present the proof of this result since it is a simple consequence of the fact that our dynamics is of exclusion type, see Section 2 of Chapter 4 of [31]. Moreover, we also have to show that the density $\rho_t(u)$ is a weak solution of the corresponding hydrodynamic equation, which we prove in Chapter 5. We divided Chapter 6 into Sections 6.1 and 6.2, where we state and prove the replacement lemmas regarding the bulk and the boundary, respectively. Then, in Chapter 7, we prove the energy estimates, which imply that $\rho^m \in L^2(0,T;\mathcal{H}^1)$. Chapter 8 aims proving the Fick's law. In Chapter 9, we prove the uniqueness of weak solutions for each hydrodynamic equation presented above. Due to this fact, we guarantee the uniqueness of the limit point.

Chapter 3

Discrete Versions of Weak solutions

In Section 4.1 we will prove that $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ has limits points, as we mentioned above. Let $\mathbb{Q}$ be one of these limit points. From Section 2 of Chapter 4 of [31], these limit points are supported on trajectories of measures that are absolutely continuous with respect to the Lebesgue measure, that is: $\mathbb{Q}(\pi_\cdot : \pi_t(du) = \rho_t(u)\,du) = 1$. We call to the function $\rho_t(\cdot)$, appearing inside last probability, the density profile. In principle this function could be random, but here we, *heuristically*, prove that $\rho_t(\cdot)$ is a weak solution of the hydrodynamic equation. For simplicity, we will present this heuristic argument for $m = 2$, but the argument for any $m > 2$ is analogous. Fix a function $G \in C^{1,2}([0,T] \times [0,1])$. We know by Dynkin's formula, see for example Lemma A1.5.1 of [31], that

$$M_t^n(G) = \langle \pi_t^n, G_t\rangle - \langle \pi_0^n, G_0\rangle - \int_0^t (\partial_s + n^2 L_P^2 + n^a L_{\mathcal{S}} + n^2 L_B)\langle \pi_s^n, G_s\rangle\, ds \tag{3.1}$$

is a martingale with respect to the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_t = \{\sigma(\eta_{sn^2}) : s \leq t\}$. Note that $\partial_s\langle \pi_s^n, G_s\rangle = \langle \pi_s^n, \partial_s G_s\rangle$, for any function $G \in C^{1,2}([0,T]\times[0,1])$ and, macroscopically, this extra term will give rise to the term involving $\partial_s G$ in (2.16) and (2.18). Since this term does not have any information about the dynamics of the model, and for simplicity of the presentation, we consider here test functions G only space-dependent, that is, $G \in C^2([0,1])$. Then, in (3.1) the term ∂_s can be suppressed. Let us now compute the other term. From (2.9), (2.10), and (2.11), we have that $n^2 L_n^2\langle \pi_s^n, G\rangle$ is given by

$$\begin{aligned}\frac{1}{n}\sum_{x=1}^{n-1}\Delta_n G\left(\tfrac{x}{n}\right)\tau_x h^2(\eta_{sn^2}) &+ \nabla_n^+ G(0)\tau_1 h^2(\eta_{sn^2}) - \nabla_n^- G(1)\tau_{n-1} h^2(\eta_{sn^2}) \\ &+ nG\left(\tfrac{1}{n}\right)\tfrac{\kappa}{n^\theta}\big(\alpha - \eta_{sn^2}(1)\big) + nG\left(\tfrac{n-1}{n}\right)\tfrac{\kappa}{n^\theta}\big(\beta - \eta_{sn^2}(n-1)\big),\end{aligned} \tag{3.2}$$

where for $x \in \Sigma_n$, the discrete Laplacian is given by

$$\Delta_n G\left(\tfrac{x}{n}\right) = n^2\Big(G\left(\tfrac{x-1}{n}\right) - 2G\left(\tfrac{x}{n}\right) + G\left(\tfrac{x+1}{n}\right)\Big),$$

and the discrete derivatives are given by

$$\nabla_n^+ G\left(\tfrac{x}{n}\right) = n\Big(G\left(\tfrac{x+1}{n}\right) - G\left(\tfrac{x}{n}\right)\Big) \quad \text{and} \quad \nabla_n^- G\left(\tfrac{x}{n}\right) = n\Big(G\left(\tfrac{x}{n}\right) - G\left(\tfrac{x-1}{n}\right)\Big).$$

Since the function G is time independent and using the convention (2.8), the martingale in (3.1) is equal to

$$
\begin{aligned}
M_t^n(G) = \langle \pi_t^n, G\rangle - \langle \pi_0^n, G\rangle &- \int_0^t \frac{1}{n}\sum_{x=1}^{n-1} \Delta_n G\left(\tfrac{x}{n}\right)\tau_x h^2(\eta_{sn^2})\,ds \\
&- \int_0^t \nabla_n^+ G(0)\tau_1 h^2(\eta_{sn^2})\,ds + \int_0^t \nabla_n^- G(1)\tau_{n-1} h^2(\eta_{sn^2})\,ds \\
&- \kappa\frac{n}{n^\theta}\int_0^t \Big\{ G\left(\tfrac{1}{n}\right)\big(\alpha-\eta_{sn^2}(1)\big) + G\left(\tfrac{n-1}{n}\right)\big(\beta-\eta_{sn^2}(n-1)\big)\Big\}\,ds.
\end{aligned}
\tag{3.3}
$$

Remark 3.0.1. *By the mean value theorem, for all $x\in\Sigma_n$ we have*

$$
\big|\Delta_n G\left(\tfrac{x}{n}\right)\big| \le 2\|G''\|_\infty, \quad |\nabla_n^+G(0)|\le \|G'\|_\infty, \quad \textit{and} \quad |\nabla_n^- G(1)| \le \|G'\|_\infty,
$$

where $\|G\|_\infty := \sup_{x\in[0,1]}|G(x)|$

From the previous remark and the fact that $|\eta_{sn^2}(x)|\le 1$, for all $s\ge 0$ and $x\in\Sigma_n$, the terms that come from the SSEP jumps vanish when $n\to\infty$. From here on we ignore them and we look only at the other terms in (3.3).

Case $\theta<1$: Since in this regime we take $G\in C_0^2([0,1])$, we can write expression (3.3) as

$$
\begin{aligned}
M_t^n(G) = \langle \pi_t^n, G\rangle - \langle \pi_0^n, G\rangle &- \int_0^t \frac{1}{n}\sum_{x=1}^{n-1} \Delta_n G(\tfrac{x}{n})\tau_x h^2(\eta_{sn^2})\,ds \\
&- \int_0^t \Big\{\nabla_n^+ G(0)\tau_1 h^2(\eta_{sn^2}) - \nabla_n^- G(1)\tau_{n-1}h^2(\eta_{sn^2})\Big\}\,ds + O(n^{-\theta}).
\end{aligned}
\tag{3.4}
$$

As usual, see [16], the notations O, o and $\sim$ have the following meaning: for functions φ and ψ depending on a parameter n, which tends to infinity and $\psi>0$, we write

$$
\left.\begin{array}{c}
\varphi = O(\psi)\\
\varphi = o(\psi)\\
\varphi \sim \psi
\end{array}\right\}
\quad \text{if} \quad \frac{\varphi}{\psi} \quad
\left\{\begin{array}{l}
\text{remains bounded}\\
\to 0\\
\to 1.
\end{array}\right.
$$

Note that the term $O(n^{-\theta})$ in (3.4) comes from last integral in (3.3), by using the fact that G is a function vanishing at the boundary and that $|\eta_{sn^2}(x)|\le 1$, for all $s\ge 0$ and $x\in\Sigma_n$. From Remark 3.0.1 and Lemma 6.2.1, we get

$$
\begin{aligned}
M_t^n(G) = \langle \pi_t^n, G\rangle - \langle \pi_0^n, G\rangle &- \int_0^t \frac{1}{n}\sum_{x=1}^{n-1} \Delta_n G(\tfrac{x}{n})\tau_x h^2(\eta_{sn^2})\,ds \\
&- \int_0^t \Big\{\nabla_n^+ G(0)\alpha^2 - \nabla_n^- G(1)\beta^2\Big\}\,ds + O(n^{-\theta}).
\end{aligned}
\tag{3.5}
$$

Now, from Theorem 6.1.1, we have that

$$
\begin{aligned}
M_t^n(G) = \langle \pi_t^n, G\rangle - \langle \pi_0^n, G\rangle - \int_0^t \frac{1}{n}\sum_{x=1}^{n-1} \Delta_n G(\tfrac{x}{n}) \overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x)\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1)\, ds \\
- \int_0^t \Big\{\nabla_n^+ G(0)\alpha^2 - \nabla_n^- G(1)\beta^2\Big\}\, ds + O(n^{-\theta}) + o(1).
\end{aligned}
$$

The sense of convergence for the term $o(1)$ is the one stated in Theorem 6.1.1. Above, $\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x)$ and $\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x)$ are the empirical densities in a box of size εn to the left (resp. to the right) of site x, which are given on $x \in \Sigma_n$ by

$$
\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x) = \frac{1}{\varepsilon n}\sum_{y=x-\varepsilon n+1}^{x} \eta_{sn^2}(y) \text{ and } \overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x) = \frac{1}{\varepsilon n}\sum_{y=x}^{x+\varepsilon n-1} \eta_{sn^2}(y). \tag{3.6}
$$

Above and below εn should be understood as $\lfloor \varepsilon n\rfloor$. Note that $\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x) = \langle \pi_s^n, \overleftarrow{\iota}^{x}_{\varepsilon}\rangle$ and $\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x) = \langle \pi_s^n, \overrightarrow{\iota}^{x}_{\varepsilon}\rangle$, where for $v \in [0,1]$

$$
\overleftarrow{\iota}^{x}_{\varepsilon}(v) = \frac{1}{\varepsilon}1_{(x-\varepsilon,x]}(v) \text{ and } \overrightarrow{\iota}^{x}_{\varepsilon}(v) = \frac{1}{\varepsilon}1_{[x,x+\varepsilon)}(v).
$$

Then, *heuristically*, we have that $\langle \pi_s^n, \overleftarrow{\iota}^{x}_{\varepsilon}\rangle$ and $\langle \pi_s^n, \overrightarrow{\iota}^{x}_{\varepsilon}\rangle$ converges, when $n \to \infty$, to

$$
\langle \pi_s, \overleftarrow{\iota}^{x}_{\varepsilon}\rangle = \int_0^1 \rho_s(u)\overleftarrow{\iota}^{x}_{\varepsilon}(u)\, du \text{ and } \langle \pi_s, \overrightarrow{\iota}^{x}_{\varepsilon}\rangle = \int_0^1 \rho_s(u)\overrightarrow{\iota}^{x}_{\varepsilon}(u)\, du,
$$

where $\rho_s(\cdot)$ is the density profile. Taking the limit when $\varepsilon \to 0$ we obtain that both $\langle \pi_s, \overleftarrow{\iota}^{x}_{\varepsilon}\rangle$ and $\langle \pi_s, \overrightarrow{\iota}^{x+1}_{\varepsilon}\rangle$ converge to $\rho_s(\frac{x}{n})$. From the observation above we say that

$$
\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x)\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1) \sim \rho_s^2(\tfrac{x}{n}).
$$

Finally, since $M_0^n(G) = 0$ and $\mathbb{E}_{\mu_n}[M_t^n(G)] = \mathbb{E}_{\mu_n}[M_0^n(G)] = 0$, taking $n \to \infty$ and $\varepsilon \to 0$ in the last display we obtain:

$$
0 = \langle \rho_t, G\rangle - \langle \rho_0, G\rangle - \int_0^t \langle \Delta G, (\rho_s)^2\rangle\, ds - \int_0^t \partial_u G(0)\alpha^2 - \partial_u G(1)\beta^2 ds,
$$

from where we see (2.16).

Remark 3.0.2. *Note that above we used* (2.8)*. If we had assumed that* $\eta(0)$ *is any positive constant, let us call it* r*, then in the second line of* (3.3) *we would have*

$$
\int_0^t \nabla_n^+ G(0)\big(\eta_{sn^2}(1)r + \eta_{sn^2}(1)\eta_{sn^2}(2) - \eta_{sn^2}(2)r + n^{a-2}\eta_{sn^2}(1)\big)ds.
$$

The last integrand function only comes from the bulk dynamics. Now to get the Dirichlet boundary conditions as above, we would have to prove that we can replace $\eta(1)$ *by* $\eta(2)$*, which can be done by using Corollary 6.2.6 and then we can use Theorem 6.1.1 of Section 6.1, to replace* $\eta_{sn^2}(1)$ *by* α*. This* *could give us some freedom to take other rates for the bulk dynamics. Here we stick to the choice* (2.8)*.*

Case $\theta = 1$: In this case, taking test functions $G \in C^2([0,1])$, we get

$$
\begin{aligned}
M_t^n(G) = \langle \pi_t^n, G\rangle - \langle \pi_0^n, G\rangle - \int_0^t \frac{1}{n}\sum_{x=1}^{n-1} \Delta_n G(\tfrac{x}{n})\tau_x h^2(\eta_{sn^2})\, ds \\
- \int_0^t \nabla_n^+ G(0)\tau_1 h^2(\eta_{sn^2})\, ds + \int_0^t \nabla_n^- G(1)\tau_{n-1} h^2(\eta_{sn^2})\, ds \\
- \kappa \int_0^t G(\tfrac{1}{n})(\alpha - \eta_{sn^2}(1)) + G(\tfrac{n-1}{n})(\beta - \eta_{sn^2}(n-1))\, ds.
\end{aligned}
$$

Since Theorem 6.1.1 allows replacing products of the form $\eta(x)\eta(x+1)$ by $\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x)\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1)$ in the bulk, and Theorem 6.2.3 replacing $\eta(1)\eta(2)$ (resp. $\eta(n-2)\eta(n-1)$) by $\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(1)\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(1+\varepsilon n)$ (resp. $\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(n-1-\varepsilon n)\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(n-1)$) at the boundary, taking the limit in $n \to \infty$ and $\varepsilon \to 0$ in last expression, we obtain

$$
\begin{aligned}
0 = \langle \rho_t, G\rangle - \langle \rho_0, G\rangle - \int_0^t \langle \Delta G, (\rho_s)^2\rangle\, ds - \int_0^t \partial_u G(0)(\rho_s(0))^2 - \partial_u G(1)(\rho_s(1))^2 ds \\
- \kappa \int_0^t G(0)(\alpha - \rho_s(0)) + G(1)(\beta - \rho_s(1))\, ds,
\end{aligned}
$$

from where we get (2.18).

Case $\theta > 1$: In this case we take the same space of test functions as in the case $\theta = 1$, but since $\theta > 1$ the last term on the right-hand side of (3.3) vanishes, as $n \to \infty$. Moreover, since Theorems 6.1.1 and 6.2.3 hold, when taking the limit in $n \to \infty$ and $\varepsilon \to 0$ in (3.3) we obtain

$$
0 = \langle \rho_t, G\rangle - \langle \rho_0, G\rangle - \int_0^t \langle \Delta G, (\rho_s)^2\rangle\, ds - \int_0^t \partial_u G(0)(\rho_s(0))^2 - \partial_u G(1)(\rho_s(1))^2 ds,
$$

which corresponds to (2.18) for $\kappa = 0$.

Chapter 4

Tightness and Auxiliary Results

This chapter is divided into two sections. In Section 4.1 we prove that the sequence of probability measures $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$, defined in Section 2.3, is tight. In order to prove the result, we use Aldous' criterium stated in Lemma 4.1.1. In Section 4.2, we state and prove some estimates on the Dirichlet forms that will be crucial along the proofs of the replacement lemmas in Chapter 6 and the energy estimates in Chapter 7.

4.1 Tightness

We start this section by stating Aldous' criterium. Before stating it let $\mathcal{D}([0,T],\mathcal{S})$ be the space of *càdlàg* functions with values in $\mathcal{S}$, where $\mathcal{S}$ is a separable metric space endowed with a distance δ. Let $\Lambda = \{\lambda : [0,T] \to [0,T] \,|\, \lambda$ is a continuous and strictly increasing function$\}$. If $\lambda \in \Lambda$, we define

$$\|\lambda\| = \sup_{t\neq s} \left| \log\left(\frac{\lambda(t)-\lambda(s)}{t-s} \right) \right|.$$

If $\mu, \nu \in \mathcal{S}$, we define the Skorokhod distance as

$$d(\mu,\nu) = \inf_{\lambda\in\Lambda} \max \left\{ \|\lambda\|, \sup_{0\leq t\leq T} \delta\left(\mu_t, \nu_{\lambda(t)}\right) \right\}.$$

Lemma 4.1.1. *Let $\{P_n\}_{n\geq 1}$ be a sequence of probability measures defined in $\mathcal{D}([0,T],\mathcal{S})$. The sequence is tight if the following conditions hold:*

1. *For every $t \in [0,T]$ and $\varepsilon > 0$, there exists a compact set $K(t,\varepsilon) \subset \mathcal{S}$, such that*

$$\sup_{n\geq 1} P_n\Big(\pi_\cdot \in \mathcal{D}([0,T],\mathcal{S}) : \pi_t \notin K(t,\varepsilon) \Big) < \varepsilon;$$

2. *For every $\varepsilon > 0$*

$$\lim_{\gamma\to 0} \varlimsup_{n\to\infty} \sup_{\tau\in\mathcal{T}_T, \sigma\leq\gamma} P_n\Big(\pi_\cdot \in \mathcal{D}([0,T],\mathcal{S}) : d(\pi_{\tau+\sigma}, \pi_\tau) > \varepsilon \Big) = 0,$$

where $\mathcal{T}_T$ is the set of stopping times bounded by T and d is the metric in $\mathcal{S}$ defined above.

Proposition 4.1.2. *The sequence of measures $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ is tight with respect to the Skorokhod topology of $\mathcal{D}([0,T],\mathcal{M}_+)$.*

Proof. We start the proof by recalling that from Proposition 4.1.7 of [31], it is enough to show that for every function G in a dense subset of $C([0,1])$, with respect to the uniform topology of $C([0,1])$, the real-valued process $\{\langle \pi_t^n, G\rangle\}_{0\le t\le T}$ is tight. Therefore, in this context, the metric d in the statement of previous lemma is the usual distance in $\mathbb{R}$.

To prove the first condition of Lemma 4.1.1, fix $G\in C([0,1])$ and $\varepsilon>0$, and note that

$$|\langle \pi_t^n, G\rangle| = \left|\frac{1}{n}\sum_{x\in\Sigma_n}\eta_{tn^2}(x)G\left(\tfrac{x}{n}\right)\right| \le \|G\|_\infty,$$

where $\forall x\in\Sigma_n$, $|\eta_{tn^2}(x)|\le 1$, $\forall t\in[0,T]$. Thus, taking $K(t,\varepsilon)=\overline{B_\varepsilon(0)}$ with $\varepsilon>\|G\|_\infty$, we conclude that

$$\mathbb{P}_{\mu_n}\Big(\langle \pi_\cdot^n, G\rangle \in \mathcal{D}([0,T],\mathbb{R}) : \langle \pi_t^n, G\rangle \notin K(t,\varepsilon)\Big) = 0 < \varepsilon.$$

Now, let us prove the second condition of Lemma 4.1.1. As we mentioned above, it is enough to show tightness of the real-valued process $\{\langle \pi_t^n, G\rangle\}_{0\le t\le T}$ for a time independent function $G\in C([0,1])$. We claim that for each $\varepsilon>0$,

$$\lim_{\gamma\to 0}\varlimsup_{n\to\infty}\sup_{\tau\in\mathcal{T}_T,\sigma\le\gamma}\mathbb{P}_{\mu_n}\Big(\big|\langle \pi_{\tau+\sigma}^n, G\rangle - \langle \pi_\tau^n, G\rangle\big| > \varepsilon\Big) = 0, \tag{4.1}$$

where $\mathcal{T}_T$ is the set of stopping times bounded by T, thus, $\tau+\sigma$ should be understood as $(\tau+\sigma)\wedge T$. From (3.1), Markov's and Chebyshev's inequalities, the probability in (4.1) can be bounded from above by

$$\begin{aligned}
&\mathbb{P}_{\mu_n}\left(\big|M_{\tau+\sigma}^n(G) - M_\tau^n(G)\big| > \frac{\varepsilon}{2}\right) + \mathbb{P}_{\mu_n}\left(\left|\int_\tau^{\tau+\sigma} n^2 L_n^m\langle \pi_r^n, G\rangle\, dr\right| > \frac{\varepsilon}{2}\right)\\
\le&\frac{4}{\varepsilon^2}\mathbb{E}_{\mu_n}\left[\big|M_{\tau+\sigma}^n(G) - M_\tau^n(G)\big|^2\right] + \frac{2}{\varepsilon}\mathbb{E}_{\mu_n}\left[\left|\int_\tau^{\tau+\sigma} n^2 L_n^m\langle \pi_r^n, G\rangle\, dr\right|\right].
\end{aligned}$$

Thus, if we prove that

$$\lim_{\gamma\to 0}\varlimsup_{n\to\infty}\sup_{\tau\in\mathcal{T}_T,\sigma\le\gamma}\mathbb{E}_{\mu_n}\left[\big(M_{\tau+\sigma}^n(G) - M_\tau^n(G)\big)^2\right] = 0, \tag{4.2}$$

and

$$\lim_{\gamma\to 0}\varlimsup_{n\to\infty}\sup_{\tau\in\mathcal{T}_T,\sigma\le\gamma}\mathbb{E}_{\mu_n}\left[\left|\int_\tau^{\tau+\sigma} n^2 L_n^m\langle \pi_r^n, G\rangle\, dr\right|\right] = 0, \tag{4.3}$$

the claim follows. We have divided the proof of (4.2) and (4.3) into two cases: $\theta\ge 1$ and $\theta\in[0,1)$.

Case $\theta\ge 1$: We begin by analyzing (4.2). Let $G\in C^2([0,1])$, and observe that $C^2([0,1])$ is a dense subset of $C([0,1])$ with respect to the uniform topology. Define

$$F_t^n(G) := n^2\big(L_n^m\langle\pi_t^n, G\rangle^2 - 2\langle\pi_t^n, G\rangle L_n^m\langle\pi_t^n, G\rangle\big).$$

Note that

$$\mathbb{E}_{\mu_n}\left[\big(M_{\tau+\sigma}^n(G) - M_\tau^n(G)\big)^2\right] = \mathbb{E}_{\mu_n}\left[\int_\tau^{\tau+\sigma} F_t^n(G)\,dt\right],$$

since $\big(M_{\tau+\sigma}^n(G) - M_\tau^n(G)\big)^2 - \int_\tau^{\tau+\sigma} F_t^n(G)\,dt$ is a mean zero martingale by Dynkin's formula. Hence, (4.2) holds if we show that $\int_\tau^{\tau+\sigma} F_t^n(G)\,dt$ converges to zero uniformly in $t \in [0,T]$, when $n \to \infty$. From Remark 3.0.1, a simple computation shows that $F_t^n(G)$ is bounded from above by a constant, times

$$\frac{(m + n^{a-2})}{n}\|(G')^2\|_\infty + \frac{\kappa}{n^\theta}C(\alpha,\beta)\|G^2\|_\infty, \tag{4.4}$$

where $C(\alpha,\beta)$ is a positive real constant depending on α and β. Since $a \in (1,2)$, taking $n \to \infty$ in the previous display, the result follows. It remains to prove (4.3). Recall (3.2). From Remark 3.0.1 and since $|\eta_{tn^2}(x)| \leq 1$ for all $t \geq 0$ and $x \in \Sigma_n$, we can bound the bulk term from above by

$$\left|\Delta_n G\left(\tfrac{x}{n}\right)\tau_x h^m(\eta_{tn^2})\right| \;\leq\; 2\|G''\|_\infty, \tag{4.5}$$

and the boundary terms by

$$\begin{aligned} \left|\nabla_n^+ G(0)\tau_1 h^m(\eta_{tn^2}) + \kappa\frac{n}{n^\theta}G\left(\tfrac{1}{n}\right)\big(\alpha - \eta_{tn^2}(1)\big)\right| \;&\leq \|G'\|_\infty + \kappa\frac{n}{n^\theta}\|G\|_\infty, \\ \left|-\nabla_n^- G_s(1)\tau_{n-1} h^m(\eta_{tn^2}) + \kappa\frac{n}{n^\theta}G\left(\tfrac{n-1}{n}\right)\big(\beta - \eta_{tn^2}(n-1)\big)\right| \;&\leq \|G'\|_\infty + \kappa\frac{n}{n^\theta}\|G\|_\infty. \end{aligned} \tag{4.6}$$

Therefore, since $\theta \geq 1$, by (3.2), (4.5), and (4.6), we have that

$$\lim_{\gamma\to 0}\overline{\lim_{n\to\infty}}\sup_{\tau\in\mathcal{T}_T,\sigma\leq\gamma}\mathbb{E}_{\mu_n}\left[\left|\int_\tau^{\tau+\sigma} n^2 L_n^m\langle\pi_r^n, G\rangle\,dr\right|\right] = 0,$$

proving (4.3). The proof of (4.2) works for any $\theta > 0$, but does not work for $\theta = 0$ since the second term in (4.4) does not vanish when we take $n \to \infty$. We will treat this case below.

Case $\theta \in [0,1)$: Note that if we try to apply the strategy used above, we will have problems trying to control the expression $\int_\tau^{\tau+\sigma} n^2 L_B\langle\pi_t^n, G\rangle\,dt$. This happens because for these values of θ, the terms that come from the boundary go to infinity with n. Indeed, since these terms depend on the value of $G\left(\frac{1}{n}\right)$ and $G\left(\frac{n-1}{n}\right)$, we can get rid of them by asking the test function G to have compact support in $(0,1)$. With this assumption, we can show that (4.2) and (4.3) are still valid when $G \in C_c^2((0,1))$ only by using the computations done for $\theta \geq 1$. To finish the proof, we need to show that (4.2) and (4.3) hold for $G \in C((0,1))$. The idea then is to approximate $G \in C((0,1))$ in L_1 by functions in $C_c^2((0,1))$. To do this, we take a function $G \in C^1([0,1]) \subset L^1([0,1])$ and a sequence of functions $\{G_k\}_{k\geq 0} \in C_c^2((0,1))$ converging to G with respect to the L^1-norm as $k \to \infty$. Note that the probability in (4.1) is bounded from above by

$$\mathbb{P}_{\mu_n}\left(|\langle\pi^n_{\tau+\sigma},G_k\rangle-\langle\pi^n_\tau,G_k\rangle|>\frac{\varepsilon}{2}\right)+\mathbb{P}_{\mu_n}\left(|\langle\pi^n_{\tau+\sigma},G-G_k\rangle-\langle\pi^n_\tau,G-G_k\rangle|>\frac{\varepsilon}{2}\right).$$

Now, since G_k has compact support, the first probability above vanishes, and it remains only to check that the last probability vanishes as $n\to\infty$, $k\to\infty$, and $\gamma\to 0$. Since $|\eta(x)\leq 1|$, we have that

$$|\langle\pi^n_{\tau+\sigma},G-G_k\rangle-\langle\pi^n_\tau,G-G_k\rangle|\leq\frac{2}{n}\sum_{x\in\Sigma_n}\left|(G-G_k)\left(\tfrac{x}{n}\right)\right|,$$

from where we have the following estimate

$$\begin{aligned}\frac{1}{n}\sum_{x\in\Sigma_n}\left|(G-G_k)\left(\tfrac{x}{n}\right)\right|&\leq\sum_{x\in\Sigma_n}\int_{\frac{x}{n}}^{\frac{x+1}{n}}\left|(G-G_k)\left(\tfrac{x}{n}\right)-(G-G_k)(q)\right|dq+\int_0^1|(G-G_k)(q)|dq\\&\leq\frac{1}{n}\|(G-G_k)'\|_\infty+\int_0^1|(G-G_k)(q)|dq.\end{aligned}$$

Therefore, since $C^1([0,1])$ is a dense subset of $C([0,1])$ wrt the sup topology, the result follows by taking first the $\overline{\lim}$ in $n\to\infty$ and then in $k\to\infty$. □

4.2 Estimates on Dirichlet forms

We start this section by defining the Dirichlet forms, the *carré du champ* operator, and the Bernoulli product measure. Thereafter, we compare the Dirichlet forms and the integral of the *carré du champ* operator in Lemma 4.2.1.

Let μ be a probability measure on Ω_n, and $f:\Omega_n\to\mathbb{R}$ a density with respect to μ. The Dirichlet form of the process is defined as

$$\langle\sqrt{f},-L^m_n\sqrt{f}\rangle_\mu=\langle\sqrt{f},-L^m_P\sqrt{f}\rangle_\mu+n^{a-2}\langle\sqrt{f},-L_S\sqrt{f}\rangle_\mu+\langle\sqrt{f},-L_B\sqrt{f}\rangle_\mu,\tag{4.7}$$

where the inner product $\langle\cdot,\cdot\rangle_\mu$ is the one of $L^2(\Omega_n,\mu)$. Moreover, we define the integral of the *carré du champ* operator, denoted by D^m_n acting on functions $f:\Omega_n\to\mathbb{R}$, with respect to μ as

$$D^m_n(\sqrt{f},\mu):=(D^m_P+n^{a-2}D_S+D_B)(\sqrt{f},\mu),\tag{4.8}$$

with

$$D^m_P(\sqrt{f},\mu)\;:=\;\sum_{x=1}^{n-2}\int_{\Omega_n}p^m_{x,x+1}(\eta)(\sqrt{f(\eta^{x,x+1})}-\sqrt{f(\eta)})^2\,d\mu,\tag{4.9}$$

and

$$
\begin{aligned}
D_S(\sqrt{f},\mu) \;:=\; & \sum_{x=1}^{n-2}\int_{\Omega_n}\big\{a_{x,x+1}(\eta)+a_{x+1,x}(\eta)\big\}(\sqrt{f(\eta^{x,x+1})}-\sqrt{f(\eta)})^2\,d\mu \\
=& \sum_{x=1}^{n-2}\int_{\Omega_n}(\sqrt{f(\eta^{x,x+1})}-\sqrt{f(\eta)})^2\,d\mu.
\end{aligned}
\tag{4.10}
$$

Above $p^m_{x,x+1}(\eta):=c^m_{x,x+1}(\eta)\big\{a_{x,x+1}(\eta)+a_{x+1,x}(\eta)\big\}$, where the rates $c^m_{x,x+1}(\eta)$ and $a_{x,x+1}(\eta)$ are given in (2.5) and (2.6) respectively, and

$$
D_B(\sqrt{f},\mu) \;:=\; \tfrac{\kappa}{n^\theta}\Big(F_1^\alpha(\sqrt{f},\mu)+F_{n-1}^\beta(\sqrt{f},\mu)\Big),
$$

where for $x\in\{1,n-1\}$ and $\gamma\in\{\alpha,\beta\}$, F_x^γ is given by

$$
F_x^\gamma(\sqrt{f},\mu)=\int_{\Omega_n} I_x^\gamma(\eta)(\sqrt{f(\eta^x)}-\sqrt{f(\eta)})^2\,d\mu, \tag{4.11}
$$

with I_x^γ given in (2.7).

For a profile $\rho:[0,1]\to[0,1]$, we define the Bernoulli product measure $\nu^n_{\rho(\cdot)}$ on Ω_n with marginals given by

$$
\nu^n_{\rho(\cdot)}\{\eta:\,\eta(x)=1\}=\rho\left(\tfrac{x}{n}\right). \tag{4.12}
$$

Let f be a density with respect to the measure $\nu^n_{\rho(\cdot)}$. This section aims proving the estimate stated in Lemma 4.2.1. The idea is to estimate the Dirichlet form $\langle L_n^m\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}$ by the carré du champ operator plus an error, depending on the properties of the profile ρ. The properties that we ask for ρ depend on the parameter θ and its corresponding hydrodynamic equation. This estimate will be used many times in Sections 6.1, 6.2, and Chapter 7, and we will highlight each choice of the profile within the proofs.

Lemma 4.2.1. *Let $\rho:[0,1]\to[0,1]$ be a Lipschitz profile such that for all $u\in(0,1)$,*

$$
\alpha=\rho(0)\le\rho(u)\le\rho(1)=\beta, \tag{4.13}
$$

and which is locally constant at the boundary. Then, the Dirichlet form satisfies

$$
\langle L_n^m\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}\le-\frac{1}{4}D_n^m(\sqrt{f},\nu^n_{\rho(\cdot)})+O\left(\tfrac{1}{n}\right). \tag{4.14}
$$

In case $\rho:[0,1]\to[0,1]$ is a constant profile, then

$$
\langle L_n^m\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}\le-\frac{1}{4}D_n^m(\sqrt{f},\nu^n_{\rho(\cdot)})+O(\tfrac{1}{n^\theta}). \tag{4.15}
$$

The proof of the previous lemma follows from the following lemma.

Lemma 4.2.2. *Let $T:\eta\in\Omega_n\to T(\eta)\in\Omega_n$ be a transformation and $c:\eta\to c(\eta)$ be a positive local* *function. Let f be a density with respect to a finite positive measure μ on Ω_n. Then, we have that*

$$\begin{aligned}&\int_{\Omega_n} c(\eta)\big[\sqrt{f(T(\eta))}-\sqrt{f(\eta)}\big]\ \sqrt{f(\eta)}\,d\mu\\ &\leq -\frac{1}{4}\int_{\Omega_n} c(\eta)\left[\sqrt{f(T(\eta))}-\sqrt{f(\eta)}\right]^2\,d\mu\\ &+\frac{1}{16}\int_{\Omega_n}\frac{1}{c(\eta)}\left[c(\eta)-c(T(\eta))\frac{\mu(T(\eta))}{\mu(\eta)}\right]^2\left[\sqrt{f(T(\eta))}+\sqrt{f(\eta)}\right]^2 d\mu.\end{aligned} \tag{4.16}$$

Remark 4.2.3. *The previous lemma is stated in [4] asking the measure μ to be a probability measure, but we can ask the measure to be finite. For simplicity of the presentation, we decided not to repeat the proof of the previous lemma here. The interested reader can see a proof of it in Section 5.1 of [4].*

Proof of Lemma 4.2.1. Recall that the Dirichlet form of the process is given by the sum of the Dirichlet forms of the PMM, SSEP and Glauber dynamics

$$\langle L_n^m\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}=\langle L_P^m\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}+n^{a-2}\langle L_S\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}+\langle L_B\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}.$$

The proof will be divided into three steps, each one regarding the Dirichlet forms on the right-hand side of the previous expression. Let us begin by examining the one regarding the PMM

$$\langle L_P^m\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}=\sum_{x=1}^{n-2}\int_{\Omega_n} a_{x,x+1}(\eta)\left[\sqrt{f(\eta^{x,x+1})}-\sqrt{f(\eta)}\right]\sqrt{f(\eta)}\,d\nu^n_{\rho(\cdot)}, \tag{4.17}$$

Define $\Omega_n^x:=\{\eta\in\Omega_n;\ p^m_{x,x+1}(\eta)\neq 0\}$. Now, from Lemma 4.2.2, and the fact that $p^m_{x,x+1}(\eta^{x,x+1})=p^m_{x,x+1}(\eta)$, we can bound the integral in (4.17) from above by

$$\begin{aligned}&-\frac{1}{4}\int_{\Omega_n^x} p^m_{x,x+1}(\eta)\left[\sqrt{f(\eta^{x,x+1})}-\sqrt{f(\eta)}\right]^2\,d\nu^n_{\rho(\cdot)}\\ &+\frac{1}{16}\int_{\Omega_n^x} p^m_{x,x+1}(\eta)\left(1-\frac{\nu^n_{\rho(\cdot)}(\eta^{x,x+1})}{\nu^n_{\rho(\cdot)}(\eta)}\right)^2\left[\sqrt{f(\eta^{x,x+1})}+\sqrt{f(\eta)}\right]^2\,d\nu^n_{\rho(\cdot)}.\end{aligned}$$

Recall (4.9). Since $p^m_{x,x+1}(\eta)\leq m$ and $\left(\frac{\nu^n_{\rho(\cdot)}(\eta^{x,x+1})}{\nu^n_{\rho(\cdot)}(\eta)}-1\right)^2\leq C(\alpha,\beta)\left(\rho\left(\frac{x}{n}\right)-\rho\left(\frac{x+1}{n}\right)\right)^2$, where $C(\alpha,\beta)>0$, then the right-hand side of (4.17) can be bounded from above by

$$-\frac{1}{4}D_P^m(\sqrt{f},\nu^n_{\rho(\cdot)})+\frac{mC(\alpha,\beta)}{8}\sum_{x=1}^{n-2}\left(\rho\left(\tfrac{x}{n}\right)-\rho\left(\tfrac{x+1}{n}\right)\right)^2, \tag{4.18}$$

since f is a density with respect to $\nu^n_{\rho(\cdot)}$. Now, we look at the Dirichlet form for the SSEP. Repeating the same arguments as above, we get

$$\langle L_{\mathcal{S}}\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}\leq -\frac{1}{4}D_{\mathcal{S}}(\sqrt{f},\nu^n_{\rho(\cdot)})+\frac{C(\alpha,\beta)}{8}\sum_{x=1}^{n-2}\Big(\rho\big(\tfrac{x}{n}\big)-\rho\big(\tfrac{x+1}{n}\big)\Big)^2.$$

Finally, let us examine the one regarding the boundary dynamics. We claim that for $\theta\geq 0$ fixed, there exists a constant $C(\alpha,\rho)>0$ (independent of f and n) such that

$$\begin{aligned}\langle L_B\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}} \leq -\frac{1}{4}D_B(\sqrt{f},\nu^n_{\rho(\cdot)}) + C(\alpha,\rho)\frac{\kappa}{n^\theta}\big[(\rho(\tfrac{1}{n})-\alpha\big]^2 \\ + C(\beta,\rho)\frac{\kappa}{n^\theta}\big[(\rho(\tfrac{n-1}{n})-\beta\big]^2,\end{aligned} \tag{4.19}$$

where f is a density with respect to $\nu^n_{\rho(\cdot)}$. Since L_B is the sum of two terms, we just present the proof for one of them, namely the term which involves α, but for the other one the proof is completely analogous. To prove the result it is enough to note that from Lemma 4.2.2 and (4.11), we have that

$$\begin{aligned}&\int_{\Omega_n}\frac{\kappa}{n^\theta}\big[\alpha(1-\eta(1))+(1-\alpha)\eta(1)\big]\ \Big[\sqrt{f(\eta^1)}-\sqrt{f(\eta)}\Big]\ \sqrt{f(\eta)}\ d\nu^n_{\rho(\cdot)}\\ &\leq -\frac{1}{4}\frac{\kappa}{n^\theta}F_1^\alpha(\sqrt{f},\nu^n_{\rho(\cdot)})+\frac{\kappa}{16\,n^\theta}\int_{\Omega_n}\frac{1}{I_1^\alpha(\eta)}\left[I_1^\alpha(\eta)-I_1^\alpha(\eta^1)\tfrac{\nu^n_{\rho(\cdot)}(\eta^1)}{\nu^n_{\rho(\cdot)}(\eta)}\right]^2\Big[\sqrt{f(\eta^1)}+\sqrt{f(\eta)}\Big]^2\ d\nu^n_{\rho(\cdot)},\end{aligned}$$

where $I_1^\alpha(\eta)$ is defined in (2.7) and $F_1^\alpha(\sqrt{f},\nu^n_{\rho(\cdot)})$ is defined in (4.11). By a simple computation we see that there exists a constant $\bar{C}(\alpha,\rho)>0$ such that

$$\frac{1}{I_1^\alpha(\eta)}\left[I_1^\alpha(\eta)-I_1^\alpha(\eta^1)\tfrac{\nu^n_{\rho(\cdot)}(\eta^1)}{\nu^n_{\rho(\cdot)}(\eta)}\right]^2\leq\bar{C}(\alpha,\rho)\,\big[\rho(\tfrac{1}{n})-\alpha\big]^2,$$

uniformly on $\eta\in\Omega_n$. Finally, using the fact that f is a density with respect to $\nu^n_{\rho(\cdot)}$, and repeating the argument for the term which involves β we conclude (4.19). Putting together all the estimates that we have obtained, we see that there exists a constant $\tilde{C}=\tilde{C}(m,\alpha,\beta,\rho)$ such that

$$\begin{aligned}\langle L_n^m\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}\leq &-\frac{1}{4}D_n^m(\sqrt{f},\nu^n_{\rho(\cdot)})+\tilde{C}\sum_{x=1}^{n-2}\Big(\rho(\tfrac{x}{n})-\rho(\tfrac{x+1}{n})\Big)^2\\ &+\tilde{C}\frac{\kappa}{n^\theta}\big[\rho(\tfrac{1}{n})-\alpha\big]^2+\tilde{C}\frac{\kappa}{n^\theta}\big[\rho(\tfrac{n-1}{n})-\beta\big]^2.\end{aligned}$$

From the previous bound, and from the fact that ρ is Lipschitz and locally constant at the boundary, we get (4.14). □

Chapter 5

Characterization of limit points

We begin this section by fixing some notations that will be used in the next chapters. Recall (2.10) and (2.11) from Chapter 2. Note that

$$\begin{aligned} \tau_1 h^m(\eta) - \alpha^m &= (\eta(1) - \alpha)\mathcal{R}^\alpha_m(\eta), \\ \tau_{n-1} h^m(\eta) - \beta^m &= (\eta(n-1) - \beta)\mathcal{R}^\beta_m(\eta), \end{aligned} \tag{5.1}$$

where

$$\mathcal{R}^\alpha_m(\eta) = \sum_{i=0}^{m-1} \alpha^{m-1-i} \prod_{j=0}^{i-1} \eta(2+j) \text{ and } \mathcal{R}^\beta_m(\eta) = \sum_{i=0}^{m-1} \beta^{m-1-i} \prod_{j=0}^{i-1} \eta(n-2-j). \tag{5.2}$$

Fix $n, \ell \in \mathbb{N}$, $x \in \Sigma_n$, $\varepsilon > 0, \delta > 0$, and recall that $a \in (1,2)$. Let

$$\overleftarrow{\Lambda}^\ell_x := \{x - \ell + 1, \dots, x\} \quad \Big(\text{resp.} \quad \overrightarrow{\Lambda}^\ell_x := \{x, \dots, x + \ell - 1\}\Big) \tag{5.3}$$

be the box of size ℓ to the left (resp. right) of the site x. We denote by

$$\overleftarrow{\eta}^\ell(x) = \frac{1}{\ell} \sum_{y \in \overleftarrow{\Lambda}^\ell_x} \eta(y) \quad \text{and} \quad \overrightarrow{\eta}^\ell(x) = \frac{1}{\ell} \sum_{y \in \overrightarrow{\Lambda}^\ell_x} \eta(y) \tag{5.4}$$

the empirical densities in the boxes $\overleftarrow{\Lambda}^\ell_x$ and $\overrightarrow{\Lambda}^\ell_x$, respectively. In many proofs presented along the text, we will need to replace terms of the form (2.10) by products of terms of the form (5.4). To do so, we need to have space in our discrete set Σ_n and we define the following space

$$\Sigma^\varepsilon_{n,m} = \{1 + \tfrac{m}{2}\varepsilon n, \dots, n - 1 - \tfrac{m}{2}\varepsilon n\}, \tag{5.5}$$

where above εn denotes $\lfloor \varepsilon n \rfloor$. For m even we consider $\Sigma^\varepsilon_{n,m}$, while for m odd, we would consider $\Sigma^\varepsilon_{n,m}$ given by

$$\Sigma^\varepsilon_{n,m} = \{1 + \tfrac{m+1}{2}\varepsilon n, \dots, n - 1 - \tfrac{m-1}{2}\varepsilon n\},$$

see Figure 5.1. More specifically, we introduce the subset $\Sigma^{\varepsilon}_{n,m}$ of the bulk Σ_n since, for each $x \in \Sigma^{\varepsilon}_{n,m}$ we will need to replace the occupation at site x by its average to the left or right of x on a box of size εn, and we are allowed to do so for $x \in \Sigma^{\varepsilon}_{n,m}$ but not for x on the whole bulk.

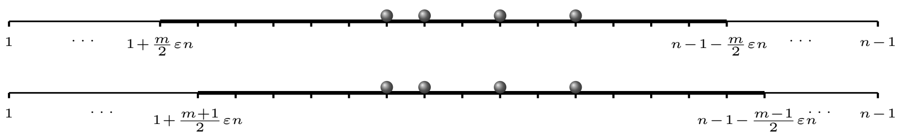


Figure 5.1: The set $\Sigma^{\varepsilon}_{n,m}$ for m even and for m odd, respectively.

Now we consider two approximations of the identity, for fixed $u \in [0,1]$, which are given on $v \in [0,1]$ by $\overleftarrow{\iota}^{u}_{\varepsilon}(v) = \dfrac{1}{\varepsilon}\mathbb{1}_{(u-\varepsilon,u]}(v)$ and $\overrightarrow{\iota}^{u}_{\varepsilon}(v) = \dfrac{1}{\varepsilon}\mathbb{1}_{[u,u+\varepsilon)}(v)$. We use the notation

$$\langle \pi_s, \overleftarrow{\iota}^{u}_{\varepsilon}\rangle = \frac{1}{\varepsilon}\int_{u-\varepsilon}^{u} \rho_s(v)\,dv \quad \text{and} \quad \langle \pi_s, \overrightarrow{\iota}^{u}_{\varepsilon}\rangle = \frac{1}{\varepsilon}\int_{u}^{u+\varepsilon} \rho_s(v)\,dv. \tag{5.6}$$

From last result, the fact that $0 \leq \rho_s(\cdot) \leq 1$, and Lebesgue differentiation Theorem, for almost every $u \in [0,1]$,

$$\lim_{\varepsilon\to 0} |\rho_s(u) - \langle \pi_s, \overleftarrow{\iota}^{u-i\varepsilon}_{\varepsilon}\rangle| = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} |\rho_s(u) - \langle \pi_s, \overrightarrow{\iota}^{u+i\varepsilon}_{\varepsilon}\rangle| = 0, \tag{5.7}$$

for any $i \in \{0,1,\ldots,m-1\}$. From (5.6) and (5.4), we have that

$$\langle \pi^n_s, \overleftarrow{\iota}^{x/n-i\varepsilon}_{\varepsilon}\rangle = \overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x - i\varepsilon n) + O\left(\tfrac{1}{\varepsilon n}\right), \quad \langle \pi^n_s, \overrightarrow{\iota}^{x/n+i\varepsilon}_{\varepsilon}\rangle = \overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1+i\varepsilon n) + O(\tfrac{1}{\varepsilon n}),$$

and

$$\langle \pi^n_s, \overrightarrow{\iota}^{j\varepsilon}_{\varepsilon}\rangle = \overrightarrow{\eta}^{\varepsilon n}_{sn^2}(2 + j\varepsilon n) + O(\tfrac{1}{\varepsilon n}),$$

for $i = 0, \ldots, \frac{m}{2} - 1$ and $j = 0, \ldots, i-1$.

From Section 4.1 we know that limit points $\mathbb{Q}$ of the sequence $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ exist. As a consequence of the exclusion rule, we now observe that they are concentrated on trajectories of measures that are absolutely continuous with respect to the Lebesgue measure, see [31] for more details. Moreover, we claim that the density $\rho_t(u)$ is a weak solution of the corresponding hydrodynamic equation. This is proved in the next proposition.

Proposition 5.0.1. *Let $\mathbb{Q}$ be a limit point of the sequence $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$. Then*

$$\mathbb{Q}\Big(\pi_\cdot \in \mathcal{D}([0,T],\mathcal{M}_+) : F_\theta(G,t,\rho,g,m) = 0, \forall t \in [0,T], \forall G \in C_\theta\Big) = 1.$$

Above $F_\theta = F_{Dir}$ and $C_\theta = C^{1,2}_0([0,T]\times[0,1])$ for $\theta < 1$; and $F_\theta = F_{Rob}$ and $C_\theta = C^{1,2}([0,T]\times[0,1])$ for $\theta \geq 1$, where F_{Dir} (resp. F_{Rob}) is defined in (2.16) *(resp.* (2.18)*).*

Proof. For simplicity of the presentation, we will present the proof for even m since the proof for m odd is analogous, and, when necessary, we explain the changes for the case m odd. The proof ends as long as we show that for any $\delta > 0$ and $G \in C_\theta$

$$\mathbb{Q}\Bigg(\pi_\cdot \in \mathcal{D}([0,T],\mathcal{M}_+) : \sup_{0\le t\le T} \big|F_\theta(G,t,\rho,g,m)\big| > \delta\Bigg) = 0, \tag{5.8}$$

for each regime of θ. We start with the case $\theta \geq 1$. Recall from item (2) of Definition 5 the definition of F_{Rob}. Since the boundary integrals are not well-defined in the Skorokhod space, the set inside last probability is not an open set in the Skorokhod space, and we can not use Portmanteaus's Theorem. To avoid this problem, we fix $\varepsilon > 0$ and we consider two approximations of the identity for fixed $u \in [0,1]$, which are given on $v \in [0,1]$ by $\overleftarrow{\iota}^{u}_{\varepsilon}(v) = \frac{1}{\varepsilon}\mathbb{1}_{(u-\varepsilon,u]}(v)$ and $\overrightarrow{\iota}^{u}_{\varepsilon}(v) = \frac{1}{\varepsilon}\mathbb{1}_{[u,u+\varepsilon)}(v)$. Recall the notations (5.6) and (5.7). By summing and subtracting proper terms, we bound the probability in (5.8) from above by the sum of

$$\begin{aligned}
&\mathbb{Q}\Bigg(\sup_{0\le t\le T}\Bigg|\langle \rho_t, G_t\rangle - \langle \rho_0, G_0\rangle + \int_0^t \int_{\varepsilon\frac{m}{2}}^{1-\varepsilon\frac{m}{2}} \prod_{i=0}^{\frac{m}{2}-1}\langle \pi_s, \overleftarrow{\iota}^{u-i\varepsilon}_{\varepsilon}\rangle \prod_{i=0}^{\frac{m}{2}-1}\langle \pi_s, \overrightarrow{\iota}^{u+i\varepsilon}_{\varepsilon}\rangle \Delta G_s(u)\,du\,ds \\
&- \int_0^t \langle \rho_s, \partial_s G_s\rangle\,ds + \int_0^t \Bigg\{\prod_{i=0}^{m-1}\langle \pi_s, \overleftarrow{\iota}^{1-i\varepsilon}_{\varepsilon}\rangle \partial_u G_s(1) - \prod_{i=0}^{m-1}\langle \pi_s, \overrightarrow{\iota}^{i\varepsilon}_{\varepsilon}\rangle \partial_u G_s(0)\Bigg\}\,ds \\
&- \kappa \int_0^t \Big\{G_s(0)\big(\alpha - \langle \pi_s, \overrightarrow{\iota}^{0}_{\varepsilon}\rangle\big) + G_s(1)\big(\beta - \langle \pi_s, \overleftarrow{\iota}^{1}_{\varepsilon}\rangle\big)\Big\}\,ds\Bigg| > \tfrac{\delta}{7}\Bigg),
\end{aligned} \tag{5.9}$$

$$\mathbb{Q}\Big(\big|\langle \rho_0 - g, G_0\rangle\big| > \tfrac{\delta}{7}\Big), \tag{5.10}$$

$$\mathbb{Q}\Bigg(\sup_{0\le t\le T}\Bigg|\int_0^t \Bigg\{\langle \rho^m_s, \Delta G_s\rangle - \int_{\varepsilon\frac{m}{2}}^{1-\varepsilon\frac{m}{2}} \prod_{i=0}^{\frac{m}{2}-1}\langle \pi_s, \overleftarrow{\iota}^{u-i\varepsilon}_{\varepsilon}\rangle \prod_{i=0}^{\frac{m}{2}-1}\langle \pi_s, \overrightarrow{\iota}^{u+i\varepsilon}_{\varepsilon}\rangle \Delta G_s(u)\,du\Bigg\}\,ds\Bigg| > \tfrac{\delta}{7}\Bigg), \tag{5.11}$$

$$\mathbb{Q}\Bigg(\sup_{0\le t\le T}\Bigg|\kappa\int_0^t G_s(0)\Big\{\langle \pi_s, \overrightarrow{\iota}^{0}_{\varepsilon}\rangle - \rho_s(0)\Big\}\,ds\Bigg| > \tfrac{\delta}{7}\Bigg), \tag{5.12}$$

$$\mathbb{Q}\Bigg(\sup_{0\le t\le T}\Bigg|\int_0^t \Big\{(\rho_s(0))^m - \prod_{i=0}^{m-1}\langle \pi_s, \overrightarrow{\iota}^{i\varepsilon}_{\varepsilon}\rangle\Big\}\partial_u G_s(0)\,ds\Bigg| > \tfrac{\delta}{7}\Bigg), \tag{5.13}$$

plus two other terms similar to (5.12) and (5.13) but with respect to the right boundary. The term (5.10) is equal to zero since $\mathbb{Q}$ is a limit point of $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ and $\mathbb{Q}_n$ is induced by μ_n, which satisfies (2.20). To treat (5.11), we use (5.6), (5.7), and the fact that $0 \leq \rho(\cdot) \leq 1$ to get

$$\begin{aligned}
&\Bigg|\int_0^t \Bigg\{\langle \rho^m_s, \Delta G_s\rangle - \int_{\varepsilon\frac{m}{2}}^{1-\varepsilon\frac{m}{2}} \prod_{i=0}^{\frac{m}{2}-1}\langle \pi_s, \overleftarrow{\iota}^{u-i\varepsilon}_{\varepsilon}\rangle \prod_{i=0}^{\frac{m}{2}-1}\langle \pi_s, \overrightarrow{\iota}^{u+i\varepsilon}_{\varepsilon}\rangle \Delta G_s(u)\,du\Bigg\}\,ds\Bigg| \\
&\leq \varepsilon C(G,T,m) + \int_0^t \int_{\varepsilon\frac{m}{2}}^{1-\varepsilon\frac{m}{2}} \Bigg\{\sum_{i=0}^{\frac{m}{2}-1}\big|\rho_s(u) - \langle \rho_s, \overleftarrow{\iota}^{u-i\varepsilon}_{\varepsilon}\rangle\big| + \sum_{i=0}^{\frac{m}{2}-1}\big|\rho_s(u) - \langle \rho_s, \overrightarrow{\iota}^{u+i\varepsilon}_{\varepsilon}\rangle\big|\Bigg\} |\Delta G_s(u)|\,du\,ds.
\end{aligned} \tag{5.14}$$

From the previous inequality and (5.7), we have that (5.11) vanishes when $\varepsilon \to 0$. In the same way, it follows that (5.12) and (5.13) vanish when $\varepsilon \to 0$. Now, to treat (5.12) and (5.13) we need the limit in (5.7) to be true for all $u \in [0,1]$ (or at least at the points of the boundary of $[0,1]$) and this is the statement of Lemma A.0.1. The probability in (5.12) vanishes when $\varepsilon \to 0$ by a simple application of Lemma A.0.1. Now, we want to show that (5.13) vanishes when $\varepsilon \to 0$. Let

$$R^i_{\varepsilon,s} := \left|\rho^i_s(0) - \prod_{j=0}^{i-1}\langle \pi_s, \overrightarrow{\iota}^{j\varepsilon}_{\varepsilon}\rangle\right|,$$

for every $i = \{0, 1, \dots, m-1\}$. In order to prove (5.13) it is enough to prove that $\lim_{\varepsilon\to 0} R^i_{\varepsilon,s} = 0$, for every $i = \{0, 1, \dots, m-1\}$. Note that $R^0_{\varepsilon,s} = 1 - 1 = 0$. For $i = 1$, $R^1_{\varepsilon,s} = |\rho_s(0) - \langle \pi_s, \overrightarrow{\iota}^{0}_{\varepsilon}\rangle|$ vanishes when $\varepsilon \to 0$ by a simple application of Lemma A.0.1. For $i = 2$,

$$\begin{aligned} R^2_{\varepsilon,s} &= \rho_s(0)\left|\rho_s(0) - \langle \pi_s, \overrightarrow{\iota}^{0}_{\varepsilon}\rangle\right| + \langle \pi_s, \overrightarrow{\iota}^{0}_{\varepsilon}\rangle\left|\rho_s(0) - \langle \pi_s, \overrightarrow{\iota}^{\varepsilon}_{\varepsilon}\rangle\right| \\ &= \rho_s(0)R^1_{\varepsilon,s} + \langle \pi_s, \overrightarrow{\iota}^{0}_{\varepsilon}\rangle\left|\rho_s(0) - \langle \pi_s, \overrightarrow{\iota}^{\varepsilon}_{\varepsilon}\rangle\right|. \end{aligned}$$

Since $\rho(0) \le 1$ and $\langle \pi_s, \overrightarrow{\iota}^{0}_{\varepsilon}\rangle \le 1$, $R^2_{\varepsilon,s}$ vanishes when $\varepsilon \to 0$ by Lemma A.0.1 for $i = 1$ for s a.s. Following the same idea as before, for $i = 3$ we have

$$R^3_{\varepsilon,s} = \rho_s(0)R^2_{\varepsilon,s} + \langle \pi_s, \overrightarrow{\iota}^{0}_{\varepsilon}\rangle\langle \pi_s, \overrightarrow{\iota}^{\varepsilon}_{\varepsilon}\rangle\left|\rho_s(0) - \langle \pi_s, \overrightarrow{\iota}^{2\varepsilon}_{\varepsilon}\rangle\right|,$$

which vanishes by Lemma A.0.1 for $i = 2$ for s a.s. Proceeding in an inductive fashion, suppose that $\lim_{\varepsilon\to 0} R^i_{\varepsilon,s} = 0$ for every $i \le m-1$ and let us prove that $\lim_{\varepsilon\to 0} R^{i+1}_{\varepsilon,s} = 0$. From the previous computations we have that

$$R^{i+1}_{\varepsilon,s} = \rho_s(0)R^i_{\varepsilon,s} + \prod_{j=0}^{i-2}\langle \pi_s, \overrightarrow{\iota}^{j\varepsilon}_{\varepsilon}\rangle\left|\rho_s(0) - \langle \pi_s, \overrightarrow{\iota}^{i\varepsilon}_{\varepsilon}\rangle\right|.$$

Thus, $\lim_{\varepsilon\to 0} R^{i+1}_{\varepsilon,s} = 0$ for every $i \le m-1$ since $\lim_{\varepsilon\to 0} R^i_{\varepsilon,s} = 0$, $\prod_{j=0}^{i-2}\langle \pi_s, \overrightarrow{\iota}^{j\varepsilon}_{\varepsilon}\rangle \le 1$ and by the application of Lemma A.0.1 for i for s a.s. Therefore, from the previous computations we have that (5.13) vanishes when $\varepsilon \to 0$.

Now, it remains only to look at (5.9). Note that we still can not use Portmanteau's Theorem, since for each $i = 0, \dots, \frac{m}{2} - 1$ the functions $\overleftarrow{\iota}^{u-i\varepsilon}_{\varepsilon}$ and $\overrightarrow{\iota}^{u+i\varepsilon}_{\varepsilon}$ are not continuous. Nevertheless, we can approximate each one of these functions by continuous functions, in such a way that the error vanishes as $\varepsilon \to 0$. Then, since for the continuous functions the set inside the probability in (5.9) is an open set with respect to the Skorokhod topology, we can use Portmanteau's Theorem, change back again from the continuous functions to $\overleftarrow{\iota}^{u-i\varepsilon}_{\varepsilon}$ and $\overrightarrow{\iota}^{u+i\varepsilon}_{\varepsilon}$, and bound (5.9) from above by

$$\begin{aligned} &\liminf_{n\to\infty} \mathbb{Q}_n\Bigg(\sup_{0\le t\le T}\Bigg|\langle \pi_t, G_t\rangle - \langle \pi_0, G_0\rangle - \int_0^t \langle \pi_s, \partial_s G_s\rangle\, ds \\ &- \int_0^t \int_{\varepsilon\frac{m}{2}}^{1-\varepsilon\frac{m}{2}} \prod_{i=0}^{\frac{m}{2}-1}\langle \pi_s, \overleftarrow{\iota}^{u-i\varepsilon}_{\varepsilon}\rangle \prod_{i=0}^{\frac{m}{2}-1}\langle \pi_s, \overrightarrow{\iota}^{u+i\varepsilon}_{\varepsilon}\rangle \Delta G_s(u)\, du\, ds \\ &+ \int_0^t \left\{\prod_{i=0}^{m-1}\langle \pi_s, \overleftarrow{\iota}^{1-i\varepsilon}_{\varepsilon}\rangle \partial_u G_s(1) - \prod_{i=0}^{m-1}\langle \pi_s, \overrightarrow{\iota}^{i\varepsilon}_{\varepsilon}\rangle \partial_u G_s(0)\right\} ds \\ &- \kappa \int_0^t \Big\{G_s(0)\big(\alpha - \langle \pi_s, \overrightarrow{\iota}^{0}_{\varepsilon}\rangle\big) + G_s(1)\big(\beta - \langle \pi_s, \overleftarrow{\iota}^{1}_{\varepsilon}\rangle\big)\Big\}\, ds\Bigg| > \tfrac{\delta}{7}\Bigg). \end{aligned} \tag{5.15}$$

Summing and subtracting $\int_0^t n^2 L_n^m \langle \pi_s^n, G_s\rangle ds$ to the term inside the supremum in (5.15), and recalling (3.3), we bound the probability in (5.15) from above by the sum of the next two terms

$$\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left|M_t^n(G)\right| > \tfrac{\delta}{14}\right), \tag{5.16}$$

and

$$\begin{aligned}\mathbb{P}_{\mu_n}\Bigg(\sup_{0\le t\le T}\Bigg|&\int_0^t n^2 L_n^m\langle \pi_s^n, G_s\rangle\, ds - \int_0^t\int_{\varepsilon\frac{m}{2}}^{1-\varepsilon\frac{m}{2}} \prod_{i=0}^{\frac{m}{2}-1}\langle \pi_s^n, \overleftarrow{\iota}_{\varepsilon}^{u-i\varepsilon}\rangle \prod_{i=0}^{\frac{m}{2}-1}\langle \pi_s^n, \overrightarrow{\iota}_{\varepsilon}^{u+i\varepsilon}\rangle \Delta G_s(u)\, du\, ds\\ &+\int_0^t\left\{\prod_{i=0}^{m-1}\langle \pi_s^n, \overleftarrow{\iota}_{\varepsilon}^{1-i\varepsilon}\rangle \partial_u G_s(1) - \prod_{i=0}^{m-1}\langle \pi_s^n, \overrightarrow{\iota}_{\varepsilon}^{i\varepsilon}\rangle\partial_u G_s(0)\right\} ds\\ &-\kappa\int_0^t\Big\{G_s(0)\big(\alpha-\langle \pi_s^n, \overrightarrow{\iota}_{\varepsilon}^{0}\rangle\big)+G_s(1)\big(\beta-\langle\pi_s^n, \overleftarrow{\iota}_{\varepsilon}^{1}\rangle\big)\Big\}\, ds\Bigg| > \tfrac{\delta}{14}\Bigg).\end{aligned} \tag{5.17}$$

From Doob's inequality and (4.4), the term (5.16) vanishes as $n\to\infty$. Finally, for $\tilde{\delta}>0$, we can bound (5.17) from above by the sum of the following terms

$$\begin{aligned}\mathbb{P}_{\mu_n}\Bigg(\sup_{0\le t\le T}\Bigg|\int_0^t\Bigg\{&\frac{1}{n}\sum_{x\in\Sigma_n}\Delta_n G_s\left(\tfrac{x}{n}\right)\tau_x h^m(\eta_{sn^2})\\ &-\int_{\varepsilon\frac{m}{2}}^{1-\varepsilon\frac{m}{2}}\prod_{i=0}^{\frac{m}{2}-1}\langle \pi_s^n, \overleftarrow{\iota}_{\varepsilon}^{u-i\varepsilon}\rangle\prod_{i=0}^{\frac{m}{2}-1}\langle\pi_s^n, \overrightarrow{\iota}_{\varepsilon}^{u+i\varepsilon}\rangle\Delta G_s(u)\, du\Bigg\} ds\Bigg| > \tilde{\delta}\Bigg),\end{aligned} \tag{5.18}$$

$$\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left|\int_0^t\left\{\nabla_n^+ G_s(0)\tau_1 h^m(\eta_{sn^2}) - \prod_{i=0}^{m-1}\langle \pi_s^n, \overrightarrow{\iota}_{\varepsilon}^{i\varepsilon}\rangle\partial_u G_s(0)\right\} ds\right| > \tilde{\delta}\right), \tag{5.19}$$

$$\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left|\int_0^t\left\{G_s(0)(\alpha-\langle \pi_s^n, \overrightarrow{\iota}_{\varepsilon}^{0}\rangle) - G_s\left(\tfrac{1}{n}\right)(\alpha-\eta_{sn^2}(1))\right\} ds\right| > \tilde{\delta}\right), \tag{5.20}$$

plus two terms which are similar to the last ones, but concerning the right boundary. Now, we show that (5.20) vanishes when $n\to\infty$ and then $\varepsilon\to 0$. By Taylor expansion on G, the terms which involve α vanish when $n\to\infty$. Recall (5.4). Observing that $\langle\pi_s^n, \overrightarrow{\iota}_{\varepsilon}^{0}\rangle = \overrightarrow{\eta}_{sn^2}^{\varepsilon n}(1)$, from Lemma 6.2.2, (5.20) goes to zero as $n\to\infty$ and $\varepsilon\to 0$. Now, we treat (5.19). Using Taylor expansion, $\partial_u G_s(0)$ can be replaced by its discrete derivative $\nabla_n^+ G_s(0)$. Recall (2.11) and note that $\tau_1 h^m(\eta_{sn^2})$ is composed by sum of products of the form $\eta(1)\eta(2)\cdots\eta(m)$ and a term $\alpha^{m-1}(\eta_{sn^2}(1)-\eta_{sn^2}(2))$. Note also that

$$\langle \pi_s^n, \overrightarrow{\iota}_{\varepsilon}^{i\varepsilon}\rangle = \overrightarrow{\eta}_{sn^2}^{\varepsilon n}(1+i\varepsilon n) + O(\tfrac{1}{\varepsilon n}),$$

for $i=0,\dots,m-1$. Thus, since Theorem 6.2.3 allow us to replace products of the form

$$\prod_{i=0}^{m-1}\eta(i+1) \text{ by } \prod_{i=0}^{m-1}\overrightarrow{\eta}_{sn^2}^{\varepsilon n}(1+i\varepsilon n),$$

and Corollary 6.2.6 allow us replacing $\eta(1)$ by $\eta(2)$, from these observations (5.19) vanishes, as $n\to\infty$ and $\varepsilon\to 0$. Finally, we treat (5.18). Recall (5.5). Note that the sum in Σ_n can be written as a sum over

$\Sigma^{\varepsilon}_{n,m}$ by paying a price of order $O(\varepsilon)$. Now, note that the error from changing the integral in the space variable by its Riemann sum is of order $O(\frac{1}{n})$, and therefore we can bound (5.18) from above by

$$\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left|\int_0^t \frac{1}{n}\sum_{x\in\Sigma^{\varepsilon}_{n,m}}\left\{\Delta_n G_s\left(\tfrac{x}{n}\right)\tau_x h^m(\eta_{sn^2})-\prod_{i=0}^{\frac{m}{2}-1}\langle\pi^n_s,\overleftarrow{\iota}^{\,x/n-i\varepsilon}_{\varepsilon}\rangle\langle\pi^n_s,\overrightarrow{\iota}^{\,x/n+i\varepsilon}_{\varepsilon}\rangle\Delta G_s\left(\tfrac{x}{n}\right)\right\}ds\right|>\tilde{\delta}\right). \tag{5.21}$$

By Taylor expansion on the test function G, we can replace its Laplacian by its discrete Laplacian, by paying a price of order $O(\frac{1}{n})$. Since for $x\in\Sigma_n$

$$\langle\pi^n_s,\overleftarrow{\iota}^{\,x/n-i\varepsilon}_{\varepsilon}\rangle=\overleftarrow{\eta}^{\,\varepsilon n}_{sn^2}(x-i\varepsilon n),\quad \langle\pi^n_s,\overrightarrow{\iota}^{\,x/n+i\varepsilon}_{\varepsilon}\rangle=\overrightarrow{\eta}^{\,\varepsilon n}_{sn^2}(x+1+i\varepsilon n)+O(\tfrac{1}{\varepsilon n}),$$

(5.21) can be bounded from above by

$$\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}\left|\int_0^t \frac{1}{n}\sum_{x\in\Sigma^{\varepsilon}_{n}}\left\{\Delta_n G_s\left(\tfrac{x}{n}\right)\tau_x h^m(\eta_{sn^2})-\prod_{i=0}^{\frac{m}{2}-1}\overleftarrow{\eta}^{\,\varepsilon n}_{sn^2}(x-i\varepsilon n)\overrightarrow{\eta}^{\,\varepsilon n}_{sn^2}(x+1+i\varepsilon n)\right\}ds\right|>\tilde{\delta}\right). \tag{5.22}$$

Then from Theorem 6.1.1, (5.22) vanishes, as $n\to\infty$ and $\varepsilon\to 0$. This ends the proof in the case $\theta=1$. We observe that the case $\theta>1$ is contained in the previous proof.

Finally, we present the proof in the case $\theta\in[0,1)$. Recall the definition of F_{Dir} from item (2) of Definition 4. Following the same ideas presented in the case $\theta=1$, we can bound (5.8) from above by the sum of

$$\begin{aligned}\mathbb{Q}\Bigg(\sup_{0\le t\le T}\Bigg|&\langle\pi_t,G_t\rangle-\langle\pi_0,G_0\rangle+\int_0^t\int_{\varepsilon\frac{m}{2}}^{1-\varepsilon\frac{m}{2}}\prod_{i=0}^{\frac{m}{2}-1}\langle\pi_s,\overleftarrow{\iota}^{\,u-i\varepsilon}_{\varepsilon}\rangle\prod_{i=0}^{\frac{m}{2}-1}\langle\pi_s,\overrightarrow{\iota}^{\,u+i\varepsilon}_{\varepsilon}\rangle\Delta G_s(u)\,du\,ds\\&-\int_0^t\langle\pi_s,\partial_s G_s\rangle\,ds+\int_0^t\left\{\beta^m\partial_u G_s(1)-\alpha^m\partial_u G_s(0)\right\}ds\Bigg|>\tfrac{\delta}{3}\Bigg),\end{aligned} \tag{5.23}$$

$$\mathbb{Q}\Big(|\langle\rho_0-g,G_0\rangle|>\tfrac{\delta}{3}\Big), \tag{5.24}$$

$$\mathbb{Q}\left(\sup_{0\le t\le T}\left|\int_0^t\left\{\langle\rho^m_s,\Delta G_s\rangle-\int_{\varepsilon\frac{m}{2}}^{1-\varepsilon\frac{m}{2}}\prod_{i=0}^{\frac{m}{2}-1}\langle\pi_s,\overleftarrow{\iota}^{\,u-i\varepsilon}_{\varepsilon}\rangle\prod_{i=0}^{\frac{m}{2}-1}\langle\pi_s,\overrightarrow{\iota}^{\,u+i\varepsilon}_{\varepsilon}\rangle\Delta G_s(u)\,du\right\}ds\right|>\tfrac{\delta}{3}\right). \tag{5.25}$$

Using the same arguments that we used above to treat (5.10) and (5.11), we can see that (5.24) and (5.25) vanish when $n\to\infty$ and $\varepsilon\to 0$. Therefore, it remains only to bound (5.23). By the same arguments used in case $\theta=1$, (5.23) is bounded from above by

$$\begin{aligned}\liminf_{n\to\infty}\mathbb{Q}_n\Bigg(\sup_{0\le t\le T}\Bigg|&\langle\pi_t,G_t\rangle-\langle\pi_0,G_0\rangle+\int_0^t\int_{\varepsilon\frac{m}{2}}^{1-\varepsilon\frac{m}{2}}\prod_{i=0}^{\frac{m}{2}-1}\langle\pi_s,\overleftarrow{\iota}^{\,u-i\varepsilon}_{\varepsilon}\rangle\prod_{i=0}^{\frac{m}{2}-1}\langle\pi_s,\overrightarrow{\iota}^{\,u+i\varepsilon}_{\varepsilon}\rangle\Delta G_s(u)\,du\,ds\\&-\int_0^t\langle\pi_s,\partial_s G_s\rangle\,ds+\int_0^t\left\{\beta^m\partial_u G_s(1)-\alpha^m\partial_u G_s(0)\right\}ds\Bigg|>\tfrac{\delta}{3}\Bigg).\end{aligned} \tag{5.26}$$

Summing and subtracting $\int_0^t n^2 L_n^m \langle \pi_s^n, G_s\rangle ds$ to the term inside the supremum in (5.26) and recalling (3.3), we can bound the probability in (5.26) from above by

$$\begin{aligned}\mathbb{P}_{\mu_n}\Bigg(\sup_{0\le t\le T}\Bigg|\int_0^t n^2 L_n^m\langle\pi_s^n, G_s\rangle ds + \int_0^t\int_{\varepsilon\frac{m}{2}}^{1-\varepsilon\frac{m}{2}}\prod_{i=0}^{\frac{m}{2}-1}\langle\pi_s^n, \overleftarrow{\iota}_{\varepsilon}^{u-i\varepsilon}\rangle\prod_{i=0}^{\frac{m}{2}-1}\langle\pi_s^n, \overrightarrow{\iota}_{\varepsilon}^{u+i\varepsilon}\rangle\Delta G_s(u)\,du\,ds\\ + \int_0^t\Big\{\beta^m\partial_u G_s(1)-\alpha^m\partial_u G_s(0)\Big\}\,ds\Bigg| > \tfrac{\delta}{6}\Bigg),\end{aligned} \tag{5.27}$$

plus $\mathbb{P}_{\mu_n}\left(\sup_{0\le t\le T}|M_t^n(G)| > \frac{\delta}{6}\right)$, which we showed above that vanishes when $n\to\infty$ without using the fact that $\theta=1$. From (5.27) and following again the steps of the case $\theta=1$, we need to bound the next terms

$$\begin{aligned}\mathbb{P}_{\mu_n}\Bigg(\sup_{0\le t\le T}\Bigg|\int_0^t\Bigg\{\frac{1}{n}\sum_{x\in\Sigma_{n,m}^{\varepsilon}}\Delta_n G_s\left(\tfrac{x}{n}\right)\tau_x h^m(\eta_{sn^2})\\ -\int_{\varepsilon\frac{m}{2}}^{1-\varepsilon\frac{m}{2}}\prod_{i=0}^{\frac{m}{2}-1}\langle\pi_s^n, \overleftarrow{\iota}_{\varepsilon}^{u-i\varepsilon}\rangle\prod_{i=0}^{\frac{m}{2}-1}\langle\pi_s^n, \overrightarrow{\iota}_{\varepsilon}^{u+i\varepsilon}\rangle\Delta G_s(u)\,du\Bigg\}\,ds\Bigg| > \tilde{\delta}\Bigg),\end{aligned} \tag{5.28}$$

$$\mathbb{P}_{\mu_n}\Bigg(\sup_{0\le t\le T}\Bigg|\kappa\tfrac{n}{n^\theta}\int_0^t G_s\left(\tfrac{1}{n}\right)\big(\alpha-\eta_{sn^2}(1)\big)+G_s\left(\tfrac{n-1}{n}\right)\big(\beta-\eta_{sn^2}(n-1)\big)\,ds\Bigg| > \tilde{\delta}\Bigg), \tag{5.29}$$

$$\mathbb{P}_{\mu_n}\Bigg(\sup_{0\le t\le T}\Bigg|\int_0^t \nabla_n^+ G_s(0)\tau_1 h^m(\eta_{sn^2})-\alpha^m\partial_u G_s(0)\,ds\Bigg| > \tilde{\delta}\Bigg), \tag{5.30}$$

plus another term similar to the last one which comes from the right boundary. Note that from the previous computations done for (5.22), we have that (5.28) vanishes, as $n\to\infty$ and $\varepsilon\to 0$. Moreover (5.29) vanishes, since from Lemma 6.2.1 we can replace $\eta_{sn^2}(1)$ by α and $\eta_{sn^2}(n-1)$ by β. Now, let us treat (5.30). Using Taylor expansion, we can replace $\partial_u G_s(0)$ by its discrete derivative $\nabla_n^+ G_s(0)$ and we get

$$\mathbb{P}_{\mu_n}\Bigg(\sup_{0\le t\le T}\Bigg|\int_0^t \nabla_n^+ G_s(0)\left(\tau_1 h^m(\eta_{sn^2})-\alpha^m\right)\,ds\Bigg| > \tilde{\delta}\Bigg).$$

From (5.1), we can rewrite the previous expression as

$$\mathbb{P}_{\mu_n}\Bigg(\sup_{0\le t\le T}\Bigg|\int_0^t \nabla_n^+ G_s(0)\left(\eta_{sn^2}(1)-\alpha\right)R_m^\alpha(\eta_{sn^2})\,ds\Bigg| > \tilde{\delta}\Bigg),$$

where $\mathcal{R}_m^\alpha(\eta)=\sum_{i=0}^{m-1}\alpha^{m-1-i}\prod_{j=0}^{i-1}\eta(2+j)$. Thus, since $\mathcal{R}_m^\alpha(\eta_{sn^2})\le m$, from the previous expression and Lemma 6.2.1, (5.30) vanishes when $n\to\infty$ and $\varepsilon\to 0$, concluding the result. □

Chapter 6

Replacement Lemmas

Recall that in Chapter 3 we presented a heuristic argument to derive the porous medium equation with different boundary conditions from the PMM with slow reservoirs, with $m = 2$. There, in order to see that the density profile $\rho_t(\cdot)$ was a weak solution of the corresponding hydrodynamic equation, we had to use some replacements, e.g., we replaced $\tau_x h^2(\eta_{sn^2})$ by $\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x)\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1)$ and α by $\eta_{sn^2}(1)$. This chapter aims to present a rigorous proof of these results.

We start this chapter by noticing one important property of the PMM, which is to be a *gradient system*. This means that the instantaneous current of the system at the bulk can be written as a discrete gradient of some local function of the dynamics, that is, $j^m_{x,x+1}(\eta) = \tau_{x+1}h^m(\eta) - \tau_x h^m(\eta)$, see (2.9) and (2.10). This function h^m is a degree m function, i.e., it is a function given by sums of terms of the form $\eta(x)\eta(x+1)\cdots\eta(x+m)$. Due to this fact, one needs a replacement lemma in the whole bulk which allows to replace

$$\prod_{i=0}^{\frac{m}{2}-1}\eta(x-i)\prod_{i=0}^{\frac{m}{2}-1}\eta(x+1+i) \text{ by } \prod_{i=0}^{\frac{m}{2}-1}\overleftarrow{\eta}^{\varepsilon n}(x-i\varepsilon n)\prod_{i=0}^{\frac{m}{2}-1}\overrightarrow{\eta}^{\varepsilon n}(x+1+i\varepsilon n),$$

for every $x \in \Sigma^{\varepsilon}_{n,m}$, as stated in Theorem 6.1.1. The idea of the argument of the proof is the following. First, we replace our general measure μ_n (which satisfies (2.20)) by a reference measure $\nu^n_{\rho(\cdot)}$ that is Bernoulli product and is defined in (4.12). Depending on the range of the parameter θ, some conditions will have to be imposed on the profile $\rho(\cdot)$. We note, however, that since we can control the entropy of μ_n with respect to this product measure, the choice on the type of profile does not impose any additional condition on the starting measure μ_n. Second, we make use of the Feynman-Kac's formula, and we have to control the error between the Dirichlet form of the process, defined in (4.7), and the integral of the *carré du champ* operator - denoted by D^m_n - defined in (4.8). We remark that D^m_n is the Dirichlet form that we would obtain if the reference measure is reversible with respect to the exchange and the Glauber dynamics. Since the reference measure that we consider below is not invariant for all these transformations, some errors appear which have to vanish in the limit, as stated in Lemma 4.2.

The idea to replace the products mentioned above consists of first removing the boundary points from the bulk, which do not allow these replacements; show that this removal is negligible in the limit, and on the remaining points, we do a step-by-step replacement. We start by presenting the argument for the function $\tau_x h^2(\eta)$, in which we need to replace terms of the form $\eta(x)\eta(y)$ for $|y-x| \leq 2$ by $\overleftarrow{\eta}^{\varepsilon n}(x)\overrightarrow{\eta}^{\varepsilon n}(y)$, as stated in Theorem 6.1.2. We do so in the following fashion: at first step fix one of the variables $\eta(x)$ and do the replacement of $\eta(x+1)$ by $\overrightarrow{\eta}^{\varepsilon n}(x+1)$. Then, fix this average and repeat the previous replacement but now for the variable $\eta(x)$ and $\overleftarrow{\eta}^{\varepsilon n}(x)$; this left-right argument is crucial so that the two boxes do not overlap and variables do not correlate, see Figure 6.1. The argument for a more general product of variables follows the same idea as the aforementioned case, the only difference is the fact that, for example, for $m = 3$, the function $\tau_x h^3(\eta)$ contains terms of the form $\eta(x)\eta(x+1)\eta(x+2)$. In order to replace them by products of averages, one first has to replace in the product above $\eta(x)$ by $\eta(x-\varepsilon n)$, this is done in another replacement, then follow the proof for the case $m = 2$ to replace $\eta(x+1)\eta(x+2)$ by $\overleftarrow{\eta}^{\varepsilon n}(x+1)\overrightarrow{\eta}^{\varepsilon n}(x+2)$, and finally fix these averages to replace $\eta(x-\varepsilon n)$ by $\overleftarrow{\eta}^{\varepsilon n}(x-\varepsilon n)$. We will explain all steps of these replacement lemmas in Section 6.1. We note that when doing all these replacements one has to use the arguments described above, in which we need to create a mobile cluster capable of making particles move. Due to the reservoir's action, we also have to control the terms that arise at the boundary and we need to derive a couple of replacements to deal with these extra terms, this in done in Section 6.2.

This chapter is divided into two sections in which we state and prove all the replacement lemmas used in Chapters 3, 5, 7 and 8. Section 6.1 aims to prove all the replacement lemmas concerning the bulk and Section 6.2 concerning the boundary.

6.1 Replacement lemmas at the bulk

As we mentioned above, one needs a replacement lemma in the whole bulk which allows writing $\tau_x h^m(\eta)$ in terms of products of averages of particles around a box of size $O(\varepsilon n)$. We remark that the sites $x \in \Sigma_n \setminus \Sigma^{\varepsilon}_{n,m}$, where $\Sigma^{\varepsilon}_{n,m}$ is defined in (5.5), are the ones where we do not have space to do the replacement and are those where we do not need to make the replacement. We stress that throughout this and the next section we will extensively use the non-cooperativity of the PMM by means of a path argument.

Theorem 6.1.1. *Let $G^n_s : [0,1] \to \mathbb{R}$ be such that $\|G^n_s\|_\infty \leq M < \infty$, for all $n \in \mathbb{N}$ and $s \in [0,T]$. For any $t \in [0,T]$, we have that*

$$\lim_{\varepsilon\to 0}\varlimsup_{n\to\infty} \mathbb{E}_{\mu_n}\left[\left|\int_0^t \frac{1}{n}\sum_{x\in\Sigma^{\varepsilon}_{n,m}} G^n_s\left(\tfrac{x}{n}\right)\left\{\tau_x h^m(\eta_{sn^2}) - \prod_{i=0}^{\frac{m}{2}-1} \overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x-i\varepsilon n)\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1+i\varepsilon n)\right\} ds\right|\right] = 0.$$

The previous theorem is the main result of this section, and it holds for every $m \geq 2$. We will divide its proof into two steps. First, we present the proof for a weaker version of Theorem 6.1.1 considering the particular case $m = 2$, i.e., replacing products of the form $\eta(x)\eta(x+1)$ by $\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x)\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1)$ using the PMM dynamics with infinitesimal generator L^m_n (for a general m), as stated in Theorem 6.1.2. Second, we present the proof of Theorem 6.1.1 in its general form, i.e., for every $m \geq 2$. Actually, the general case can be proved by combining the arguments described in Theorem 6.1.2 and the ones described in Theorem 6.2.3, that we will explain at the end of this section. Recall $\Sigma_{n,2}^{\varepsilon} = \{1+\varepsilon n, \dots, n-1-\varepsilon n\}$, the convention in (2.8), and that for every $x \in \Sigma_n$

$$\tau_x h^2(\eta) = \eta(x-1)\eta(x) + \eta(x)\eta(x+1) - \eta(x-1)\eta(x+1).$$

Since the previous function contains terms of the form $\eta(x)\eta(x+1)$, we now focus in replacing them by products of averages, and this is the content of the next theorem.

Theorem 6.1.2. *Let $G_s^n : [0,1] \to \mathbb{R}$ be such that $\|G_s^n\|_\infty \leq M < \infty$, for all $n \in \mathbb{N}$ and $s \in [0,T]$. For any $t \in [0,T]$, we have that*

$$\lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \mathbb{E}_{\mu_n} \left[\left| \int_0^t \frac{1}{n} \sum_{x \in \Sigma_{n,2}^{\varepsilon}} G_s^n\left(\tfrac{x}{n}\right) \left\{ \eta_{sn^2}(x)\eta_{sn^2}(x+1) - \overleftarrow{\eta}_{sn^2}^{\varepsilon n}(x) \overrightarrow{\eta}_{sn^2}^{\varepsilon n}(x+1) \right\} ds \right| \right] = 0.$$

In order to simplify the presentation of the previous theorem's proof, we divided it into three steps as described below and illustrated in Figure 6.1. For $x \in \Sigma_{n,2}^{\varepsilon}$, $a \in (1,2)$, $\varepsilon > 0$, and $\delta > 0$ such that $a-1-\delta \geq 0$:

1) replace $\eta(x)\eta(x+1)$ by $\overleftarrow{\eta}^{\ell}(x)\eta(x+1)$, for $\ell = n^{a-1-\delta}$; (Lemma 6.1.3)
2) replace $\overleftarrow{\eta}^{\ell}(x)\eta(x+1)$ by $\overleftarrow{\eta}^{\ell}(x)\overrightarrow{\eta}^{\varepsilon n}(x+1)$, for $\ell = n^{a-1-\delta}$; (Lemma 6.1.7)
3) replace $\overleftarrow{\eta}^{\ell}(x)\overrightarrow{\eta}^{\varepsilon n}(x+1)$ by $\overleftarrow{\eta}^{L}(x)\overrightarrow{\eta}^{\varepsilon n}(x+1)$, for $\ell = n^{a-1-\delta}$ and $L = \varepsilon n$. (Lemma 6.1.9)

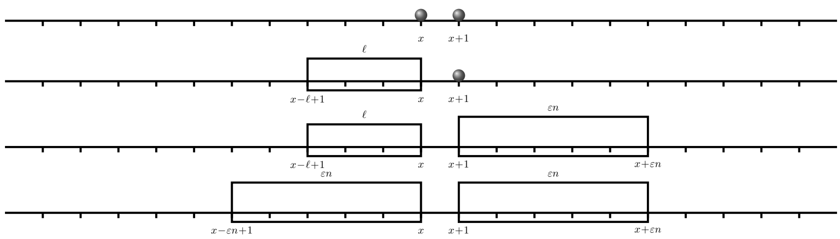


Figure 6.1: Steps to prove Theorem 6.1.2.

The steps to prove Theorem 6.1.2 are intentionally divided in this way because the proof of the theorem is based on the path argument that we will explain below. The idea is the following: since we want to replace $\eta(x)\eta(x+1)$ by $\overleftarrow{\eta}_{sn^2}^{\varepsilon n}(x)\overrightarrow{\eta}_{sn^2}^{\varepsilon n}(x+1)$ one could think that this could be accomplished easily once we can freely move particles using the SSEP dynamics. However, since their jumps are in a time scale less than the diffusive one particles can not travel to sites at a distance of order $O(\varepsilon n)$. To overcome this problem, we could try to use the PMM dynamics since now particles can travel at a distance of order $O(\varepsilon n)$. The problem is that when using only the PMM dynamics we can have locally blocked configurations so that we can not accomplish this all the time. Thus, the strategy is the following: we first create a finite size box around the jumping particle (Lemma 6.1.3) so that we create a mobile cluster inside this box. Now, since we have this box, we can use the path argument to do the replacement described in Lemma 6.1.7 and then in Lemma 6.1.9.

Lemma 6.1.3. *Let $G_s^n : [0,1] \to \mathbb{R}$ be such that $\|G_s^n\|_\infty \le M < \infty$, for all $n \in \mathbb{N}$ and $s \in [0,T]$. For any $t \in [0,T]$, $\varepsilon > 0$ and $\ell = n^{a-1-\delta}$ with $\delta > 0$ such that $a - 1 - \delta \ge 0$, we have that*

$$\varlimsup_{n\to\infty} \mathbb{E}_{\mu_n}\left[\left|\int_0^t \frac{1}{n}\sum_{x\in\Sigma_{n,2}^\varepsilon} G_s^n\left(\tfrac{x}{n}\right)\{\eta_{sn^2}(x) - \overleftarrow{\eta}^{\ell}_{sn^2}(x)\}\eta_{sn^2}(x+1)\,ds\right|\right] = 0. \tag{6.1}$$

Proof. Note that the expectation in the statement of the lemma can be written as

$$\frac{1}{n}\int_{\Omega_n} \mathbb{E}_\eta\left[n\left|\int_0^t \frac{1}{n}\sum_{x\in\Sigma_{n,2}^\varepsilon} G_s^n\left(\tfrac{x}{n}\right)\{\eta_{sn^2}(x) - \overleftarrow{\eta}^{\ell}_{sn^2}(x)\}\eta_{sn^2}(x+1)\,ds\right|\right] d\mu_n.$$

Since we do not have enough information about the measure μ_n, except the fact that it is associated with a profile (2.20), we want to change this measure to the Bernoulli product measure $\nu^n_{\rho(\cdot)}$, where $\rho(\cdot)$ satisfies (4.13). Since Ω_n is a countable state space, the entropy of μ_n with respect to $\nu^n_{\rho(\cdot)}$ (see [31] for more details) can be defined as $H\left(\mu_n|\nu^n_{\rho(\cdot)}\right) = \sum_{\eta\in\Omega_n}\mu_n(\eta)\log\left(\frac{\mu_n(\eta)}{\nu^n_{\rho(\cdot)}(\eta)}\right)$. By entropy inequality, the previous expression is bounded from above, for any $B > 0$, by $\frac{H\left(\mu_n|\nu^n_{\rho(\cdot)}\right)}{nB}$ plus

$$\frac{1}{nB}\log\int_{\Omega_n}\exp\left\{nB\,\mathbb{E}_\eta\left[\left|\int_0^t \frac{1}{n}\sum_{x\in\Sigma_{n,2}^\varepsilon} G_s^n\left(\tfrac{x}{n}\right)\{\eta_{sn^2}(x) - \overleftarrow{\eta}^{\ell}_{sn^2}(x)\}\eta_{sn^2}(x+1)\,ds\right|\right]\right\} d\nu^n_{\rho(\cdot)}.$$

Now, using Jensen's inequality, we can bound (6.1) from above by $\frac{H\left(\mu_n|\nu^n_{\rho(\cdot)}\right)}{nB}$ plus

$$\frac{1}{nB}\log\mathbb{E}_{\nu^n_{\rho(\cdot)}}\left[\exp\left\{nB\left|\int_0^t \frac{1}{n}\sum_{x\in\Sigma_{n,2}^\varepsilon} G_s^n\left(\tfrac{x}{n}\right)\{\eta_{sn^2}(x) - \overleftarrow{\eta}^{\ell}_{sn^2}(x)\}\eta_{sn^2}(x+1)\,ds\right|\right\}\right]. \tag{6.2}$$

Since $\rho(\cdot)$ satisfies (4.13), from Lemma A.0.2 we have that

$$H\left(\mu_n|\nu^n_{\rho(\cdot)}\right) \le \log\left(\frac{1}{(\alpha\wedge(1-\beta))^n}\right)\sum_{\eta\in\Omega_n}\mu_n(\eta) \le nC(\alpha,\beta). \tag{6.3}$$

Thus, we only need to treat the term in (6.2), since the term $\frac{C(\alpha,\beta)}{B}$ vanishes in the end when we take $B \to \infty$. From Feynman-Kac's formula (see, for example, Lemma A.1 of [2]), (6.2) is bounded from above by

$$\int_0^t \sup_f\left(\left|\int_{\Omega_n}\frac{1}{n}\sum_{x\in\Sigma_{n,2}^\varepsilon} G_s^n\left(\tfrac{x}{n}\right)\{\eta(x) - \overleftarrow{\eta}^{\ell}(x)\}\eta(x+1)f(\eta)\,d\nu^n_{\rho(\cdot)}\right| + \frac{n}{B}\langle L_n^m\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}\right)ds,$$

where the supremum is carried over all densities f with respect to $\nu^n_{\rho(\cdot)}$. Let us now examine

$$\left|\int_{\Omega_n}\frac{1}{n}\sum_{x\in\Sigma_{n,2}^\varepsilon} G_s^n\left(\tfrac{x}{n}\right)\{\eta(x) - \overleftarrow{\eta}^{\ell}(x)\}\eta(x+1)f(\eta)\,d\nu^n_{\rho(\cdot)}\right|. \tag{6.4}$$

Note that $\eta(x) - \overleftarrow{\eta}^{\ell}(x) = \frac{1}{\ell}\sum_{y\in\overleftarrow{\Lambda}_x^{\ell}}\eta(x) - \eta(y)$, and that $\eta(x) - \eta(y) = \sum_{z=y}^{x-1}\eta(z+1) - \eta(z)$. Hence, by summing and subtracting the term $\frac{1}{2}f(\eta^{z,z+1})$ and using the hypothesis on G, we can bound (6.4) from above by

$$
\begin{aligned}
&\frac{M}{2\ell n}\sum_{x\in\Sigma^{\varepsilon}_{n,2}}\sum_{y\in\overleftarrow{\Lambda}^{\ell}_x}\sum_{z=y}^{x-1}\Bigg|\int_{\Omega_n}\big(\eta(z+1)-\eta(z)\big)\eta(x+1)\big(f(\eta)+f(\eta^{z,z+1})\big)\,d\nu^n_{\rho(\cdot)}\Bigg|\\
&+\frac{M}{2\ell n}\sum_{x\in\Sigma^{\varepsilon}_{n,2}}\sum_{y\in\overleftarrow{\Lambda}^{\ell}_x}\sum_{z=y}^{x-1}\Bigg|\int_{\Omega_n}\big(\eta(z+1)-\eta(z)\big)\eta(x+1)\big(f(\eta)-f(\eta^{z,z+1})\big)\,d\nu^n_{\rho(\cdot)}\Bigg|.
\end{aligned}
\tag{6.5}
$$

Let $\bar\eta$ denote the configuration η removing its value at the sites z and $z+1$. Thus, we can write the first integral in (6.5) as

$$
\begin{aligned}
&\Bigg|\sum_{\bar\eta\in\Omega_{n-2}}\Bigg(\bar\eta(x+1)\big(f(\bar\eta,0,1)+f(\bar\eta,1,0)\big)\left(1-\rho\left(\tfrac{z}{n}\right)\right)\rho\left(\tfrac{z+1}{n}\right)\\
&\quad-\bar\eta(x+1)\big(f(\bar\eta,0,1)+f(\bar\eta,1,0)\big)\rho\left(\tfrac{z}{n}\right)\left(1-\rho\left(\tfrac{z+1}{n}\right)\right)\Bigg)\nu^{n-2}_{\rho(\cdot)}(\bar\eta)\Bigg|,
\end{aligned}
\tag{6.6}
$$

where the notation $f(\bar\eta,1,0)$ means that we are computing $f(\eta)$ with $\eta(z)=1$ and $\eta(z+1)=0$. Using the fact that $\rho(\cdot)$ satisfies the hypothesis of Lemma 4.2.1 so that (4.14) holds, (6.6) is bounded from above by a constant (depending on $\rho(\cdot)$) times

$$
\frac{1}{n}\sum_{\bar\eta\in\Omega_{n-2}}\big(f(\bar\eta,0,1)+f(\bar\eta,1,0)\big)\nu^{n-2}_{\rho(\cdot)}(\bar\eta).
$$

Since last term is bounded from above by

$$
\frac{2}{n}\sum_{z\in\{0,1\}}\sum_{\eta\in\Omega_n}f(\eta)\Big(\prod_{y=z,z+1}\rho\left(\tfrac{y}{n}\right)^{\eta(y)}\left(1-\rho\left(\tfrac{y}{n}\right)\right)^{1-\eta(y)}\Big)^{-1}\nu^n_{\rho(\cdot)}(\eta)
$$

and f is a density with respect to $\nu^n_{\rho(\cdot)}$, (6.6) is of order $O(\frac{1}{n})$. Thus, the first line in (6.5) is bounded from above by a constant, times $\frac{\ell}{n}$. It remains to treat the second line of (6.5). Note that for two nonnegative numbers a and b, $a-b=[\sqrt{a}-\sqrt{b}][\sqrt{a}+\sqrt{b}]$. Then, from Young's inequality we have that for any $A>0$ the absolute value of the integral in the second line of (6.5) is bounded from above by

$$
\begin{aligned}
&\frac{M}{4n\ell A}\sum_{x\in\Sigma^{\varepsilon}_{n,2}}\sum_{y\in\overleftarrow{\Lambda}^{\ell}_x}\sum_{z=y}^{x-1}\Bigg|\int_{\Omega_n}\big(\eta(z+1)-\eta(z)\big)^2\eta(x+1)^2\left(\sqrt{f(\eta)}+\sqrt{f(\eta^{z,z+1})}\right)^2d\nu^n_{\rho(\cdot)}\Bigg|\\
&+\frac{MA}{4n\ell}\sum_{x\in\Sigma^{\varepsilon}_{n,2}}\sum_{y\in\overleftarrow{\Lambda}^{\ell}_x}\sum_{z=y}^{x-1}\Bigg|\int_{\Omega_n}\left(\sqrt{f(\eta)}-\sqrt{f(\eta^{z,z+1})}\right)^2d\nu^n_{\rho(\cdot)}\Bigg|.
\end{aligned}
\tag{6.7}
$$

Now, since f is a density with respect to $\nu^n_{\rho(\cdot)}$, $|\eta(x)|\le 1$ for $x\in\Sigma_n$, and $(a+b)^2\le 2a^2+2b^2$, the first line of the previous display is bounded from above by $\frac{M\ell}{A}$. Furthermore, recalling the definition of (4.10), we can bound (6.7) from above by

$$
\frac{M\ell}{A}+\frac{MA}{4}D_S(\sqrt{f},\nu^n_{\rho(\cdot)})\,. \tag{6.8}
$$

Now, recall from (4.14) that

$$\langle L_n^m\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}} \;\leq -\frac{n^{a-2}}{4}D_S\Big(\sqrt{f},\nu^n_{\rho(\cdot)}\Big)+O(\tfrac{1}{n}).$$

Taking $A=\frac{n^{a-1}}{BM}$ in (6.8), from last inequality and the previous computations, the expectation in the statement of the lemma is bounded from above by a constant, times

$$\frac{1}{B}+T\left(\frac{\ell}{n}+\frac{B\ell}{n^{a-1}}\right). \tag{6.9}$$

Therefore, from our choice of ℓ, taking $n\to\infty$ and then $B\to\infty$ in (6.9), the proof ends. □

Remark 6.1.4. *We stress that, in the proof above and the ones below, we present the replacement lemmas using the Bernoulli product measure $\nu^n_{\rho(\cdot)}$ and asking $\rho(\cdot)$ to satisfy the conditions stated in the first part of Lemma 4.2.1. Nevertheless, in the case $\theta\geq 1$, it is enough to consider the constant profile $\rho(\cdot)$, due to the bound obtained in* (4.15)*.*

Remark 6.1.5. *We observe that the restriction imposed above Remark 2.1, that the parameters $\alpha,\beta\in(0,1)$, comes from the estimate in* (6.3)*. Since, as mentioned above, in the case $\theta\geq 1$ we can take any constant profile, that restriction on the parameters is only needed in Dirichlet case, that is, when $\theta<1$.*

Remark 6.1.6. *A simple modification of the proof of Lemma 6.1.3 also shows that, for all $\varepsilon>0$*

$$\varlimsup_{n\to\infty}\mathbb{E}_{\mu_n}\left[\left|\int_0^t\frac{1}{n}\sum_{x\in\Sigma^\varepsilon_{n,2}}G^n_s\left(\tfrac{x}{n}\right)\eta_{sn^2}(x-1)\big\{\eta_{sn^2}(x)-\overrightarrow{\eta}^{\ell}_{sn^2}(x)\big\}\,ds\right|\right]=0.$$

Lemma 6.1.7. *Let $G^n_s:[0,1]\to\mathbb{R}$ be such that $\|G^n_s\|_\infty\leq M<\infty$, for all $n\in\mathbb{N}$ and $s\in[0,T]$. For any $t\in[0,T]$ and $\ell=n^{a-1-\delta}$ with $\delta>0$ such that $a-1-\delta\geq 0$, we have that*

$$\lim_{\varepsilon\to 0}\varlimsup_{n\to\infty}\mathbb{E}_{\mu_n}\left[\left|\int_0^t\frac{1}{n}\sum_{x\in\Sigma^\varepsilon_{n,2}}G^n_s\left(\tfrac{x}{n}\right)\overleftarrow{\eta}^{\ell}_{sn^2}(x)\big\{\eta_{sn^2}(x+1)-\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1)\big\}\,ds\right|\right]=0. \tag{6.10}$$

Proof. Following the same steps as we did in the beginning of the previous lemma, for $B>0$, we can bound the expectation in (6.10) from above by $\frac{C(\alpha,\beta)}{B}$, plus

$$\int_0^t\sup_f\left(\left|\int_{\Omega_n}\frac{1}{n}\sum_{x\in\Sigma^\varepsilon_{n,2}}G^n_s\left(\tfrac{x}{n}\right)\overleftarrow{\eta}^{\ell}(x)\big\{\eta(x+1)-\overrightarrow{\eta}^{\varepsilon n}(x+1)\big\}f(\eta)\,d\nu^n_{\rho(\cdot)}\right|+\frac{n}{B}\langle L^m_n\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}\right)ds.$$

Now, we need to examine the term

$$\left|\int_{\Omega_n}\frac{1}{n}\sum_{x\in\Sigma^\varepsilon_{n,2}}G^n_s\left(\tfrac{x}{n}\right)\overleftarrow{\eta}^{\ell}(x)\big\{\eta(x+1)-\overrightarrow{\eta}^{\varepsilon n}(x+1)\big\}f(\eta)\,d\nu^n_{\rho(\cdot)}\right|. \tag{6.11}$$

Note that although the expression above is similar to (6.4), we can not use the same argument to treat it. The main difference here is that we can not estimate expressions like (6.7) by only using the SSEP dynamics. In order to overcome this, we will define a set of configurations in which we can combine both the SSEP jumps and the PMM jumps to estimate the aforementioned expression, as we will explain below.

Recall the definition of $\overleftarrow{\Lambda}_x^\ell$ in (5.3). Denote by $X_1 = \{\eta \in \Omega_n : \overleftarrow{\eta}^\ell(x) \geq \frac{m}{\ell}\}$ the set of configurations that have at least m particles in $\overleftarrow{\Lambda}_x^\ell$. Thus, we can write (6.11) as the sum of the integral over the set X_1, plus the integral over its complement X_1^c. By the hypothesis on G, the fact that $|\eta(x)| \leq 1$ for $x \in \Sigma_n$, and since f is a density with respect to $\nu_{\rho(\cdot)}^n$, the absolute value of the integral over X_1^c is bounded from above by a constant, times $\frac{1}{\ell}$. By the hypothesis on ℓ, the integral over X_1^c vanishes as n goes to infinity. Hence, since $\eta(x+1) - \overrightarrow{\eta}^{\varepsilon n}(x+1) = \frac{1}{\varepsilon n}\sum_{y \in \overrightarrow{\Lambda}_{x+1}^{\varepsilon n}} \eta(x+1) - \eta(y)$, we can bound (6.11) from above by

$$\left| \frac{1}{n^2\varepsilon} \sum_{x \in \Sigma_{n,2}^\varepsilon} \sum_{y \in \overrightarrow{\Lambda}_{x+1}^{\varepsilon n}} \int_{X_1} G_s^n\left(\tfrac{x}{n}\right) \overleftarrow{\eta}^\ell(x) \{\eta(x+1) - \eta(y)\} f(\eta)\, d\nu_{\rho(\cdot)}^n \right|.$$

From the hypothesis on G and by summing and subtracting the term $\frac{1}{2}f(\eta^{x+1,y})$ in the previous expression, we can bound from above the last expression by

$$\begin{aligned} &\frac{M}{2n^2\varepsilon} \sum_{x \in \Sigma_{n,2}^\varepsilon} \sum_{y \in \overrightarrow{\Lambda}_{x+1}^{\varepsilon n}} \left| \int_{X_1} \overleftarrow{\eta}^\ell(x) \{\eta(x+1) - \eta(y)\} \big(f(\eta) + f(\eta^{x+1,y})\big)\, d\nu_{\rho(\cdot)}^n \right| \\ +&\frac{M}{2n^2\varepsilon} \sum_{x \in \Sigma_{n,2}^\varepsilon} \sum_{y \in \overrightarrow{\Lambda}_{x+1}^{\varepsilon n}} \left| \int_{X_1} \overleftarrow{\eta}^\ell(x) \{\eta(x+1) - \eta(y)\} \big(f(\eta) - f(\eta^{x+1,y})\big)\, d\nu_{\rho(\cdot)}^n \right|. \end{aligned} \tag{6.12}$$

We begin by estimating the first line in the previous display. We use the notation $\bar{\eta}$ for the configuration η removing its value at the sites $x+1$ and y. Since $x+1$ and y do not intersect $\overleftarrow{\Lambda}_x^\ell$, the term inside the absolute value in the first equation in (6.12), can be written as

$$\left| \sum_{\bar{\eta} \in \Omega_{n-2}} \mathbb{1}_{\bar{\eta} \in X_1} \overleftarrow{\eta}^\ell(x) \left\{ \Big(f(\bar{\eta},0,1) + f(\bar{\eta},1,0)\Big) \Big(\left(1 - \rho\left(\tfrac{y}{n}\right)\right) \rho\left(\tfrac{x+1}{n}\right) - \rho\left(\tfrac{y}{n}\right)\left(1 - \rho\left(\tfrac{x+1}{n}\right)\right) \Big) \right\} \nu_{\rho(\cdot)}^{n-2}(\bar{\eta}) \right|.$$

Using the fact that $\rho(\cdot)$ satisfies the hypothesis of Lemma 4.2.1 so that (4.14) holds, the first term in (6.12) can be bounded from above by a constant, times

$$\frac{M}{2n^2\varepsilon} \sum_{x \in \Sigma_{n,2}^\varepsilon} \sum_{y \in \overrightarrow{\Lambda}_{x+1}^{\varepsilon n}} \left| \tfrac{x+1-y}{n} \right|,$$

which is of order $O(\varepsilon)$. To bound the second line in (6.12) we need to be careful. Recall that the idea behind this lemma is to replace a particle at the site $x+1$ by the empirical density in the box $\overrightarrow{\Lambda}_{x+1}^{\varepsilon n}$. To accomplish this we have to construct a path (with allowed jumps from the SSEP and the PMM dynamics), in such a way that we can send a particle from the site $x+1$ to the site y, for any $y \in \overrightarrow{\Lambda}_{x+1}^{\varepsilon n}$. This path depends on the creation of a *mobile cluster*. Thus, to present the argument in a more didactic way, will explain in the next paragraph how to construct the aforementioned path by using the SSEP jumps and the PMM jumps restricted to the case $m = 2$, and on Remark 6.1.8 we present what changes in the general case.

Recall that in the case $m = 2$ we are integrating over $X_1 = \{\eta \in \Omega_n : \overleftarrow{\eta}^{\ell}(x) \geq \frac{2}{\ell}\}$, so that we have at least two particles in $\overleftarrow{\Lambda}_x^{\ell}$. Suppose, without loss of generality, that we have a particle at site $x_1 \in \overleftarrow{\Lambda}_x^{\ell}$, and another one at site $x_2 \in \overleftarrow{\Lambda}_x^{\ell}$, with $x_1 < x_2$. Using the SSEP jumps, we can take the particle from the site x_1 close to the particle at the site x_2, in such a way that the distance between them is less than or equal to 2. Denoting by $\bullet$ an occupied site and by $\circ$ an empty site, this approximation is done by the SSEP jumps and at the end we get one of the following structures ($\bullet\ \circ\ \bullet$ or $\bullet\ \bullet\ \circ$). When we reach a structure of the previous form, we say that a *mobile cluster* has been created. Now, since we have a mobile cluster, there exists a sequence of nearest-neighbor jumps (with the PMM dynamics) which allow us to move the mobile cluster to any position on the box $\overrightarrow{\Lambda}_{x+1}^{\varepsilon n}$. Note that the SSEP jumps are used to approximate particles inside a box of size ℓ, with the choice of ℓ as in the statement of this lemma. However, the PMM jumps can be used in the presence of the mobile cluster, to take a particle from a site $x+1$ to a site y at a distance at most εn. After the creation of the mobile cluster with SSEP jumps, we move it to a vicinity of the site $x+1$ until the distance between them is less than or equal to 2. Then, using the PMM jumps we take a particle to the site y and we bring back the mobile cluster to the same position where it was created. When we reach this step, we use the SSEP jumps again to put the particles back to their initial positions, x_1 and x_2, respectively. To have a picture of all the steps mentioned above see Figure 6.2.

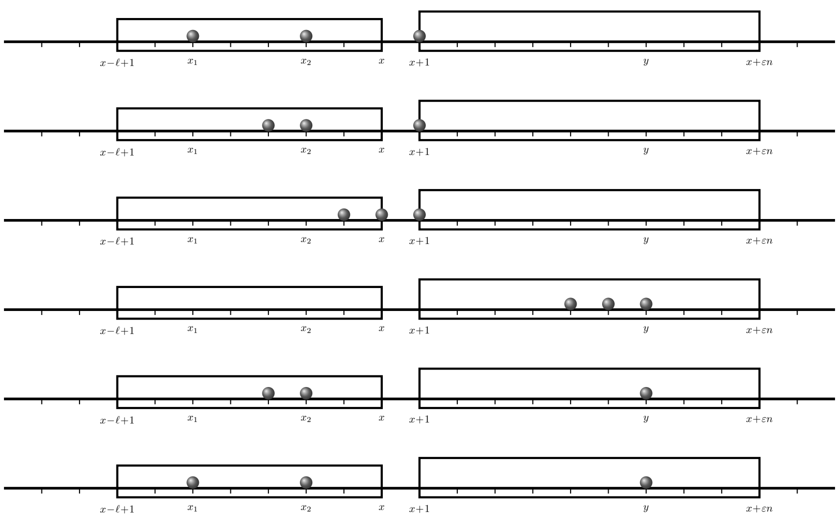


Figure 6.2: Path used to send a particle from site $x+1$ to y inside the box of size εn, combining SSEP jumps and PMM jumps (with $m = 2$).

Note that, in this path, we use at most 4ℓ jumps from the SSEP and $6(\ell + \varepsilon n)$ jumps from the PMM. From this, it follows that for any configuration $\eta \in X_1$, if x_1 and x_2 denote the position of the two closest particles to $x+1$, then there exist $N(x_1) \leq \ell + \varepsilon n$ and a sequence of allowed moves $\{x(i)\}_{i=0,\dots,N(x_1)}$, which takes values in the set of points $\{x_1, \dots, y\}$, such that $\eta^{(0)} = \eta, \eta^{(i+1)} = (\eta^{(i)})^{x(i),x(i)+1}$ and the final configuration is $\eta^{(N(x_1))} = \eta^{x+1,y}$. Note that the rates for each exchange is strictly positive. With this in mind, we can rewrite the exchange $f(\eta)-f(\eta^{x+1,y})$ as

$$f(\eta)-f(\eta^{x+1,y})=\sum_{i=1}^{N(x_1)}f(\eta^{(i-1)})-f(\eta^{(i)})=\sum_{i\in I^{\mathsf{exc}}}f(\eta^{(i-1)})-f(\eta^{(i)})+\sum_{i\in I^{\mathsf{pmm}}}f(\eta^{(i-1)})-f(\eta^{(i)}),\tag{6.13}$$

where I^{exc} (resp. I^{pmm}) are the sets of indexes that count the bonds used with SSEP jumps (resp. PMM jumps) along the path. Take into account the fact that the SSEP jumps are used only to create and to destroy the mobile cluster, while all the rest of the path is done with PMM jumps. Now, substituting (6.13) in the second line of (6.12) and using the triangular inequality, we need to estimate the following expressions

$$\begin{aligned}&\frac{M}{2n^2\varepsilon}\sum_{x\in\Sigma^{\varepsilon}_{n,2}}\sum_{y\in\overrightarrow{\Lambda}^{\varepsilon n}_{x+1}}\sum_{i\in I^{\mathsf{exc}}}\left|\int_{X_1}\overleftarrow{\eta}^{\ell}(x)\big(\eta(x+1)-\eta(y)\big)\big(f(\eta^{(i-1)})-f(\eta^{(i)})\,d\nu^n_{\rho(\cdot)}\right|\\&+\frac{M}{2n^2\varepsilon}\sum_{x\in\Sigma^{\varepsilon}_{n,2}}\sum_{y\in\overrightarrow{\Lambda}^{\varepsilon n}_{x+1}}\sum_{i\in I^{\mathsf{pmm}}}\left|\int_{X_1}\overleftarrow{\eta}^{\ell}(x)\big(\eta(x+1)-\eta(y)\big)\big(f(\eta^{(i-1)})-f(\eta^{(i)})\big)\,d\nu^n_{\rho(\cdot)}\right|.\end{aligned}\tag{6.14}$$

Since $\eta^{(i)}=(\eta^{(i-1)})^{x(i-1),x(i-1)+1}$, the way to estimate the first line above is the same as it is done in (6.7). For the sake of completeness, for each $A>0$, by Young's inequality, the definitions of $\overrightarrow{\Lambda}^{\varepsilon n}_{x+1}$ and $\Sigma^{\varepsilon}_{n,2}$ (see (5.5) and (5.3), respectively), the first sum of (6.14) is bounded from above by

$$\frac{1}{2A}\sum_{i\in I^{\mathsf{exc}}}\int_{X_1}\left(\sqrt{f(\eta^{(i-1)})}+\sqrt{f(\eta^{(i)})}\right)^2d\nu^n_{\rho(\cdot)}+\frac{A}{2}\sum_{i\in I^{\mathsf{exc}}}\int_{X_1}\left(\sqrt{f(\eta^{(i-1)})}-\sqrt{f(\eta^{(i)})}\right)^2d\nu^n_{\rho(\cdot)}\,.$$

Now, remember that the indexes in I^{exc} count the number of SSEP jumps to move the two particles that are in the box $\overleftarrow{\Lambda}^{\ell}_{x}$ (which exist due to the fact that the integral is over X_1) close to the bond $(x-1,x)$ and the path back to the initial position of these particles. Then, the first line in (6.14) is bounded from above by

$$\frac{2M\ell}{A}+\frac{AM}{4}D_S(\sqrt{f},\nu^n_{\rho(\cdot)}).$$

In order to estimate the second line in (6.14), we repeat the argument above using the PMM jump rates. Recall that $p^2_{x,x+1}(\eta)$ is introduced below (4.10). Thus, for all $\tilde{A}>0$, the sum involving I^{pmm} in (6.14) is bounded from above by

$$\begin{aligned}&\frac{1}{2\tilde{A}}\sum_{i\in I^{\mathsf{pmm}}}\int_{X_1}\frac{1}{p^2_{x(i-1),x(i-1)+1}(\eta)}\left(\sqrt{f(\eta^{(i-1)})}+\sqrt{f(\eta^{(i)})}\right)^2d\nu^n_{\rho(\cdot)}\\&+\frac{\tilde{A}}{2}\sum_{i\in I^{\mathsf{pmm}}}\int_{X_1}p^2_{x(i-1),x(i-1)+1}(\eta)\left(\sqrt{f(\eta^{(i-1)})}-\sqrt{f(\eta^{(i)})}\right)^2d\nu^n_{\rho(\cdot)}\,.\end{aligned}$$

Observe that for $\eta\in X_1$ and $i\in I^{\mathsf{pmm}}$, $p^2_{x(i-1),x(i-1)+1}(\eta)$ is either equal to 1 or 2. Therefore, the second line in (6.14) is bounded from above by

$$\frac{6M\varepsilon n}{\tilde{A}}+\frac{\tilde{A}M}{4}D^2_P(\sqrt{f},\nu^n_{\rho(\cdot)}).$$

Taking $A = \frac{n^{a-1}}{MB}$ and $\tilde{A} = \frac{n}{MB}$, from the previous computations, the expectation in the statement of the lemma is bounded from above by a constant, times

$$\frac{1}{B} + T\left(\frac{1}{\ell} + \varepsilon + \frac{\ell B}{n^{a-1}} + \varepsilon B\right). \tag{6.15}$$

Taking $n \to \infty$, the second and fourth term of (6.15) vanish by the choice of ℓ. Taking $\varepsilon \to 0$, the third and fifth terms of (6.15) vanish. To finish, we send $B \to \infty$ and the remaining term vanishes, concluding the proof. $\square$

Remark 6.1.8. *We decided to use the PMM with generator L_P^2 in the middle of the theorem above because it is the easiest way to illustrate the path argument's ideas. However, this argument can be generalized with some adaptations considering the PMM with generator L_P^m for $m > 2$, since as we increase the degree of m, the jump rates degree increase, and therefore the number of particles to create the mobile cluster. Indeed, in the case $m = 3$, the main changes in the argument will be the following: to estimate* (6.11) *we consider $X_1 = \left\{\eta \in \Omega_n : \overleftarrow{\eta}^{\ell}(x) \geq \frac{3}{\ell}\right\}$; the mobile cluster will have the forms (• • ◦ •) or (• ◦ • •) or (• • • ◦); and the path to move a particle from the site $x+1$ to the site y will have at most 6ℓ jumps from the SSEP, and $8(\ell + \varepsilon n)$ jumps from the PMM dynamics. Thus, for the general case $m \geq 2$, we will have: $X_1 = \left\{\eta \in \Omega_n : \overleftarrow{\eta}^{\ell}(x) \geq \frac{m}{\ell}\right\}$, a path with at most $2m\ell$ jumps from the SSEP, and $2(m+1)(\ell + \varepsilon n)$ jumps from the PMM dynamics, and the mobile clusters will have the following forms*

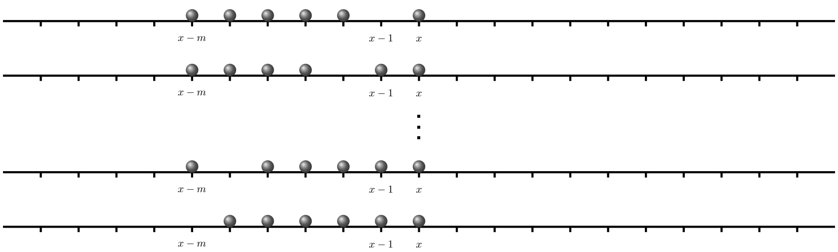


Figure 6.3: Possible mobile clusters used in the case $m \geq 2$.

Lemma 6.1.9. *Let $G_s^n : [0,1] \to \mathbb{R}$ be such that $\|G_s^n\|_\infty \leq M < \infty$, for all $n \in \mathbb{N}$ and $s \in [0,T]$. For any $t \in [0,T]$, $L = \varepsilon n$ and $\ell = n^{a-1-\delta}$ with $\delta > 0$ such that $a - 1 - \delta \geq 0$, we have that*

$$\lim_{\varepsilon \to 0} \varlimsup_{n \to \infty} \mathbb{E}_{\mu_n}\left[\left|\int_0^t \frac{1}{n} \sum_{x \in \Sigma_{n,2}^{\varepsilon}} G_s^n\left(\tfrac{x}{n}\right)\left\{\overleftarrow{\eta}^{\ell}_{sn^2}(x) - \overleftarrow{\eta}^{L}_{sn^2}(x)\right\}\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1)\, ds\right|\right] = 0. \tag{6.16}$$

Proof. The proof of this lemma is similar to one of Lemma 6.1.7, with some modifications due to the sizes of the boxes involved here. Again, we can bound the expectation in the statement of the lemma from above by $\frac{C(\alpha,\beta)}{B}$ plus

$$\int_0^t \sup_f \left(\left|\int_{\Omega_n} \frac{1}{n} \sum_{x \in \Sigma_{n,2}^{\varepsilon}} G_s^n\left(\tfrac{x}{n}\right)\left\{\overleftarrow{\eta}^{\ell}(x) - \overleftarrow{\eta}^{L}(x)\right\}\overrightarrow{\eta}^{\varepsilon n}(x+1) f(\eta)\, d\nu_{\rho(\cdot)}^n\right| + \frac{n}{B}\langle L_n^m \sqrt{f}, \sqrt{f}\rangle_{\nu_{\rho(\cdot)}^n}\right) ds, \tag{6.17}$$

for any $B > 0$, where f is a density with respect to $\nu^n_{\rho(\cdot)}$. Take $L = k\ell$ with $k = \frac{\varepsilon n}{\ell}$, and note that

$$\overleftarrow{\eta}^{\ell}(x) - \overleftarrow{\eta}^{L}(x) = \frac{1}{k}\sum_{j=1}^{k-1}\Big(\overleftarrow{\eta}^{\ell}(x) - \overleftarrow{\eta}^{\ell}(x - j\ell)\Big).$$

From the last identity and the hypothesis on G, to bound the first integral inside the supremum in (6.17), it is enough to estimate the term

$$\frac{M}{kn}\sum_{x\in\Sigma^{\varepsilon}_{n,2}}\sum_{j=1}^{k-1}\left|\int_{\Omega_n}\{\overleftarrow{\eta}^{\ell}(x) - \overleftarrow{\eta}^{\ell}(x - j\ell)\}\overrightarrow{\eta}^{\varepsilon n}(x+1)f(\eta)\,d\nu^n_{\rho(\cdot)}\right|.$$

For $j = 1, \dots, k-1$, let $X^j_2 = \{\eta \in \Omega_n : \overleftarrow{\eta}^{\ell}(x) \geq \frac{m+1}{\ell}\} \cup \{\eta \in \Omega_n : \overleftarrow{\eta}^{\ell}(x - j\ell) \geq \frac{m}{\ell}\}$. Thus, the integral in the previous display can be written as the integral over X^j_2 plus the integral over its complement $(X^j_2)^c$. We observe that the integral over $(X^j_2)^c$ is of order $O(\frac{1}{\ell})$. Since $\overleftarrow{\eta}^{\ell}(x) - \overleftarrow{\eta}^{\ell}(x-j\ell) = \frac{1}{\ell}\sum_{z\in\overleftarrow{\Lambda}^{\ell}_x}\eta(z) - \eta(z - j\ell)$, we can write the integral over X^j_2 as

$$\frac{M}{kn}\sum_{x\in\Sigma^{\varepsilon}_{n,2}}\sum_{j=1}^{k-1}\left|\int_{X^j_2}\frac{1}{\ell}\sum_{z\in\overleftarrow{\Lambda}^{\ell}_x}\big(\eta(z) - \eta(z - j\ell)\big)\overrightarrow{\eta}^{\varepsilon n}(x+1)f(\eta)\,d\nu^n_{\rho(\cdot)}\right|. \tag{6.18}$$

Basically the idea above is to send a particle $z \in \overleftarrow{\Lambda}^{\ell}_x$ to a site inside a box of size $j\ell$, given that we have at least $m+1$ particles in $\overleftarrow{\Lambda}^{\ell}_x$ or m particles in $\overleftarrow{\Lambda}^{\ell}_{x-j\ell}$. In Figure 6.4 we illustrate these two cases for $m = 3$. We stress that the path argument used here is the same used above to prove Lemma 6.1.7.

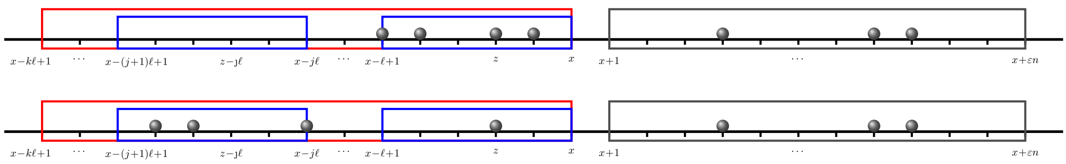


Figure 6.4: Two possible situations in which it is possible to send a particle from site z to $z - j\ell$. (with $m = 3$)

Summing and subtracting $\frac{1}{2}f(\eta^{z-j\ell,z})$ in (6.18), we get

$$\begin{aligned}&\frac{M}{2kn\ell}\sum_{x\in\Sigma^{\varepsilon}_{n,2}}\sum_{j=1}^{k-1}\sum_{z\in\overleftarrow{\Lambda}^{\ell}_x}\left|\int_{X^j_2}\big(\eta(z) - \eta(z-j\ell)\big)\overrightarrow{\eta}^{\varepsilon n}(x+1)\big(f(\eta) + f(\eta^{z-j\ell,z})\big)\,d\nu^n_{\rho(\cdot)}\right|\\ +&\frac{M}{2kn\ell}\sum_{x\in\Sigma^{\varepsilon}_{n}}\sum_{j=1}^{k-1}\sum_{z\in\overleftarrow{\Lambda}^{\ell}_x}\left|\int_{X^j_2}\big(\eta(z) - \eta(z-j\ell)\big)\overrightarrow{\eta}^{\varepsilon n}(x+1)\big(f(\eta) - f(\eta^{z-j\ell,z})\big)\,d\nu^n_{\rho(\cdot)}\right|.\end{aligned} \tag{6.19}$$

Note that, as in Lemma 6.1.7, we can write the integral in the first line of (6.19) as

$$\frac{M}{2kn\ell}\sum_{x\in\Sigma^{\varepsilon}_{n,2}}\sum_{j=1}^{k-1}\sum_{z\in\overleftarrow{\Lambda}^{\ell}_x}\left|\sum_{\tilde{\eta}\in\Omega_{n-2}}\mathbb{1}_{\tilde{\eta}\in X^j_2}\overrightarrow{\eta}^{\varepsilon n}(x+1)\Big(f(0,1,\tilde{\eta}) + f(1,0,\tilde{\eta})\Big)\Big(\rho\left(\tfrac{z}{n}\right) - \rho\left(\tfrac{z-j\ell}{n}\right)\Big)\nu^{n-2}_{\rho(\cdot)}(\tilde{\eta})\right|,$$

where $\tilde{\eta}$ denotes the configuration η removing its value at the sites $z - j\ell$ and z. Since f is a density with respect to $\nu^n_{\rho(\cdot)}$ and $\rho(\cdot)$ satisfies the hypothesis of Lemma 4.2.1, we can bound the first line in (6.19) from above by

$$\frac{M}{kn\ell}\sum_{x\in\Sigma^{\varepsilon}_{n,2}}\sum_{j=1}^{k-1}\sum_{z\in\overleftarrow{\Lambda}^{\ell}_{x}}\Big|\frac{j\ell}{n}\Big|\leq\frac{Mk\ell}{n}.$$

Since $k=\frac{\varepsilon n}{\ell}$, that term is of order $O(\varepsilon)$. It remains to estimate the second term in (6.19). The idea is to send a particle from the site z to the site $z-j\ell$. This can be done since we are restricted to the set X^j_2, so that we know that there are at least m particles either in the box $\overleftarrow{\Lambda}^{\ell}_{x}$ or in the box $\overleftarrow{\Lambda}^{\ell}_{x-j\ell}$ apart from the particle at site z. With this in mind, we can again construct a path using the SSEP jumps to create a mobile cluster in the box where there are for sure m particles apart from the site z. Now, we use the PMM jumps to move the mobile cluster close to the particle at site z, and to send it to the site $z-j\ell$. Then, we put the mobile cluster back to its starting point using the PMM jumps, and we then put the m particles back to their initial position using the SSEP jumps. As in the previous lemma, for $A,\ \tilde{A}>0$, we can bound the second line in (6.19) from above by a constant, times

$$\frac{\ell}{A}+AD_S(\sqrt{f},\nu^n_{\rho(\cdot)})+\frac{\ell k}{\tilde{A}}+\tilde{A}D^m_P(\sqrt{f},\nu^n_{\rho(\cdot)}).$$

By choosing $A=\frac{n^{a-1}}{B}$ and $\tilde{A}=\frac{n}{B}$, we can bound (6.17) from above by a constant, times

$$\frac{1}{B}+T\left(\varepsilon+\frac{\ell B}{n^{a-1}}+\frac{\ell k B}{n}\right). \tag{6.20}$$

From the choice of ℓ and k, (6.20) is bounded from above by $\frac{1}{B}+T(\varepsilon+n^{-\delta}B+\varepsilon B)$, which vanishes when we take $n\to\infty$, then $\varepsilon\to 0$ and finally $B\to\infty$. □

Proof of Theorem 6.1.1. The proof of this general case is similar to the proof of Theorem 6.1.2. The only difference is that, for example, for $m=4$, the function $\tau_x h^4(\eta)$ contains terms of the form $\eta(x)\eta(x+1)\eta(x+2)\eta(x+3)$ and one must have space to do the replacement. The idea is the following: one first has to replace $\eta(x)$ by $\eta(x+1-\ell)$ in the product above, then $\eta(x+3)$ by $\eta(x+2+\ell)$, which can be done by using Lemma 6.2.5 for $\ell=n^{a-1-\delta}$, and finally we combine Lemmas 6.2.7, 6.2.8, and 6.2.9 to replace these terms by boxes of size $O(\varepsilon n)$, see Figure 6.5.

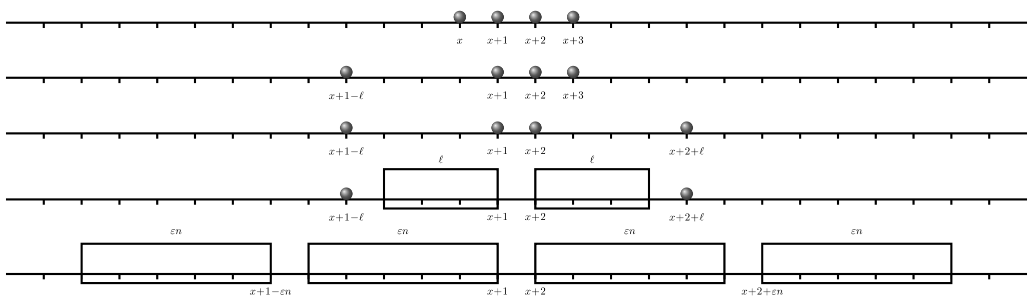


Figure 6.5: Replacing the occupation sites x, $x+1$, $x+2$, and $x+3$ by occupation averages on boxes of size εn.

In the general case, the function $\tau_x h^m(\eta)$ contains terms of the form $\eta(x)\eta(x+1)\cdots\eta(x+m-1)$.

Note that we can write this product as

$$\prod_{i=0}^{\frac{m}{2}-1} \eta(x+i) \prod_{i=0}^{\frac{m}{2}-1} \eta(x+\tfrac{m}{2}+i).$$

Thus, following the same idea as in the previous case, the result follows from the application of Theorem 6.2.4 twice to replace:

$$\prod_{i=0}^{\frac{m}{2}-1} \eta(x+i) \prod_{i=0}^{\frac{m}{2}-1} \eta(x+\tfrac{m}{2}+i) \text{ by } \prod_{i=0}^{\frac{m}{2}-1} \eta(x+i) \prod_{i=0}^{\frac{m}{2}-1} \overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1+i\varepsilon n),$$

and

$$\prod_{i=0}^{\frac{m}{2}-1} \eta(x+i) \prod_{i=0}^{\frac{m}{2}-1} \overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1+i\varepsilon n) \text{ by } \prod_{i=0}^{\frac{m}{2}-1} \overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x-i\varepsilon n) \prod_{i=0}^{\frac{m}{2}-1} \overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1+i\varepsilon n).$$

$\square$

6.2 Replacement lemmas at the boundary

In this section, we prove the different replacement lemmas regarding the boundary. Throughout this section $\rho(\cdot)$ will also be a profile satisfying the hypothesis of Lemma 4.2.1 so that (4.14) holds, as in the previous section. We stress that we had to define the available sets in the last section in which we can prove the replacement lemmas, namely $\Sigma^{\varepsilon}_{n,m}$. Thus, since the replacement lemmas here concern the boundary, we have to define two different subsets of our discrete space Σ_n, one regarding the available sites in which we can prove the replacement lemma for the left boundary $\Sigma^{\varepsilon,l}_{n,m}$, and the other for the right boundary $\Sigma^{\varepsilon,r}_{n,m}$, that are represented below:

$$\Sigma^{\varepsilon,l}_{n,m} := \{1,2,\dots,n-1-m\varepsilon n\} \quad \text{and} \quad \Sigma^{\varepsilon,r}_{n,m} := \{1+m\varepsilon n,\dots,n-2,n-1\}.$$

The replacement lemmas at the boundary say that we can replace:

- $\eta(1)$ (resp. $\eta(n-1)$) by α (resp. β); (Lemma 6.2.1)

- For every $x \in \Sigma^{\varepsilon,l}_{n,1}$ (resp. $x \in \Sigma^{\varepsilon,r}_{n,1}$)

$$\eta(x) \text{ by } \overrightarrow{\eta}^{\varepsilon n}(x), \quad \text{(resp. } \eta(x) \text{ by } \overleftarrow{\eta}^{\varepsilon n}(x)); \quad \text{(Lemma 6.2.2)}$$

- For every $x \in \Sigma^{\varepsilon,l}_{n,m}$

$$\prod_{i=0}^{m-1} \eta(x+1+i) \text{ by } \prod_{i=0}^{m-1} \overrightarrow{\eta}^{\varepsilon n}(x+1+i\varepsilon n), \quad \text{(Theorem 6.2.3)}$$

- For every $x \in \Sigma^{\varepsilon,r}_{n,m}$

$$\prod_{i=0}^{m-1} \eta(x-i) \text{ by } \prod_{i=0}^{m-1} \overleftarrow{\eta}^{\varepsilon n}(x-i\varepsilon n). \quad \text{(Theorem 6.2.3)}$$

Note that the index l in $\Sigma^{\varepsilon,l}_{n,m}$ regards the word 'left", and ℓ regards the size of the box. Before proving the main theorem of this section, namely, Theorem 6.2.3, let us prove the results presented in the first and second items above.

Lemma 6.2.1. *Fix $\theta < 1$. Let $\varphi : \Omega_n \to \Omega_n$ be a positive and bounded function which does not depend on the value of the configuration η at the site 1. For any $t \in [0,T]$, we have that*

$$\overline{\lim_{n\to\infty}} \mathbb{E}_{\mu_n}\left[\left|\int_0^t \varphi(\eta_{sn^2})(\alpha - \eta_{sn^2}(1))\, ds\right|\right] = 0.$$

The same is true replacing α by β, 1 by $n-1$ and requiring φ not to depend on η at the site $n-1$.

Proof. As in the replacement lemmas of the previous section, the expectation in the statement of the lemma is bounded from above by $\frac{C(\alpha,\beta)}{B}$, plus

$$t \sup_f \left(\left|\int_{\Omega_n} \varphi(\eta)(\alpha-\eta(1))f(\eta)\, d\nu^n_{\rho(\cdot)}\right| + \frac{n}{B}\langle L^m_n \sqrt{f}, \sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}\right), \tag{6.21}$$

where $B > 0$ and the supremum is carried over all the densities f with respect to $\nu^n_{\rho(\cdot)}$. Summing and subtracting $\frac{1}{2}f(\eta^1)$ in the first integral term inside the supremum in (6.21), we can bound this integral term from above by

$$\frac{1}{2}\left|\int_{\Omega_n} \varphi(\eta)(\alpha-\eta(1))\big(f(\eta)-f(\eta^1)\big)\, d\nu^n_{\rho(\cdot)}\right| + \frac{1}{2}\left|\int_{\Omega_n} \varphi(\eta)(\alpha-\eta(1))\big(f(\eta)+f(\eta^1)\big)\, d\nu^n_{\rho(\cdot)}\right|. \tag{6.22}$$

From Young's inequality and computations similar to the ones used to treat (6.5), the first term in (6.22) is bounded from above by

$$\frac{A}{4}\int_{\Omega_n} \frac{(\varphi(\eta)(\alpha-\eta(1)))^2}{I^\alpha_1(\eta)}\left(\sqrt{f(\eta^1)}+\sqrt{f(\eta)}\right)^2 d\nu^n_{\rho(\cdot)} + \frac{1}{4A}\int_{\Omega_n} I^\alpha_1(\eta)\left(\sqrt{f(\eta)}-\sqrt{f(\eta^1)}\right)^2 d\nu^n_{\rho(\cdot)},$$

for any $A > 0$, where $I^\alpha_1(\eta) = \alpha(1-\eta(1)) + (1-\alpha)\eta(1)$. Recalling the definition of $F^\alpha_1(\sqrt{f}, \nu^n_{\rho(\cdot)})$ in (4.11), and since φ is bounded, we can bound the previous display from above by

$$CA + \frac{1}{4A}F^\alpha_1(\sqrt{f}, \nu^n_{\rho(\cdot)}),$$

where $C := C(\alpha,\varphi)$. Now let us treat the remaining term in (6.22). Denoting by $\bar{\eta}$ the configuration η removing its value at the site 1, we can rewrite the second term of (6.22) as

$$\frac{1}{2}\left|\sum_{\bar{\eta}\in\Omega_{n-1}} \Big((\alpha-1)\varphi(\bar{\eta})(f(1,\bar{\eta})+f(0,\bar{\eta}))\rho\left(\tfrac{1}{n}\right) + \alpha\varphi(\bar{\eta})(f(0,\bar{\eta})+f(1,\bar{\eta}))\left(1-\rho\left(\tfrac{1}{n}\right)\right)\Big)\nu^{n-1}_{\rho(\cdot)}(\bar{\eta})\right|,$$

which is equal to

$$\frac{1}{2}\Bigg|\sum_{\bar\eta\in\Omega_{n-1}}\left(\alpha-\rho\left(\tfrac{1}{n}\right)\right)\varphi(\bar\eta)(f(0,\bar\eta)+f(1,\bar\eta))\nu^{n-1}_{\rho(\cdot)}(\bar\eta)\Bigg|,$$

where the notation $f(j,\bar\eta)$ means that we are computing $f(\eta)$ with $\eta(1)=j$ with $j\in\{0,1\}$. Since φ is bounded and f is a density with respect to $\nu^n_{\rho(\cdot)}$, we can bound the previous expression from above by

$$\tilde C\left|\alpha-\rho\left(\tfrac{1}{n}\right)\right|\sum_{\bar\eta\in\Omega_{n-1}}\Big(\rho(\tfrac{1}{n})f(1,\bar\eta)\nu^{n-1}_{\rho(\cdot)}(\bar\eta)+\left(1-\rho(\tfrac{1}{n})\right)f(0,\bar\eta)\nu^{n-1}_{\rho(\cdot)}(\bar\eta)\Big),$$

which is equal to $\tilde C\left|\alpha-\rho\left(\frac{1}{n}\right)\right|$, where $\tilde C:=\tilde C(\varphi,\rho)$. Now, from the previous computations, (6.22) is bounded from above by

$$CA+\frac{1}{4A}F_1^\alpha(\sqrt{f},\nu^n_{\rho(\cdot)})+\tilde C\left|\alpha-\rho\left(\tfrac{1}{n}\right)\right|.$$

Thus, taking $A=\frac{Bn^{\theta-1}}{\kappa}$, from (4.14) we have that (6.21) is bounded from above by a constant, times

$$\frac{Bn^{\theta-1}}{\kappa}+\left|\alpha-\rho(\tfrac{1}{n})\right|.$$

Taking $n\to\infty$ and using the fact that $\rho(\cdot)$ satisfies (4.13), we have that these terms vanish since $\theta<1$. □

Lemma 6.2.2. *For any $t\in[0,T]$ and $x\in\Sigma^{\varepsilon,l}_{n,1}$, we have*

$$\lim_{\varepsilon\to0}\overline{\lim_{n\to\infty}}\,\mathbb{E}_{\mu_n}\left[\left|\int_0^t\left\{\eta_{sn^2}(x)-\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x)\right\}ds\right|\right]=0.$$

In the same way, for any $t\in[0,T]$ and $x\in\Sigma^{\varepsilon,r}_{n,1}$, we have

$$\lim_{\varepsilon\to0}\overline{\lim_{n\to\infty}}\,\mathbb{E}_{\mu_n}\left[\left|\int_0^t\left\{\eta_{sn^2}(x)-\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x)\right\}ds\right|\right]=0.$$

Proof. We will explain the idea to prove the second equality since the first one is analogous. This proof can be done in two steps. The first one is to replace $\eta(x)$ by $\overleftarrow{\eta}^{\ell}_{sn^2}(x)$ using Lemma 6.1.3 and the second one is to replace $\overleftarrow{\eta}^{\ell}_{sn^2}(x)$ by $\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x)$ using Lemma 6.1.9. □

Now, we state the main theorem of this section.

Theorem 6.2.3. *For any $t\in[0,T]$, $m\in\mathbb{N}$, and $x\in\Sigma^{\varepsilon,l}_{n,m}$, we have*

$$\overline{\lim_{\varepsilon\to0}}\lim_{n\to\infty}\mathbb{E}_{\mu_n}\left[\left|\int_0^t\left\{\prod_{i=0}^{m-1}\eta_{sn^2}(x+i)-\prod_{i=0}^{m-1}\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+i\varepsilon n)\right\}ds\right|\right]=0.$$

In the same way, for any $t\in[0,T]$, $m\in\mathbb{N}$, and $x\in\Sigma^{\varepsilon,r}_{n,m}$, we have

$$\overline{\lim_{\varepsilon\to0}}\lim_{n\to\infty}\mathbb{E}_{\mu_n}\left[\left|\int_0^t\left\{\prod_{i=0}^{m-1}\eta_{sn^2}(x-i)-\prod_{i=0}^{m-1}\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x-i\varepsilon n)\right\}ds\right|\right]=0.$$

As in the previous section, we will divide the proof of previous theorem into two steps. First, we will present the proof for replacing products of the form $\eta(x)\eta(x+1)$ (resp. $\eta(x-1)\eta(x)$) by $\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x)\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+\varepsilon n)$ (resp. $\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x-\varepsilon n)\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x)$) as stated in Theorem 6.2.4. Second, we will present the proof of Theorem 6.2.3 for any $m > 2$.

Theorem 6.2.4. *For any $t \in [0,T]$ and $x \in \Sigma^{\varepsilon,l}_{n,2}$, we have*

$$\lim_{\varepsilon\to 0}\varlimsup_{n\to\infty}\mathbb{E}_{\mu_n}\left[\left|\int_0^t \left\{\eta_{sn^2}(x)\eta_{sn^2}(x+1) - \overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x)\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+\varepsilon n)\right\} ds\right|\right] = 0. \tag{6.23}$$

In the same way, for any $t \in [0,T]$ and $x \in \Sigma^{\varepsilon,r}_{n,2}$, we have

$$\lim_{\varepsilon\to 0}\varlimsup_{n\to\infty}\mathbb{E}_{\mu_n}\left[\left|\int_0^t \left\{\eta_{sn^2}(x)\eta_{sn^2}(x+1) - \overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x-\varepsilon n+1)\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x+1)\right\} ds\right|\right] = 0. \tag{6.24}$$

For simplicity of the presentation, we will only prove (6.23), that is, the left boundary part. We note that the result concerning the right boundary in (6.24) can be proved with an analogous argument. Let $x \in \Sigma^{\varepsilon,l}_{n,m}$, $a \in (1,2)$, $\varepsilon > 0$, and $\delta > 0$ such that $a-1-\delta \geq 0$. Thus, in the same way that we did in the proof Theorem 6.1.1, we will divide the proof of the previous theorem into fours steps as follows (see also Figure 6.6):

1) replace $\eta(x)\eta(x+1)$ by $\eta(x)\eta(x+\ell)$, for $\ell = n^{a-1-\delta}$; (Lemma 6.2.5)

2) replace $\eta(x)\eta(x+\ell)$ by $\overrightarrow{\eta}^{\ell}(x)\eta(x+\ell)$, for $\ell = n^{a-1-\delta}$; (Lemma 6.2.7)

3) replace $\overrightarrow{\eta}^{\ell}(x)\eta(x+\ell)$ by $\overrightarrow{\eta}^{\ell}(x)\overrightarrow{\eta}^{\varepsilon n}(x+\varepsilon n)$, for $\ell = n^{a-1-\delta}$; (Lemma 6.2.8)

4) replace $\overrightarrow{\eta}^{\ell}(x)\overrightarrow{\eta}^{\varepsilon n}(x+\varepsilon n)$ by $\overrightarrow{\eta}^{L}(x)\overrightarrow{\eta}^{\varepsilon n}(x+\varepsilon n)$, for $\ell = n^{a-1-\delta}$ and $L = \varepsilon n$. (Lemma 6.2.9)

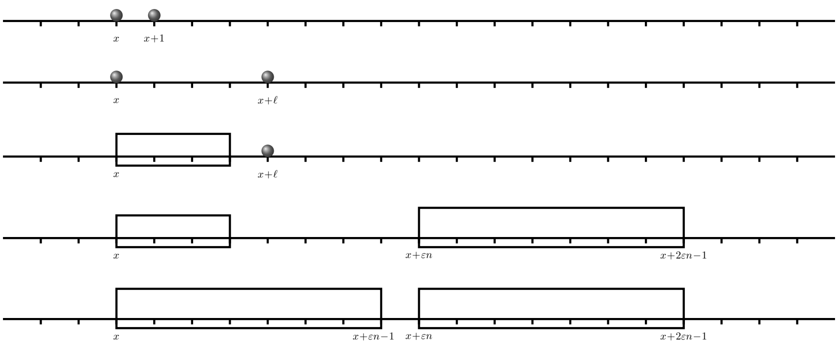


Figure 6.6: Steps to prove Theorem 6.2.4.

Lemma 6.2.5. *For any $t \in [0,T]$, $x \in \Sigma^{\varepsilon,l}_{n,1}$, $\ell = n^{a-1-\delta}$ with $\delta > 0$ such that $a-1-\delta \geq 0$, we have*

$$\varlimsup_{n\to\infty}\mathbb{E}_{\mu_n}\left[\left|\int_0^t \eta_{sn^2}(x)\left\{\eta_{sn^2}(x+1) - \eta_{sn^2}(x+\ell)\right\} ds\right|\right] = 0.$$

Proof. Following the same steps of the proof of Lemma 6.1.3, the expectation in the statement of the lemma is bounded from above by $\frac{C(\alpha,\beta)}{B}$, plus

$$T\sup_f\left(\left|\int_{\Omega_n}\eta(x)\{\eta(x+1)-\eta(x+\ell)\}f(\eta)\,d\nu^n_{\rho(\cdot)}\right|+\frac{n}{B}\langle L^m_n\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}\right),\tag{6.25}$$

where $B>0$ and the supremum is carried over all the densities f with respect to $\nu^n_{\rho(\cdot)}$. Write $\eta(x)-\eta(x+\ell)=\sum_{y=x}^{x+\ell-1}\eta(y)-\eta(y+1)$. Using the same strategy that we used to bound the term in (6.5), for $A>0$, the first term inside the supremum in (6.25) is bounded from above by a constant, times

$$\frac{\ell}{n}+\frac{\ell}{A}+AD_S(\sqrt{f},\nu^n_{\rho(\cdot)}).\tag{6.26}$$

With the choice $A=\frac{n^{a-1}}{B}$, from (4.14), (6.25), and (6.26), we have that the expectation in the statement of the lemma is bounded from above by a constant times

$$\frac{1}{B}+T\left(\frac{\ell}{n}+\frac{\ell B}{n^{a-1}}\right).$$

From the choice of ℓ, taking $n\to\infty$ we have that the right-hand side of last expression vanishes. By sending $B\to\infty$, we conclude the proof. ☐

Corollary 6.2.6. *For any $t\in[0,T]$ and $x\in\Sigma^{\varepsilon,l}_{n,1}$, we have*

$$\overline{\lim_{n\to\infty}}\,\mathbb{E}_{\mu_n}\left[\left|\int_0^t\big\{\eta_{sn^2}(x)-\eta_{sn^2}(x+1)\big\}\,ds\right|\right]=0.$$

In the same way, for any $t\in[0,T]$ and $x\in\Sigma^{\varepsilon,r}_{n,2}$, we have

$$\overline{\lim_{n\to\infty}}\,\mathbb{E}_{\mu_n}\left[\left|\int_0^t\big\{\eta_{sn^2}(x+1)-\eta_{sn^2}(x)\big\}\,ds\right|\right]=0.$$

Proof. To prove this corollary it is enough to repeat the proof of Lemma 6.2.5 taking $\ell=1$ and replacing $\eta(x+1)$ by $\eta(x)$. ☐

Lemma 6.2.7. *For any $t\in[0,T]$, $x\in\Sigma^{\varepsilon,l}_{n,1}$, $\ell=n^{a-1-\delta}$ with $\delta>0$ such that $a-1-\delta\geq0$, we have*

$$\overline{\lim_{n\to\infty}}\,\mathbb{E}_{\mu_n}\Big[\Big|\int_0^t\{\eta_{sn^2}(x)-\overrightarrow{\eta}^{\ell}_{sn^2}(x)\}\eta_{sn^2}(x+\ell)\,ds\Big|\Big]=0.\tag{6.27}$$

Proof. Following the same steps of previous lemmas, the expectation in (6.27) is bounded from above by $\frac{C(\alpha,\beta)}{B}$, plus

$$T\sup_f\left(\left|\int_{\Omega_n}\{\eta(x)-\overrightarrow{\eta}^{\ell}(x)\}\eta(x+\ell)f(\eta)\,d\nu^n_{\rho(\cdot)}\right|+\frac{n}{B}\langle L^m_n\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}\right),$$

where $B>0$ and the supremum is carried over all densities f with respect to $\nu^n_{\rho(\cdot)}$. Now, following exactly the same computations done in the proof of Lemma 6.1.3, the expectation in the statement of the lemma is bounded from above by a constant times

$$\frac{1}{B} + T\bigg(\frac{\ell}{n} + \frac{\ell B}{n^{a-1}}\bigg).$$

Taking $n \to \infty$ and then $B \to \infty$, the expression above vanishes due to our choice of ℓ. $\square$

Lemma 6.2.8. *For any $t \in [0,T]$, $x \in \Sigma_{n,2}^{\varepsilon,l}$, $\ell = n^{a-1-\delta}$ with $\delta > 0$ such that $a-1-\delta \geq 0$, we have*

$$\lim_{\varepsilon \to 0} \varlimsup_{n \to \infty} \mathbb{E}_{\mu_n}\left[\left|\int_0^t \overrightarrow{\eta}_{sn^2}^{\ell}(x)\big\{\eta_{sn^2}(x+\ell) - \overrightarrow{\eta}_{sn^2}^{\varepsilon n}(x+\varepsilon n)\big\}\, ds\right|\right] = 0.$$

Proof. Following the same steps of the proof of Lemma 6.1.3, the expectation in the statement of the lemma is bounded from above by $\frac{C(\alpha,\beta)}{B}$, plus

$$T \sup_f \left(\left|\int_{\Omega_n} \overrightarrow{\eta}^{\ell}(x)\big\{\eta(x+\ell) - \overrightarrow{\eta}^{\varepsilon n}(x+\varepsilon n)\big\} f(\eta)\, d\nu_{\rho(\cdot)}^n\right| + \frac{n}{B}\langle L_n^m \sqrt{f}, \sqrt{f}\rangle_{\nu_{\rho(\cdot)}^n}\right),$$

where $B > 0$ and the supremum is carried over all densities f with respect to $\nu_{\rho(\cdot)}^n$. Now, we want to examine the first term inside the supremum above. In order to do it we will use the path argument again. First, let $X_1 = \{\eta \in \Omega_n : \overrightarrow{\eta}^{\ell}(x) \geq \frac{m}{\ell}\}$ and note that the first integral inside the supremum above can be written as the integral over the set X_1 plus the integral over its complement X_1^c. The integral over X_1^c is of order $O\left(\frac{1}{\ell}\right)$ so that it vanishes in the limit due to our choice of ℓ. To treat the integral over X_1, we start by noticing that

$$\eta(x+\ell) - \overrightarrow{\eta}^{\varepsilon n}(x+\varepsilon n) = \frac{1}{\varepsilon n}\sum_{y=x+\varepsilon n}^{x+2\varepsilon n-1} \eta(x+\ell) - \eta(y).$$

Thus, we need to examine

$$\frac{1}{\varepsilon n}\sum_{y=x+\varepsilon n}^{x+2\varepsilon n-1} \left|\int_{X_1} \overrightarrow{\eta}^{\ell}(x)\big\{\eta(x+\ell) - \eta(y)\big\} f(\eta)\, d\nu_{\rho(\cdot)}^n\right|. \tag{6.28}$$

Following the same computations done in the proof of Lemma 6.1.7, we can bound the previous expression by the sum

$$\begin{aligned}
&\frac{1}{2\varepsilon n}\sum_{y=x+\varepsilon n}^{x+2\varepsilon n-1} \left|\int_{X_1} \overrightarrow{\eta}^{\ell}(x)\big\{\eta(x+\ell) - \eta(y)\big\} \left(f(\eta) + f(\eta^{x+\ell,y})\right)\, d\nu_{\rho(\cdot)}^n\right| \\
+&\frac{1}{2\varepsilon n}\sum_{y=x+\varepsilon n}^{x+2\varepsilon n-1} \left|\int_{X_1} \overrightarrow{\eta}^{\ell}(x)\big\{\eta(x+\ell) - \eta(y)\big\} \left(f(\eta) - f(\eta^{x+\ell,y})\right)\, d\nu_{\rho(\cdot)}^n\right|.
\end{aligned}$$

Since $\rho(\cdot)$ is Lipschitz, the first line of the previous expression is bounded from above by

$$\frac{1}{\varepsilon n}\sum_{y=x+\varepsilon n}^{x+2\varepsilon n-1} \left|\rho\left(\tfrac{x+\ell}{n}\right) - \rho\left(\tfrac{y}{n}\right)\right| \leq \frac{\ell + 1 + 2\varepsilon n}{n}.$$

Now, since we are integrating over X_1, we will use again the path's argument to examine the remaining term. The idea is to move a particle at site $x+\ell$ to a site y inside a box of size εn combining the SSEP and the PMM jumps. See Figure 6.7 to have a picture of the aforementioned path in the case $m=3$. Combining the path argument with the computations above, for $A, \tilde{A} > 0$ the expectation in the statement of the lemma is bounded from above by a constant times

$$\frac{1}{B} + T\Big(\frac{\ell}{n} + \frac{1}{n} + \varepsilon + \frac{\ell}{A} + \left(\frac{A}{4} - \frac{n^{a-1}}{4B}\right) D_S(\sqrt{f}, \nu^n_{\rho(\cdot)}) + \frac{\varepsilon n}{\tilde{A}} + \left(\frac{\tilde{A}}{4} - \frac{n}{4B}\right) D_P^m(\sqrt{f}, \nu^n_{\rho(\cdot)})\Big).$$

With the choice $A = \frac{n^{a-1}}{B}$ and $\tilde{A} = \frac{n}{B}$ we get

$$\frac{1}{B} + T\left(\frac{\ell}{n} + \frac{1}{n} + \varepsilon + \frac{\ell B}{n^{a-1}} + \varepsilon B\right).$$

Taking $n \to \infty$, then $\varepsilon \to 0$, and finally $B \to \infty$, the result follows due to our choice of ℓ.

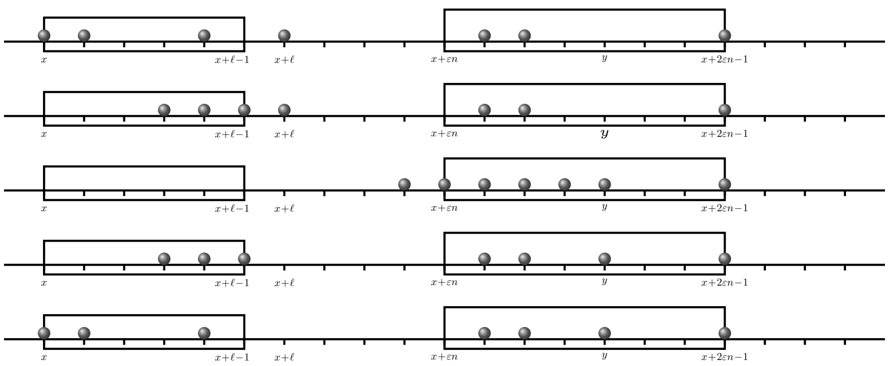


Figure 6.7: Path used to send a particle from site $x+\ell$ to y inside a box of size εn.

□

Lemma 6.2.9. *For any $t \in [0,T]$, $x \in \Sigma^{\varepsilon,l}_{n,2}$, $L = \varepsilon n$ and $\ell = n^{a-1-\delta}$ with $\delta > 0$ such that $a-1-\delta \geq 0$, we have*

$$\lim_{\varepsilon \to 0} \varlimsup_{n\to\infty} \mathbb{E}_{\mu_n}\left[\left|\int_0^t \{\overrightarrow{\eta}^{\ell}_{sn^2}(x) - \overrightarrow{\eta}^{L}_{sn^2}(x)\} \overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+\varepsilon n)\, ds\right|\right] = 0. \tag{6.29}$$

Proof. Following the same steps of Lemma 6.1.9, the expectation in (6.29) is bounded from above by $\frac{C(\alpha,\beta)}{B}$ plus

$$T \sup_f \left(\left|\int_{\Omega_n} \{\overrightarrow{\eta}^{\ell}(x) - \overrightarrow{\eta}^{L}(x)\} \overrightarrow{\eta}^{\varepsilon n}(x+\varepsilon n) f(\eta)\, d\nu^n_{\rho(\cdot)}\right| + \frac{n}{B}\langle L^m_n \sqrt{f}, \sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}\right),$$

where $B > 0$ and the supremum is carried over all the densities f with respect to $\nu^n_{\rho(\cdot)}$. Take $L = k\ell$ with $k = \frac{\varepsilon n}{\ell}$. As in Lemma 6.1.9, let $X_2^j = \{\eta \in \Omega_n : \overrightarrow{\eta}^{\ell}(x) \geq \frac{m}{\ell}\} \cup \{\eta \in \Omega_n : \overrightarrow{\eta}^{\ell}(x+j\ell) \geq \frac{m}{\ell}\}$. Now, following exactly the same computations done in the proof of that lemma, we have that the expectation in (6.29) is bounded from above by a constant times

$$\frac{1}{B} + T\left(\varepsilon + \frac{\ell B}{n^{a-1}} + B\varepsilon\right).$$

Taking $n \to \infty$, then $\varepsilon \to 0$, and $B \to \infty$, the result follows due to our choice of ℓ and k. $\square$

Proof of Theorem 6.2.3. Note that the cases $m = 1$ and $m = 2$ were proved in Lemma 6.2.2 and Theorem 6.2.3, respectively. The only difference between these cases and the general one is the fact that, for example, for $m = 4$, the function $\tau_x h^4(\eta)$ contain terms of the form $\eta(x)\eta(x+1)\eta(x+2)\eta(x+3)$. Since one must have to have space to replace this product by a product of averages of particles in a box of size $O(\varepsilon n)$, we are allowed to do so only for sites x in $\Sigma^{\varepsilon}_{n,4}$. The idea is the following: First, we use Lemma 6.2.5 three times in order to have space to replace the product by the product of boxes of size $O(\varepsilon n)$ so that they do not overlap and variables do not correlate, i.e., replace $\eta(x+3)$ by $\eta(x+3\ell)$, $\eta(x+2)$ by $\eta(x+2\ell)$ and $\eta(x+1)$ by $\eta(x+\ell)$. Thus, we use Lemma 6.2.7 to replace $\eta(x)$ by $\overrightarrow{\eta}^{\ell}(x)$ in order to be able to use the path argument. Then, we use Lemma 6.2.8, three times to replace $\eta(x+3\ell)$ by $\overrightarrow{\eta}^{\varepsilon n}(x+3\varepsilon n)$, $\eta(x+2\ell)$ by $\overrightarrow{\eta}^{\varepsilon n}(x+2\varepsilon n)$, and $\eta(x+\ell)$ by $\overrightarrow{\eta}^{\varepsilon n}(x+\varepsilon n)$. Finally, we use Lemma 6.2.9 to replace $\overrightarrow{\eta}^{\ell}(x)$ by $\overrightarrow{\eta}^{\varepsilon n}(x)$ and we are done. Note that the proof has 8 steps, and we summarize them in Figure 6.8 below.

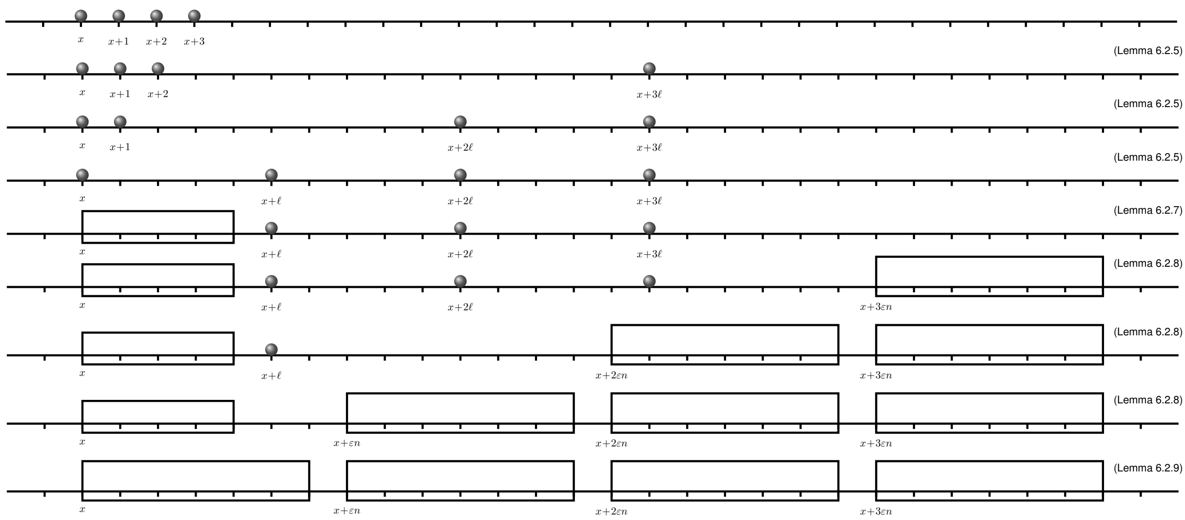


Figure 6.8: Steps to prove Theorem 6.2.3.

For the general case we follow the same idea as before, but here we need to use Lemma 6.2.5 $m-1$ times, Lemma 6.2.7 once, Lemma 6.2.8 $m-1$ times, and Lemma 6.2.9 once. $\square$

6.3 Fixing the profile at the boundary (The Dirichlet case)

In this section we will fix the profile at the boundary for the case $\theta < 1$, i.e., we intend to prove item (3) in Definition 4, that is, $\rho_t(0) = \alpha$ and $\rho_t(1) = \beta$ for a.e. $t \in (0, T]$. We present the proof for the term concerning the left boundary since the other one is analogous. Recall from Section 2.3 that $\mathbb{Q}$ is a limit point of the sequence $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ and note that

$$\mathbb{E}_{\mu_n}\left[\left|\int_0^1 (\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(1) - \alpha)\, ds\right|\right] = \mathbb{E}_n\left[\left|\int_0^1 \langle \pi_s, \overrightarrow{\iota_\varepsilon}^0\rangle - \alpha)\, ds\right|\right].$$

From Markov's inequality, for any $\delta > 0$ we have

$$\mathbb{Q}_n\left(\left|\int_0^1 (\langle \pi_s, \overrightarrow{\iota_\varepsilon}^0\rangle - \alpha)\, ds\right| > \delta\right) \le \frac{1}{\delta}\mathbb{E}_{\mu_n}\left[\left|\int_0^1 (\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(1) - \alpha)\, ds\right|\right].$$

Now, since $\overrightarrow{\iota_\varepsilon}^0$ is not a continuous function, we can not use Portmanteus's Theorem in the previous expression. Nevertheless, we can approximate this function by a continuous function so that the error vanishes as $\varepsilon \to 0$. Then, after this approximation and recalling the definition of $\mathbb{Q}_n$, we can use Portmanteau's Theorem to conclude that

$$\mathbb{Q}\left(\left|\int_0^1 (\langle \pi_s, \overrightarrow{\iota_\varepsilon}^0\rangle - \alpha)\, ds\right| > \delta\right) \le \frac{1}{\delta}\liminf_{n\to\infty}\mathbb{E}_{\mu_n}\left[\left|\int_0^1 (\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(1) - \alpha)\, ds\right|\right].$$

Combining both Lemma 6.2.1 with $\varphi \equiv 1$ and Lemma 6.2.2, the right-hand side of the previous inequality vanishes as $\varepsilon \to 0$. Therefore, since $\mathbb{Q}$ a.s. $\pi_t(du) = \rho_t(u)du$ with $\rho_t(\cdot)$ a continuous function in 0 for a.e. t, taking $\varepsilon \to 0$ we conclude that $\mathbb{Q}$ a.s. $\rho_t(0) = \alpha$ for a.e. t.

Chapter 7

Energy Estimates

This chapter aims to deduce the energy estimate stated in Theorem 7.0.4. From the hydrodynamic limit perspective, we can see this chapter as being devoted to proving that any limit point $\mathbb{Q}$ of the sequence $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ is concentrated on trajectories $\rho_t(u)du$, so that ρ^m belongs to $L^2(0,T;\mathcal{H}^1)$. This is a crucial step to guarantee the uniqueness of weak solutions of our hydrodynamic equations, and it is a consequence of Theorem 7.0.4. Since our model is an exclusion process, it is standard to check that $\mathbb{Q}$ is supported on trajectories of measures that are absolutely continuous with respect to the Lebesgue measure, that is, $\pi_t(du) = \rho_t(u)du$, for all $t \in [0,T]$ where $\rho : [0,T] \times [0,1] \to [0,1]$. The energy estimate stated and proved in this chapter is based on the Robin case, once the energy estimate for the Dirichlet and Neumann cases follows from the Robin one. We will explain how to proceed in each one of these cases in the end of this chapter. Thus, we can see the result of this chapter as strong energy estimates, which allow obtaining detailed information about the boundary behavior of the weak solutions of (2.17).

Although the proof of energy estimates is based on interacting particle systems techniques, it can be used for example, in the proof of convergence results for weak solutions of partial differential equations of parabolic type, see [9, 22]. In [9], Theorem 7.0.4 was the most important step in the proof of the following result about the convergence of weak solutions of (2.17):

Theorem 7.0.1 ([9]). *Let $g : [0,1] \to [0,1]$ be a measurable function. For each $\kappa > 0$, let $\rho^\kappa : [0,T] \times [0,1] \to [0,1]$ be the unique weak solution of* (2.17) *with initial condition g. Then,*

$$\lim_{\kappa\to 0} \rho^\kappa = \rho^0 \quad \textit{and} \quad \lim_{\kappa\to\infty} \rho^\kappa = \rho^\infty$$

in $L^2([0,T]\times[0,1])$, where ρ^0 is the unique weak solution of (2.17) *(with $\kappa = 0$), and ρ^∞ is the unique weak solution of* (2.15)*, both with initial condition g.*

Before deducing the energy estimates, recall the definitions, notations and results of the previous sections. We start this chapter by introducing a proper weighted L^2 space, and from this space we define an energy functional. If the energy of a function ξ is finite, then the functional captures a lot of information about ξ^m, see Proposition 7.0.2. Then, we will state that the solution ρ^κ of (2.17) has finite energy, and this is the content of Theorem 7.0.4.

Recall that the parameters α, β and m are fixed as in (2.14). Let $\kappa > 0$ and $a, b \geq 0$. We define a measure $W^{\alpha,\beta}_{\kappa,a,b}$ on $[0,1]$ by

$$W^{\alpha,\beta}_{\kappa,a,b}(du) := du + \frac{1}{k}P^{\alpha}_{m}(a)\delta_0(du) + \frac{1}{k}P^{\beta}_{m}(b)\delta_1(du),$$

where $\delta_z(du)$, with $z \in \{0,1\}$, is the Dirac measure and

$$P^{\gamma}_{m}(\rho) = \sum_{i=0}^{m-1} \gamma^{m-1-i}\rho^i, \text{ for } \gamma \in \{\alpha, \beta\} \text{ and } \rho \geq 0\,. \tag{7.1}$$

Thus,

$$P^{\gamma}_{m}(\rho) \geq \gamma^{m-1} > 0, \tag{7.2}$$

because $\rho \geq 0$ and $\gamma > 0$, since $\gamma \in \{\alpha, \beta\}$ and the restrictions in (2.14). If $\gamma = 0$, then $P^0_m(\rho) = \rho^{m-1}$ so that the inequality (7.2) would not hold in general. Observe that, since

$$\gamma^m - \rho^m = (\gamma - \rho)\sum_{i=0}^{m-1} \gamma^{m-1-i}\rho^i = (\gamma-\rho)P^{\gamma}_{m}(\rho)\,, \tag{7.3}$$

then

$$\gamma - \rho = \frac{\gamma^m - \rho^m}{P^{\gamma}_{m}(\rho)}, \tag{7.4}$$

for $\gamma \in \{\alpha, \beta\}$ and for all $\rho \geq 0$. The measure $W^{\alpha,\beta}_{\kappa,a,b}$ is the sum of the Lebesgue measure and Dirac measures concentrated on 0 and 1 with weights $\frac{1}{k}P^{\alpha}_{m}(a)$ and $\frac{1}{k}P^{\beta}_{m}(b)$, respectively. For $g \in L^2([0,1])$ such that $g(0)$ and $g(1)$ are both well-defined, we denote

$$\|g\|^2_{W^{\alpha,\beta}_{\kappa,a,b}} := \int_0^1 g^2(u)\, W^{\alpha,\beta}_{\kappa,a,b}(du)\,. \tag{7.5}$$

Definition 6. *Let $\mathcal{B}$ be the space of measurable functions $\xi : [0,T] \times [0,1] \to [0,\infty)$ such that the applications $s \mapsto \xi_s(0)$ and $s \mapsto \xi_s(1)$ are measurable and bounded.*

Definition 7. *Let $\xi \in \mathcal{B}$. For any $\kappa > 0$, we denote by $L^2_{\kappa,\xi}([0,T] \times [0,1])$ the Hilbert space composed of all measurable functions $H : [0,T] \times [0,1] \to \mathbb{R}$ such that*

$$\begin{aligned}\langle\!\langle H, H \rangle\!\rangle^{\alpha,\beta}_{\kappa,\xi} &:= \int_0^T \|H_s\|^2_{W^{\alpha,\beta}_{\kappa,\xi_s(0),\xi_s(1)}}\, ds \\ &= \langle\!\langle H, H \rangle\!\rangle + \int_0^T \left\{ \frac{P^{\alpha}_{m}(\xi_s(0))}{\kappa} H^2_s(0) + \frac{P^{\beta}_{m}(\xi_s(1))}{\kappa} H^2_s(1) \right\} ds < \infty,\end{aligned} \tag{7.6}$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is defined in (2.12) *and for $\gamma \in \{\alpha, \beta\}$, P^{γ}_{m} is defined in* (7.1)*. Moreover, $L^2_{\kappa,\xi}([0,T] \times [0,1]) \subseteq L^2([0,T] \times [0,1])$.*

Definition 8. *For a function ξ such that $\xi^m \in L^2([0,T] \times [0,1])$, we define the functional $\mathcal{T}^{\alpha,\beta}_{\xi,m}$ on*

$C^{0,1}([0,T]\times[0,1])$ *by*

$$\mathcal{T}^{\alpha,\beta}_{\xi,m}(H) := \langle\!\langle\, \xi^m,\, \partial_u H \rangle\!\rangle + \int_0^T \left\{ \alpha^m H_s(0) - \beta^m H_s(1) \right\} ds. \tag{7.7}$$

Let us define the energy functional:

Definition 9 (Energy functional)**.** *For each $\kappa > 0$ and $c > 0$ fixed, we define the functional $\mathcal{E}^{\alpha,\beta}_{m,\kappa,c}$ which acts on functions $\xi \in \mathcal{B}$ such that $\xi^m \in L^2([0,T]\times[0,1])$ as*

$$\mathcal{E}^{\alpha,\beta}_{m,\kappa,c}(\xi) := \sup_{H\in C^{0,1}([0,T]\times[0,1])} \left\{ \mathcal{T}^{\alpha,\beta}_{\xi,m}(H) - c\langle\!\langle H, H \rangle\!\rangle^{\alpha,\beta}_{\kappa,\xi} \right\}. \tag{7.8}$$

Fortunately, by means of estimating the energy functional $\mathcal{E}^{\alpha,\beta}_{m,\kappa,c}(\cdot)$ we are able to obtain a lot of information about ξ^m, which is given by the next proposition.

Proposition 7.0.2. *Let $\xi \in \mathcal{B}$ such that $\xi^m \in L^2([0,T]\times[0,1])$ and $\mathcal{E}^{\alpha,\beta}_{m,\kappa,c}(\xi) \le M_0 < \infty$, for some $\kappa > 0$, $c > 0$, and $M_0 > 0$. Then, there exists $\partial_u \xi^m \in L^2_{\kappa,\xi}([0,T]\times[0,1])$ such that for all $H \in C^{0,1}([0,T]\times[0,1])$*

$$\mathcal{T}^{\alpha,\beta}_{\xi,m}(H) = -\langle\!\langle \partial_u \xi^m, H \rangle\!\rangle^{\alpha,\beta}_{\kappa,\xi}, \tag{7.9}$$

and $\xi^m \in L^2(0,T;\mathcal{H}^1)$. Moreover,

$$\mathcal{E}^{\alpha,\beta}_{m,\kappa,c}(\xi) = \frac{1}{4c} \langle\!\langle \partial_u \xi^m, \partial_u \xi^m \rangle\!\rangle^{\alpha,\beta}_{\kappa,\xi}, \tag{7.10}$$

and

$$\partial_u(\xi_s)^m(0)\ P^{\alpha}_m(\xi_s(0)) = \kappa\left((\xi_s)^m(0) - \alpha^m\right) \quad \textit{and} \quad \partial_u(\xi_s)^m(1)\ P^{\alpha}_m(\xi_s(0)) = \kappa\left(\beta^m - (\xi_s)^m(1)\right), \tag{7.11}$$

for almost every $s \in (0,T]$.

We observe that last result holds for $\alpha, \beta \in [0,1]$.

Remark 7.0.3. *In particular, as we assume that $\alpha, \beta > 0$ (see* (2.14)*),* (7.4) *and* (7.2)*, the boundary conditions in* (7.11) *become*

$$\partial_u(\xi_s)^m(0) = \kappa(\xi_s(0) - \alpha) \quad \textit{and} \quad \partial_u(\xi_s)^m(1) = \kappa(\beta - \xi_s(1)), \tag{7.12}$$

for almost every $s \in (0,T]$. This is one of the major consequences of the energy estimates.

Properties of the weak solution of (2.17)**.**

In the next theorem, we state that the unique weak solution ρ^κ of (2.17) has finite energy, and from the last proposition we obtain information about $(\rho^\kappa)^m$. The proof of the next theorem is a consequence of Proposition 7.0.6, and since its proof is quite long, we will present it afterward. The idea is to consider the PMM with slow reservoirs, whose hydrodynamic limit is ruled by the weak solution of (2.17), and from its properties prove the energy bound (7.13) below.

Theorem 7.0.4 (Energy estimate)**.** *For any $\kappa > 0$, there exists a constant $c > 0$ such that the unique weak solution $\rho^\kappa : [0,T] \times [0,1] \to [0,1]$ of* (2.17) *satisfies the energy estimate:*

$$\mathcal{E}^{\alpha,\beta}_{m,\kappa,c}(\rho^\kappa) \leq M_0, \tag{7.13}$$

where M_0 is a constant that does not depend on κ. As a consequence, for all $\kappa > 0$, the weak solution ρ^κ satisfies the boundary conditions:

$$\partial_u(\rho^\kappa_s)^m(0) = \kappa(\rho^\kappa_s(0) - \alpha) \quad \textit{and} \quad \partial_u(\rho^\kappa_s)^m(1) = \kappa(\beta - \rho^\kappa_s(1)), \tag{7.14}$$

for almost every $s \in (0,T]$ and the set $\{(\rho^\kappa)^m : \kappa > 0\}$ is bounded in $L^2(0,T;\mathcal{H}^1)$.

Remark 7.0.5. *If we do not assume that $\alpha, \beta > 0$, the boundary conditions* (7.14) *would be*

$$\partial_u(\rho^\kappa_s)^m(0)\ P^\alpha_m(\rho^\kappa_s(0)) = \kappa\left((\rho^\kappa_s)^m(0) - \alpha^m\right) \quad \textit{and} \quad \partial_u(\rho^\kappa_s)^m(1)\ P^\alpha_m(\rho^\kappa_s(0)) = \kappa\left(\beta^m - (\rho^\kappa_s)^m(1)\right), \tag{7.15}$$

for almost every $s \in (0,T]$.

Proof of Theorem 7.0.4. The result follows as a consequence of the next proposition, which states that the expectation with respect to $\mathbb{Q}$ of $\mathcal{E}^{\alpha,\beta}_{m,\kappa,c}(\rho^\kappa)$ is bounded. To remove the expectation in (7.17), we use Proposition 2.3.3, and then, the first result in Theorem 7.0.4 follows. Now, for (7.14) we start by observing that $(\rho^\kappa)^m \in L^2([0,T] \times [0,1])$. Moreover, since $\mathcal{E}^{\alpha,\beta}_{m,\kappa,c}(\rho^\kappa) < \infty$, from (7.11) the identities in (7.14) follow. Finally, to prove the boundedness of $\{(\rho^\kappa)^m : \kappa > 0\}$ in $L^2(0,T;\mathcal{H}^1)$, we argue as follows. By Definition 7, (7.10), and (7.13), it holds that

$$\begin{aligned} \langle\langle \partial_u(\rho^\kappa)^m,\, \partial_u(\rho^\kappa)^m \rangle\rangle &\leq \langle\langle \partial_u(\rho^\kappa)^m,\, \partial_u(\rho^\kappa)^m \rangle\rangle^{\alpha,\beta}_{\kappa,\rho^\kappa} \\ &= 4c\,\mathcal{E}^{\alpha,\beta}_{m,\kappa,c}(\rho^\kappa) \leq 4c\,M_0, \end{aligned} \tag{7.16}$$

for all $\kappa > 0$. Therefore, from the definition of the norm in $L^2(0,T;\mathcal{H}^1)$ given in (2.13), and the fact that $0 \leq (\rho^\kappa)^m \leq 1$, we have

$$\|(\rho^\kappa)^m\|^2_{L^2(0,T;\mathcal{H}^1)} = \langle\langle (\rho^\kappa)^m, (\rho^\kappa)^m \rangle\rangle + \langle\langle \partial_u(\rho^\kappa)^m, \partial_u(\rho^\kappa)^m \rangle\rangle \leq T + 4cM_0,$$

concluding the proof. To finish, we are only left to prove the next proposition.

□

Proposition 7.0.6. *For any $\kappa > 0$ and any $m \in \mathbb{N}$, there exist constants M_0 and c, that do not depend on κ, such that*

$$\mathbb{E}_{\mathbb{Q}}\left[\mathcal{E}^{\alpha,\beta}_{m,\kappa,c}(\rho^\kappa)\right] \leq M_0 < \infty, \tag{7.17}$$

where ρ^κ is the unique weak solution of (2.17)*, $\mathbb{Q}$ is a limit point of $\mathbb{Q}_n$ and $\mathcal{E}^{\alpha,\beta}_{m,\kappa,c}$ is defined in* (7.8)*.*

Remark 7.0.7. *If we restrict the supremum in the definiton of $\mathcal{E}^{\alpha,\beta}_{m,\kappa,c}$, see* (7.8)*, to functions $H \in C^{0,1}_c([0,T]\times(0,1))$, the statement of Proposition 7.0.6 reduces to*

$$\mathbb{E}_{\mathbb{Q}}\Big[\sup_{H\in C^{0,1}_c([0,T]\times(0,1))}\Big\{\langle\langle(\rho^\kappa)^m,\partial_u H\rangle\rangle - c\langle\langle H,H\rangle\rangle\Big\}\Big] \;\le\; M_0<\infty.$$

Proof of Proposition 7.0.6. For simplicity of the presentation, we will present the proof for even m since the proof for m odd is analogous and, when necessary, we explain the changes for the case m odd. We begin by noticing that the space $C^{0,1}\left([0,T]\times[0,1]\right)$ is separable with respect to the norm $\|H\|_\infty + \|\partial_u H\|_\infty$. Thus, it is enough to restrict the supremum inside the expectation in the statement of the proposition to functions H on a countable dense subset $\{H^q\}_{q\in\mathbb{N}}$ of $C^{0,1}([0,T]\times[0,1])$. In addition, since

$$\max_{l\le q}\left\{\mathcal{T}^{\alpha,\beta}_{\rho^\kappa,m}(H^l) - c\langle\langle H^l,H^l\rangle\rangle^{\alpha,\beta}_{\kappa,\rho^\kappa}\right\}\;\uparrow\;\sup_{H\in C^{0,1}([0,T]\times[0,1])}\left\{\mathcal{T}^{\alpha,\beta}_{\rho^\kappa,m}(H) - c\langle\langle H,H\rangle\rangle^{\alpha,\beta}_{\kappa,\rho^\kappa}\right\},$$

as $l\to\infty$, then by Monotone Convergence Theorem

$$\mathbb{E}_{\mathbb{Q}}\left[\max_{l\le q}\left\{\mathcal{T}^{\alpha,\beta}_{\rho^\kappa,m}(H^l) - c\langle\langle H^l,H^l\rangle\rangle^{\alpha,\beta}_{\kappa,\rho^\kappa}\right\}\right]\to\mathbb{E}_{\mathbb{Q}}\left[\sup_{H\in C^{0,1}([0,T]\times[0,1])}\left\{\mathcal{T}^{\alpha,\beta}_{\rho^\kappa,m}(H) - c\langle\langle H,H\rangle\rangle^{\alpha,\beta}_{\kappa,\rho^\kappa}\right\}\right],$$

when $q\to\infty$. Therefore, we are left to show that

$$\mathbb{E}_{\mathbb{Q}}\left[\max_{l\le q}\left\{\mathcal{T}^{\alpha,\beta}_{\rho^\kappa,m}(H^l) - c\langle\langle H^l,H^l\rangle\rangle^{\alpha,\beta}_{\kappa,\rho^\kappa}\right\}\right]\le M_0, \tag{7.18}$$

for any q and for some M_0 independent from q and κ. From (7.6) and (7.7), we have

$$\begin{aligned}\mathcal{T}^{\alpha,\beta}_{\rho^\kappa,m}(H^l) - c\langle\langle H^l,H^l\rangle\rangle^{\alpha,\beta}_{\kappa,\rho^\kappa} = \int_0^T\Bigg\{&\int_0^1\left(\rho^\kappa_s\right)^m(u)\,\partial_u H^l_s(u)\,du + \alpha^m H^l_s(0) - \beta^m H^l_s(1)\\ &- c\int_0^1 (H^l_s(u))^2du \;-\; \frac{c}{\kappa}P^\alpha_m\left(\rho^\kappa_s(0)\right)(H^l_s(0))^2 \;-\; \frac{c}{\kappa}P^\beta_m\left(\rho^\kappa_s(1)\right)(H^l_s(1))^2\Bigg\}\,ds.\end{aligned} \tag{7.19}$$

Now, from the same computations done to treat (5.13), we conclude that

$$\left|\int_0^T\frac{c}{\kappa}P^\alpha_m\left(\rho^k_s(0)\right)(H^l_s(0))^2\,ds - \int_0^T\frac{c}{\kappa}\sum_{i=0}^{m-1}\alpha^{m-1-i}\prod_{j=0}^{i-1}\langle\rho^\kappa_s,\overrightarrow{\iota}^{\,j\varepsilon}_{\varepsilon}\rangle(H^l_s(0))^2\,ds\right|$$

vanishes when $\varepsilon\to0$. By the definition of $P^\alpha_m(\cdot)$ given in (7.1), the product above is understood as being equal to one for $i=0$. Following the same computations done to treat (5.11) we can also conclude, for m even, that

$$\left|\int_0^T\int_0^1(\rho^\kappa_s)^m(u)\,\partial_u H^l_s(u)\,du\,ds - \int_0^T\int_{\varepsilon\frac{m}{2}}^{1-\varepsilon\frac{m}{2}}\prod_{i=0}^{\frac{m}{2}-1}\langle\rho^\kappa_s,\overleftarrow{\iota}^{\,u-i\varepsilon}_{\varepsilon}\rangle\prod_{i=0}^{\frac{m}{2}-1}\langle\rho^\kappa_s,\overrightarrow{\iota}^{\,u+i\varepsilon}_{\varepsilon}\rangle\partial_u H^l_s(u)\,du\,ds\right|$$

vanishes when $\varepsilon \to 0$. For m odd, we would replace the previous display by

$$\left| \int_0^T \int_0^1 (\rho_s^\kappa)^m(u)\, \partial_u H_s^l(u)\, du\, ds - \int_0^T \int_{\varepsilon \frac{m+1}{2}}^{1-\varepsilon \frac{m-1}{2}} \prod_{i=0}^{\frac{m+1}{2}-1} \langle \rho_s^\kappa, \overleftarrow{\iota}_\varepsilon^{u-i\varepsilon} \rangle \prod_{i=0}^{\frac{m-1}{2}-1} \langle \rho_s^\kappa, \overrightarrow{\iota}_\varepsilon^{u+i\varepsilon} \rangle \partial_u H_s^l(u)\, du\, ds \right|.$$

Observe that in the last two displays we changed the function $(\rho_s^\kappa)^m$ by a choice of products of the form (5.6), and despite not being the obvious change, it will be useful when we move to the microscopic system due to the result of Theorem 6.1.1.

To treat the boundary term at the right-hand side of (7.19), we can do exactly the same argument as we did to control the left boundary term. For simplicity of the presentation, we just present the arguments for the left boundary, but for the right, it is completely analogous. From here on, we neglect all the contributions from the right boundary.

From previous results, we are left to prove that there exist constants $c > 0$ and $M_0 > 0$ such that

$$\begin{aligned} \overline{\lim_{\varepsilon \to 0}}\, \mathbb{E}_{\mathbb{Q}} \Bigg[\max_{l \leq q} \Bigg\{ \int_0^T \Bigg(& \int_{\varepsilon \frac{m}{2}}^{1-\varepsilon \frac{m}{2}} \prod_{i=0}^{\frac{m}{2}-1} \langle \rho_s^\kappa, \overleftarrow{\iota}_\varepsilon^{u-i\varepsilon} \rangle \prod_{i=0}^{\frac{m}{2}-1} \langle \rho_s^\kappa, \overrightarrow{\iota}_\varepsilon^{u+i\varepsilon} \rangle \partial_u H_s^l(u)\, du + \alpha^m H_s^l(0) \\ & -c \int_0^1 (H_s^l(u))^2 du - \frac{c}{\kappa} \sum_{i=0}^{m-1} \alpha^{m-1-i} \prod_{j=0}^{i-1} \langle \rho_s^\kappa, \overrightarrow{\iota}_\varepsilon^{j\varepsilon} \rangle (H_s^l(0))^2 \Bigg) ds \Bigg\} \Bigg] \leq M_0. \end{aligned} \tag{7.20}$$

Now, we define the application $\Phi : \mathcal{D}([0,T], \mathcal{M}_+) \to \mathbb{R}$ by

$$\begin{aligned} \Phi(\pi_\cdot) = \max_{l \leq q} \Bigg\{ \int_0^T \Bigg(& \int_{\varepsilon \frac{m}{2}}^{1-\varepsilon \frac{m}{2}} \prod_{i=0}^{\frac{m}{2}-1} \langle \pi_s, \overleftarrow{\iota}_\varepsilon^{u-i\varepsilon} \rangle \prod_{i=0}^{\frac{m}{2}-1} \langle \pi_s, \overrightarrow{\iota}_\varepsilon^{u+i\varepsilon} \rangle\, \partial_u H_s^l(u)\, du + \alpha^m H_s^l(0) \\ & -c \int_0^1 (H_s^l(u))^2 du - \frac{c}{\kappa} \sum_{i=0}^{m-1} \alpha^{m-1-i} \prod_{j=0}^{i-1} \langle \pi_s, \overrightarrow{\iota}_\varepsilon^{j\varepsilon} \rangle (H_s^l(0))^2 \Bigg)\, ds \Bigg\}. \end{aligned}$$

The function Φ is lower semi-continuous and bounded with respect to the Skorokhod topology of $\mathcal{D}([0,T], \mathcal{M}_+)$. Therefore, recalling that $\mathbb{E}_n$ is the expectation with respect to the measure $\mathbb{Q}_n$, we can bound the expectation in (7.20) from above by

$$\begin{aligned} \underline{\lim_{n \to \infty}}\, \mathbb{E}_n \Bigg[\max_{l \leq q} \Bigg\{ \int_0^T \Bigg(& \int_{\varepsilon \frac{m}{2}}^{1-\varepsilon \frac{m}{2}} \prod_{i=0}^{\frac{m}{2}-1} \langle \pi_s^n, \overleftarrow{\iota}_\varepsilon^{u-i\varepsilon} \rangle \prod_{i=0}^{\frac{m}{2}-1} \langle \pi_s^n, \overrightarrow{\iota}_\varepsilon^{u+i\varepsilon} \rangle\, \partial_u H_s^l(u)\, du + \alpha^m H_s^l(0) \\ & -c \int_0^1 (H_s^l(u))^2 du - \frac{c}{\kappa} \sum_{i=0}^{m-1} \alpha^{m-1-i} \prod_{j=0}^{i-1} \langle \pi_s^n, \overrightarrow{\iota}_\varepsilon^{j\varepsilon} \rangle (H_s^l(0))^2 \Bigg)\, ds \Bigg\} \Bigg]. \end{aligned}$$

Now we want to compare this expression to its analogue at the microscopic level. To that end, fix $n \in \mathbb{N}$, $x \in \Sigma_n$, $\varepsilon > 0$, and recall that the definition of $\Sigma_{n,m}^\varepsilon$ in (5.5), where εn denotes $\lfloor \varepsilon n \rfloor$. Since the error from changing the integral in the space variable by its Riemann sum is of order $O(\frac{1}{n})$, we can rewrite last display as

$$\lim_{n\to\infty}\mathbb{E}_n\Bigg[\max_{l\leq q}\Bigg\{\int_0^T\Bigg(\frac{1}{n}\sum_{x\in\Sigma^\varepsilon_{n,m}}\prod_{i=0}^{\frac{m}{2}-1}\langle\pi^n_s,\overleftarrow{\iota}_\varepsilon^{x/n-i\varepsilon}\rangle\prod_{i=0}^{\frac{m}{2}-1}\langle\pi^n_s,\overrightarrow{\iota}_\varepsilon^{x/n+i\varepsilon}\rangle\,\partial_u H^l_s(\tfrac{x}{n})+\alpha^m H^l_s(0)$$
$$-c\int_0^1(H^l_s(u))^2du-\frac{c}{\kappa}\sum_{i=0}^{m-1}\alpha^{m-1-i}\prod_{j=0}^{i-1}\langle\pi^n_s,\overrightarrow{\iota}_\varepsilon^{j\varepsilon}\rangle(H^l_s(0))^2\Bigg)ds\Bigg\}\Bigg]. \tag{7.21}$$

Recall that the boxes of size ℓ to the left and to the right of site x are given by $\overleftarrow{\Lambda}^\ell_x:=\{x-\ell+1,\dots,x\}$ and $\overrightarrow{\Lambda}^\ell_x:=\{x,\dots,x+\ell-1\}$, respectively. Recall also that the empirical densities in the boxes $\overleftarrow{\Lambda}^\ell_x$ and $\overrightarrow{\Lambda}^\ell_x$ are given by

$$\overleftarrow{\eta}^\ell(x)=\frac{1}{\ell}\sum_{y\in\overleftarrow{\Lambda}^\ell_x}\eta(y)\quad\text{and}\quad\overrightarrow{\eta}^\ell(x)=\frac{1}{\ell}\sum_{y\in\overrightarrow{\Lambda}^\ell_x}\eta(y). \tag{7.22}$$

Now, just as we did in Chapter 5 we need to introduce the subset $\Sigma^\varepsilon_{n,m}$ of the bulk Σ_n. We use this set since, for each $x\in\Sigma^\varepsilon_{n,m}$ we will need to replace the occupation at site x by its average to the left or right of x on a box of size εn, and we are allowed to do so for $x\in\Sigma^\varepsilon_{n,m}$ but not for x on the whole bulk.

Recall from Chapter 5 that

$$\langle\pi^n_s,\overleftarrow{\iota}_\varepsilon^{x/n-i\varepsilon}\rangle=\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x-i\varepsilon n),\quad\langle\pi^n_s,\overrightarrow{\iota}_\varepsilon^{x/n+i\varepsilon}\rangle=\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1+i\varepsilon n)+O(\tfrac{1}{\varepsilon n}),$$

and

$$\langle\pi^n_s,\overrightarrow{\iota}_\varepsilon^{j\varepsilon}\rangle=\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(2+j\varepsilon n)+O(\tfrac{1}{\varepsilon n}),$$

for $i=0,\dots,\frac{m}{2}-1$ and $j=0,\dots,m-2$. Then, we can rewrite the expectation, now with respect to $\mathbb{P}_{\mu_n}$, in (7.21) as

$$\mathbb{E}_{\mu_n}\Bigg[\max_{l\leq q}\Bigg\{\int_0^T\Bigg(\frac{1}{n}\sum_{x\in\Sigma^\varepsilon_{n,m}}\prod_{i=0}^{\frac{m}{2}-1}\overleftarrow{\eta}^{\varepsilon n}_{sn^2}(x-i\varepsilon n)\prod_{i=0}^{\frac{m}{2}-1}\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(x+1+i\varepsilon n)\,\partial_u H^l_s(\tfrac{x}{n})+\alpha^m H^l_s(0)$$
$$-c\int_0^1(H^l_s(u))^2du-\frac{c}{\kappa}\sum_{i=0}^{m-1}\alpha^{m-1-i}\prod_{j=0}^{i-1}\overrightarrow{\eta}^{\varepsilon n}_{sn^2}(2+j\varepsilon n)(H^l_s(0))^2\Bigg)ds\Bigg\}\Bigg]$$

plus terms that vanish as $n\to\infty$. Now, recall the definition of $\tau_x h^m$ in (2.10). By putting together Theorem 6.1.1 and Theorem 6.2.3, we are left to show that

$$\varlimsup_{\varepsilon\to 0}\varlimsup_{n\to\infty}\mathbb{E}_{\mu_n}\Bigg[\max_{l\leq q}\Bigg\{\int_0^T\Bigg(\frac{1}{n}\sum_{x=1}^{n-2}\partial_u H^l_s(\tfrac{x}{n})\tau_x h^m(\eta_{sn^2})+\alpha^m H^l_s(0)$$
$$-c\int_0^1(H^l_s(u))^2du-\frac{c}{\kappa}\mathcal{R}^\alpha_m(\eta_{sn^2})(H^l_s(0))^2\Bigg)ds\Bigg\}\Bigg]\leq M_0, \tag{7.23}$$

where $\mathcal{R}^\alpha_m(\eta_{sn^2})$ is defined in (5.2). Observe that above we are back to the whole bulk since the replacement from $\Sigma^\varepsilon_{n,m}$ to Σ_n vanishes as $\varepsilon\to 0$. Now we want to change the initial measure μ_n to a suitable measure, here being the Bernoulli product measure $\nu^n_{\rho(\cdot)}$ satisfying the conditions of Lemma 4.2.1. We observe that to derive the inequality above we need to restrict $\alpha,\beta\in(0,1)$. Therefore, by entropy's and

Jensen's inequality, Proposition A.0.2, and the fact that $\exp\{\max_{l\leq q} a_l\} \leq \sum_{l=1}^{q} \exp\{a_l\}$, the expectation in (7.23) is bounded from above by

$$
\begin{aligned}
C(\alpha,\beta) + \frac{1}{n}\log \mathbb{E}_{\nu^n_{\rho(\cdot)}}\Bigg[\sum_{l=1}^{q}\exp\Bigg\{\int_0^T\Bigg(\sum_{x=1}^{n-2}\partial_u H^l_s(\tfrac{x}{n})\tau_x h^m(\eta_{sn^2}) + n\alpha^m H^l_s(0) \\
- n\,c\int_0^1 (H^l_s(u))^2 du - n\frac{c}{\kappa}\mathcal{R}^\alpha_m(\eta_{sn^2})(H^l_s(0))^2\Bigg)ds\Bigg\}\Bigg].
\end{aligned}
\tag{7.24}
$$

From the identity

$$
\overline{\lim_{n\to\infty}}\, n^{-1}\log(a_n+b_n) = \max\left\{\overline{\lim_{n\to\infty}}\, n^{-1}\log(a_n),\ \overline{\lim_{n\to\infty}}\, n^{-1}\log(b_n)\right\},
$$

in order to estimate (7.24), it is enough to bound

$$
\begin{aligned}
\frac{1}{n}\log \mathbb{E}_{\nu^n_{\rho(\cdot)}}\Bigg[\exp\Bigg\{\int_0^T\Bigg(\sum_{x=1}^{n-2}\partial_u H_s(\tfrac{x}{n})\tau_x h^m(\eta_{sn^2}) - n\frac{c}{\kappa}\mathcal{R}^\alpha_m(\eta_{sn^2})\,(H_s(0))^2 \\
- n\,c\int_0^1 (H_s(u))^2\,du + n\alpha^m H_s(0)\Bigg)ds\Bigg\}\Bigg],
\end{aligned}
$$

for a fixed function $H \in C^{0,1}([0,T]\times[0,1])$. Now, by Feynman-Kac's formula (see, for example, Lemma A.1 of [2]), we can bound the previous display from above by

$$
\begin{aligned}
\int_0^T \sup_f \Bigg\{\frac{1}{n}\int_{\Omega_n}\sum_{x=1}^{n-2}\partial_u H_s(\tfrac{x}{n})\tau_x h^m(\eta)f(\eta)\,d\nu^n_{\rho(\cdot)} - \frac{c}{\kappa}\,(H_s(0))^2\int_{\Omega_n}\mathcal{R}^\alpha_m(\eta)f(\eta)\,d\nu^n_{\rho(\cdot)} \\
- c\int_0^1 (H_s(u))^2\,du + \alpha^m H_s(0) + n\,\langle L^m_n\sqrt{f},\sqrt{f}\rangle_{\nu^n_{\rho(\cdot)}}\Bigg\}\,ds,
\end{aligned}
\tag{7.25}
$$

where the supremum is carried over all the densities f with respect to $\nu^n_{\rho(\cdot)}$. Now, recall the results related to the Dirichlet forms in Section 4.2. From a Taylor expansion on H, we can replace its space derivative by the discrete gradient $\nabla^-_n H_s\left(\frac{x}{n}\right) = n\left(H_s\left(\frac{x}{n}\right) - H_s\left(\frac{x-1}{n}\right)\right)$, by paying a price of order $O\left(\frac{1}{n}\right)$. Then, from a summation by parts, the first integral inside the supremum in (7.25) is equal to

$$
\begin{aligned}
&\int_{\Omega_n}\sum_{x=1}^{n-2} H_s\left(\tfrac{x}{n}\right)\left\{\tau_x h^m(\eta) - \tau_{x+1}h^m(\eta)\right\}f(\eta)\,d\nu^n_{\rho(\cdot)} \\
-&\int_{\Omega_n}\left\{H_s(0)\tau_1 h^m(\eta) - H_s\left(\tfrac{n-2}{n}\right)\tau_{n-1}h^m(\eta)\right\}f(\eta)\,d\nu^n_{\rho(\cdot)}.
\end{aligned}
\tag{7.26}
$$

Now, we need to bound both terms in expression (7.26) separately. We will call the term on the first line, the bulk term, and the one on the second line, the boundary term. Let us start by examining the bulk term. Recall (2.5), (2.9), (2.10), and (2.11). Thus, we can write the first line of (7.26) as

$$
\int_{\Omega_n}\sum_{x=1}^{n-2} H_s\left(\tfrac{x}{n}\right)c^m_{x,x+1}(\eta)(\eta(x)-\eta(x+1))f(\eta)\,d\nu^n_{\rho(\cdot)}. \tag{7.27}
$$

Writing the previous expression as one half of it plus one half of it, and by summing and subtracting

$\frac{1}{2}f(\eta^{x,x+1})$, we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{x=1}^{n-2} H_s\left(\tfrac{x}{n}\right)\int_{\Omega_n} c^m_{x,x+1}(\eta)(\eta(x)-\eta(x+1))(f(\eta)+f(\eta^{x,x+1})\,d\nu^n_{\rho(\cdot)}\\
&+\frac{1}{2}\sum_{x=1}^{n-2} H_s\left(\tfrac{x}{n}\right)\int_{\Omega_n} c^m_{x,x+1}(\eta)(\eta(x)-\eta(x+1))(f(\eta)-f(\eta^{x,x+1})\,d\nu^n_{\rho(\cdot)}.
\end{aligned}\tag{7.28}$$

Let $\bar{\eta}$ be the configuration η removing its value at the site x and $x+1$. Thus, we can rewrite the integral on the first line of (7.28) as

$$\begin{aligned}
\sum_{\bar{\eta}\in\Omega_{n-2}} &\Big(c^m_{x,x+1}(\eta)(f(\bar{\eta},1,0)+f(\bar{\eta},0,1))\rho\left(\tfrac{x}{n}\right)\left(1-\rho\left(\tfrac{x+1}{n}\right)\right)\\
&-c^m_{x,x+1}(\eta)(f(\bar{\eta},0,1)+f(\bar{\eta},1,0))\left(1-\rho\left(\tfrac{x}{n}\right)\right)\rho\left(\tfrac{x+1}{n}\right)\Big)\nu^{n-2}_{\rho(\cdot)}(\bar{\eta}),
\end{aligned}$$

which is equal to

$$\sum_{\bar{\eta}\in\Omega_{n-2}} c^m_{x,x+1}(\eta)(f(\bar{\eta},1,0)+f(\bar{\eta},0,1))\left(\rho\left(\tfrac{x}{n}\right)-\rho\left(\tfrac{x+1}{n}\right)\right)\nu^{n-2}_{\rho(\cdot)}(\bar{\eta}),$$

where the notation $f(\bar{\eta},0,1)$ means that we are computing $f(\eta)$ with $\eta(x)=0$ and $\eta(x+1)=1$. From the previous computation, we can bound the first line of (7.28) from above by

$$\frac{1}{2}\sum_{x=1}^{n-2} H_s\left(\tfrac{x}{n}\right)\left(\rho\left(\tfrac{x}{n}\right)-\rho\left(\tfrac{x+1}{n}\right)\right)\sum_{\bar{\eta}\in\Omega_{n-2}} c^m_{x,x+1}(\eta)(f(\bar{\eta},1,0)+f(\bar{\eta},0,1))\nu^{n-2}_{\rho(\cdot)}(\bar{\eta}).$$

Then, from Young's inequality, we get that the first line of (7.28) is bounded from above by

$$\frac{1}{4A}\sum_{x=1}^{n-2}\left(H_s\left(\tfrac{x}{n}\right)\right)^2+\frac{AC(\rho)m^2}{n},\tag{7.29}$$

for $A>0$, where $C(\rho)>0$. Let us now examine the second line of (7.28). Since $a-b=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ and $p^m_{x,x+1}(\eta)=c^m_{x,x+1}(\eta)\{a_{x,x+1}(\eta)+a_{x+1,x}(\eta)\}\neq 0$, from Young's inequality we can bound the second line of (7.28) from above by

$$\begin{aligned}
&\frac{B}{4}\sum_{x=1}^{n-2}\int_{\Omega_n} p^m_{x,x+1}(\eta)(\sqrt{f(\eta)}-\sqrt{f(\eta^{x,x+1})})^2\,d\nu^n_{\rho(\cdot)}\\
&+\frac{1}{4B}\sum_{x=1}^{n-2}\left(H_s\left(\tfrac{x}{n}\right)\right)^2\int_{\Omega_n}\frac{\left(c^m_{x,x+1}(\eta)\right)^2(\eta(x)-\eta(x+1))^2}{p^m_{x,x+1}(\eta)}(\sqrt{f(\eta)}+\sqrt{f(\eta^{x,x+1})})^2\,d\nu^n_{\rho(\cdot)},
\end{aligned}$$

for $B>0$. From (4.9) and the definition of $p^m_{x,x+1}(\eta)$, we can bound the previous expression from above by

$$\frac{B}{4}D^m_P\left(\sqrt{f},\nu^n_{\rho(\cdot)}\right)+\frac{1}{4B}\sum_{x=1}^{n-2}\left(H_s\left(\tfrac{x}{n}\right)\right)^2\int_{\Omega_n} c^m_{x,x+1}(\eta)\left(\sqrt{f(\eta)}+\sqrt{f(\eta^{x,x+1})}\right)^2\,d\nu^n_{\rho(\cdot)}.$$

Since $(a+b)^2\leq 2a^2+2b^2$, $c^m_{x,x+1}(\eta)\leq m$, and f is a density with respect to $\nu^n_{\rho(\cdot)}$, we can bound the previous expression by

$$\frac{B}{4}D_P^m\left(\sqrt{f},\nu_{\rho(\cdot)}^n\right)+\frac{1}{2B}\sum_{x=1}^{n-2}\left(H_s\left(\tfrac{x}{n}\right)\right)^2\left(m+\int_{\Omega_n}c_{x,x+1}^m(\eta^{x,x+1})f(\eta)\frac{d\,\nu_{\rho(\cdot)}^n(\eta^{x,x+1})}{d\,\nu_{\rho(\cdot)}^n(\eta)}\,d\nu_{\rho(\cdot)}^n\right).$$

From Proposition A.0.3, last expression is bounded from above by

$$\frac{B}{4}D_P^m\left(\sqrt{f},\nu_{\rho(\cdot)}^n\right)+\frac{1}{2B}\sum_{x=1}^{n-2}\left(H_s\left(\tfrac{x}{n}\right)\right)^2\left(m+m\hat{C}(\alpha,\beta)\right),\tag{7.30}$$

where $\hat{C}(\alpha,\beta)$ is a positive constant. Therefore, combining (7.29) and (7.30), we can bound (7.27) from above by

$$\frac{1}{4A}\sum_{x=1}^{n-2}\left(H_s\left(\tfrac{x}{n}\right)\right)^2+\frac{AC(\rho)m^2}{n}+\frac{B}{4}D_P^m\left(\sqrt{f},\nu_{\rho(\cdot)}^n\right)+\frac{1}{2B}\sum_{x=1}^{n-2}\left(H_s\left(\tfrac{x}{n}\right)\right)^2\left(m+m\hat{C}(\alpha,\beta)\right).$$

Taking $A=\frac{n}{m^2}$ and $B=n$, we get

$$\frac{n}{4}D_P^m\left(\sqrt{f},\nu_{\rho(\cdot)}^n\right)+\sum_{x=1}^{n-2}\left(H_s\left(\tfrac{x}{n}\right)\right)^2\left(\frac{m^2}{4n}+\frac{m(1+\hat{C}(\alpha,\beta))}{2n}\right)+C(\rho),$$

which is bounded by

$$\frac{n}{4}D_P^m(\sqrt{f},\nu_{\rho(\cdot)}^n)+\frac{1}{n}\sum_{x=1}^{n-2}\left(H_s\left(\tfrac{x}{n}\right)\right)^2\left(m+\tilde{C}(m,\alpha,\beta)\right)+C(\rho),\tag{7.31}$$

with $\tilde{C}(m,\alpha,\beta)=m^2+m\hat{C}(\alpha,\beta)$, where $\hat{C}(\alpha,\beta)$ is a positive constant. Now we need to examine the second line of (7.26). Let us begin by examining the leftmost term given by

$$-\int_{\Omega_n}H_s\left(0\right)\tau_1h^m(\eta)f(\eta)\,d\nu_{\rho(\cdot)}^n.\tag{7.32}$$

Recall that we neglected above all the terms from the right boundary so that we will not treat the contribution from the rightmost term on the second line of (7.26), but it is completely analogous to what we do for the left boundary. Recall from (2.10) and (5.1) that

$$\tau_1h^m(\eta)=\sum_{k=0}^{m-1}\alpha^k\prod_{j=1}^{m-k}\eta(j)-\sum_{k=1}^{m-1}\alpha^k\prod_{j=2}^{m+1-k}\eta(j),$$
$$\tau_1h^m(\eta)-\alpha^m=(\eta(1)-\alpha)\mathcal{R}_m^\alpha(\eta).$$

Summing and subtracting α^m in (7.32), and since f is a density with respect to $\nu_{\rho(\cdot)}^n$, we can rewrite (7.32) as

$$H_s\left(0\right)\left(\int_{\Omega_n}(\alpha-\eta(1))\mathcal{R}_m^\alpha(\eta)f(\eta)\,d\nu_{\rho(\cdot)}^n-\alpha^m\right).\tag{7.33}$$

The argument to estimate the leftmost term in (7.33) is similar, in essence, to the one used to treat the first term of (7.26). We write the leftmost term in (7.33) as one half of it plus one half of it, and by summing and subtracting $\frac{1}{2}f(\eta^1)$, we obtain

$$\frac{1}{2}H_s(0)\int_{\Omega_n}(\alpha-\eta(1))\mathcal{R}_m^\alpha(\eta)(f(\eta)+f(\eta^1))\,d\nu_{\rho(\cdot)}^n+\frac{1}{2}H_s(0)\int_{\Omega_n}(\alpha-\eta(1))\mathcal{R}_m^\alpha(\eta)(f(\eta)-f(\eta^1))\,d\nu_{\rho(\cdot)}^n. \tag{7.34}$$

Denoting by $\tilde{\eta}$ the configuration η removing its value at site 1, and noticing that $\mathcal{R}_m^\alpha(\eta)$ does not depend on $\eta(1)$, we can write the first term in (7.34) as

$$\begin{aligned}\frac{1}{2}H_s(0)\sum_{\tilde{\eta}\in\Omega_{n-1}}\Bigg(&\alpha\mathcal{R}_m^\alpha(\eta)(f(0,\tilde{\eta})+f(1,\tilde{\eta}))\left(1-\rho\left(\tfrac{1}{n}\right)\right)\\&+(\alpha-1)\mathcal{R}_m^\alpha(\eta)(f(1,\tilde{\eta})+f(0,\tilde{\eta}))\rho\left(\tfrac{1}{n}\right)\Bigg)\nu_{\rho(\cdot)}^{n-1}(\tilde{\eta}),\end{aligned}$$

where the notation $f(1,\tilde{\eta})$ (resp. $f(0,\tilde{\eta})$) means that we are computing $f(\eta)$ with $\eta(1)=1$ (resp. $\eta(1)=0$). Hence, the previous expression is equal to

$$\frac{1}{2}H_s(0)\left(\alpha-\rho\left(\tfrac{1}{n}\right)\right)\sum_{\tilde{\eta}\in\Omega_{n-1}}\mathcal{R}_m^\alpha(\eta)(f(0,\tilde{\eta})+f(1,\tilde{\eta}))\nu_{\rho(\cdot)}^{n-1}(\tilde{\eta}).$$

Since $\mathcal{R}_m^\alpha(\eta)\le m$, ρ satisfies the conditions we imposed, and since f is a density with respect to $\nu_{\rho(\cdot)}^n$, the previous expression vanishes when $n\to\infty$. Let us now estimate the second term of (7.34). Combining the identity $a-b=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ and Young's inequality, we can bound it from above by

$$\begin{aligned}&\frac{A}{2}\int_{\Omega_n}I_1^\alpha(\eta)\left(\sqrt{f(\eta)}-\sqrt{f(\eta^1)}\right)^2d\nu_{\rho(\cdot)}^n\\+&\frac{(H_s(0))^2}{8A}\int_{\Omega_n}\frac{(\alpha-\eta(1))^2\left(\mathcal{R}_m^\alpha(\eta)\right)^2}{I_1^\alpha(\eta)}\left(\sqrt{f(\eta)}+\sqrt{f(\eta^1)}\right)^2d\nu_{\rho(\cdot)}^n,\end{aligned}$$

where $A>0$ and $I_1^\alpha(\eta)$ is defined in (2.7). Recall the definition of $F_1^\alpha(\sqrt{f},\nu_{\rho(\cdot)}^n)$ in (4.11). Using the inequality $(a+b)^2\le 2a^2+2b^2$ and the identity $I_1^\alpha(\eta)=\frac{(\alpha-\eta(1))^2}{I_1^\alpha(\eta)}$, last expression can be bounded from above by $\frac{A}{2}F_1^\alpha(\sqrt{f},\nu_{\rho(\cdot)}^n)$ plus

$$\frac{(H_s(0))^2}{4A}\left(\int_{\Omega_n}\left(\mathcal{R}_m^\alpha(\eta)\right)^2I_1^\alpha(\eta)f(\eta)\,d\nu_{\rho(\cdot)}^n+\int_{\Omega_n}\left(\mathcal{R}_m^\alpha(\eta)\right)^2I_1^\alpha(\eta)f(\eta^1)\,d\nu_{\rho(\cdot)}^n\right).$$

After a change of variables in the second integral of the previous expression, we get

$$\begin{aligned}\frac{A}{2}F_1^\alpha(\sqrt{f},\nu_{\rho(\cdot)}^n)+\frac{(H_s(0))^2}{4A}\Bigg(&\int_{\Omega_n}\left(\mathcal{R}_m^\alpha(\eta)\right)^2I_1^\alpha(\eta)f(\eta)\,d\nu_{\rho(\cdot)}^n\\&+\int_{\Omega_n}\left(\mathcal{R}_m^\alpha(\eta)\right)^2I_1^\alpha(\eta^1)\tfrac{d\nu_{\rho(\cdot)}^n(\eta^1)}{d\nu_{\rho(\cdot)}^n(\eta)}f(\eta)\,d\nu_{\rho(\cdot)}^n\Bigg).\end{aligned}$$

Since $I_1^\alpha(\eta) = I_1^\alpha(\eta^1)\frac{d\nu^n_{\rho(\cdot)}(\eta^1)}{d\nu^n_{\rho(\cdot)}(\eta)}$, last expression is equal to

$$\frac{A}{2}F_1^\alpha(\sqrt{f},\nu^n_{\rho(\cdot)}) + \frac{(H_s(0))^2}{2A}\int_{\Omega_n}(\mathcal{R}^\alpha_m(\eta))^2 I_1^\alpha(\eta)f(\eta)\,d\nu^n_{\rho(\cdot)}.$$

Thus, taking $A = \frac{\kappa n}{2n^\theta}$ and since $(\mathcal{R}^\alpha_m(\eta))^2 I_1^\alpha(\eta) \leq (\mathcal{R}^\alpha_m(\eta))^2 \leq m\mathcal{R}^\alpha_m(\eta)$, the first boundary term of (7.26) is bounded from above by

$$\frac{n}{4}\frac{\kappa}{n^\theta}F_1^\alpha(\sqrt{f},\nu^n_{\rho(\cdot)}) + \frac{m}{\kappa}\frac{n^\theta}{n}(H_s(0))^2\int_{\Omega_n}\mathcal{R}^\alpha_m(\eta)f(\eta)\,d\nu^n_{\rho(\cdot)} - \alpha^m H_s(0)\,. \tag{7.35}$$

Thus, combining (7.31) and (7.35), we bound (7.26) from above by

$$\begin{aligned}&\frac{n}{4}D_P^m(\sqrt{f},\nu^n_{\rho(\cdot)}) + \Big(m+\tilde{C}(m,\alpha,\beta)\Big)\frac{1}{n}\sum_{x=1}^{n-2}\left(H_s\left(\tfrac{x}{n}\right)\right)^2 + C(\rho)\\ &\qquad+\frac{n}{4}\frac{\kappa}{n^\theta}F_1^\alpha(\sqrt{f},\nu^n_{\rho(\cdot)}) + \frac{m}{\kappa}\frac{n^\theta}{n}(H_s(0))^2\int_{\Omega_n}\mathcal{R}^\alpha_m(\eta)f(\eta)\,d\nu^n_{\rho(\cdot)} - \alpha^m H_s(0)\,,\end{aligned}$$

plus the terms that come from the right boundary and which are very similar to the ones we obtained for the left boundary. Therefore, taking $c = m+\tilde{C}(m,\alpha,\beta)$ in (7.25), from last expression and (4.14), we can bound (7.24) from above by

$$\begin{aligned}C(\alpha,\beta)+\int_0^T\Bigg\{&\Big(m+\tilde{C}(m,\alpha,\beta)\Big)\frac{1}{n}\sum_{x=1}^{n-2}\left(H_s\left(\tfrac{x}{n}\right)\right)^2 - \Big(m+\tilde{C}(m,\alpha,\beta)\Big)\int_0^1(H_s(u))^2\,du\\ &+C(\rho)+\frac{(H_s(0))^2}{\kappa}\left(m\left(\frac{n^\theta}{n}-1\right)-\tilde{C}(m,\alpha,\beta)\right)\int_{\Omega_n}\mathcal{R}^\alpha_m(\eta)f(\eta)\,d\nu^n_{\rho(\cdot)}\Bigg\}ds.\end{aligned}$$

Above $C(\alpha,\beta)$ is the one in (7.24). Noting that $\tilde{C}(m,\alpha,\beta)$ and $\mathcal{R}^\alpha_m(\eta)$ as defined in (5.2) are non-negative we can bound from above the last display by

$$\begin{aligned}C(\alpha,\beta)+\int_0^T\Bigg\{&\Big(m+\tilde{C}(m,\alpha,\beta)\Big)\left(\frac{1}{n}\sum_{x=1}^{n-2}\left(H_s\left(\tfrac{x}{n}\right)\right)^2 - \int_0^1(H_s(u))^2\,du\right)\\ &+C(\rho)+\frac{m}{\kappa}(H_s(0))^2\left(\tfrac{n^\theta}{n}-1\right)\int_{\Omega_n}\mathcal{R}^\alpha_m(\eta)f(\eta)\,d\nu^n_{\rho(\cdot)}\Bigg\}ds.\end{aligned} \tag{7.36}$$

Now, recall that $\theta = 1$. Therefore, the previous expression converges to $TC(\rho)$, as $n\to\infty$. From all this, we conclude that the expectation in (7.17) is bounded from above by $M_0 := C(\alpha,\beta)+TC(\rho)$. This ends the proof. $\square$

Remark 7.0.8. *We note that due to the rightmost term in* (7.36)*, the energy estimate with test functions without compact support can only be obtained for $\theta = 1$. Even in the case $\theta < 1$, where the factor $\frac{m}{\kappa}\frac{n^\theta}{n}(H_s(0))^2\int_{\Omega_n}\mathcal{R}^\alpha_m(\eta)f(\eta)\,d\nu^n_{\rho(\cdot)}$ would simply vanish as $n\to\infty$, there would remain the term $\frac{m}{\kappa}(H_s(0))^2\int_{\Omega_n}\mathcal{R}^\alpha_m(\eta)f(\eta)\,d\nu^n_{\rho(\cdot)}ds$ which would blow up since we are taking the supremum over functions f.*

Remark 7.0.9. *The proof of the energy estimate in the Dirichlet and Neumann cases follows from re-*

stricting the supremum of $\mathcal{E}^{\alpha,\beta}_{m,\kappa,c}$ in Proposition 7.0.6 to functions $H \in C^{0,1}_c([0,T] \times (0,1))$, such that the statement of Proposition 7.0.6 reduces to

$$\mathbb{E}_{\mathbb{Q}}\Big[\sup_{H \in C^{0,1}_c([0,T]\times(0,1))} \Big\{\langle\langle(\rho)^m, \partial_u H\rangle\rangle - c\langle\langle H, H\rangle\rangle\Big\}\Big] \leq M_0 < \infty,$$

where ρ is the unique weak solution of (2.15) *or* (2.17) *(with $\kappa = 0$). Since in these cases we have functions with compact support, the proof follows the same steps done above and noticing that the boundary term of* (7.28) *is negligible (order $O(\frac{1}{n})$).*

Chapter 8

Fick's law

This chapter aims examining one of the questions that arises when studying diffusive systems out of equilibrium. This chapter regards the Fick's law of diffusion, namely Theorem 8.2.1, derived by Adolf Fick in [18], that says that the rate of the flux of particles is proportional to the density gradient. The Robin boundary conditions that we have in our hydrodynamic equation, i.e., $\partial_t \rho^m(t,0) = \kappa(\alpha - \rho(t,0))$ and $\partial_t \rho^m(t,1) = \kappa(\rho(t,1) - \beta)$, give us an idea of the transport of mass through the macroscopic points 0 and 1 - the rate at which mass crosses borders is proportional to the difference in the density. Although we studied scaling limits for the empirical density of the PMM with slow reservoirs in the previous sections, we study scaling limits for the empirical currents of this model. The motivation comes from [5], in which the authors derived the large deviation principle for the empirical currents of the SSEP in the domain $\{-n, n\}$ with creation and annihilation of particles in the bulk and a Glauber dynamics at the boundaries.

The result we prove here is a consequence of the hydrodynamic limit proved through the previous sections. Then, recall the definitions and results from that part to connect with the one we will present here. We start this chapter by defining some important quantities of interest, namely, the integrated currents and the empirical measures associated with these currents. After defining them, we present the proof of Theorem 8.2.1.

8.1 Integrated currents

Recall the definition of the instantaneous current between sites x and $x+1$ in (2.9). The idea of this section is to define the integrated current. For $T > 0$, let $t \in [0,T]$. For any $x \in \Sigma_n \cup \{0\}$, we denote by $N_t^n(x)$ the total number of particles that jumped from site x to $x+1$ in an interval of time $[0, tn^2]$, and by $\tilde{N}_t^n(x)$ the total number of particles that jumped from site $x+1$ to x in the same time interval. Thus, we define the integrated current at time t and location x by

$$J_t^n(x) := N_t^n(x) - \tilde{N}_t^n(x), \quad \text{for } x \in \Sigma_n \cup \{0\}. \tag{8.1}$$

In other words, $J_t^n(x)$ denotes the flux of particles through the bond $(x, x+1)$ in an interval of time $[0, tn^2]$. The integrated current (8.1) can be written in terms of its conservative and non-conservative parts. We denote by $Q_t^n(x)$ the conservative integrated current at time t and location x, which records the particles jump from the diffusive part of the dynamics (PMM and SSEP) and it is given by

$$Q_t^n(x) := J_t^n(x), \quad \text{for } x \in \Sigma_{n-1}. \tag{8.2}$$

In the same way, we denote by $K_t^n(x)$ the non-conservative integrated current at time t and location x, which records the particles inserted and removed from the system at sites 1 or $n-1$ (Glauber dynamics)

$$K_t^n(x) := J_t^n(x), \quad \text{for } x \in \{0, n-1\}. \tag{8.3}$$

Having disposed of this preliminary step, for $x \in \{0, n-1\}$ we can now define the infinitesimal generator of the joint process $\{\eta_t, J_t^n(x)\}_{t\geq 0}$ as

$$\tilde{L}_n^m f(\eta, J^n(x)) = \tilde{L}_P^m f(\eta, J^n(x)) + n^{a-2}\tilde{L}_S f(\eta, J^n(x)) + \tilde{L}_\alpha f(\eta, J^n(x)) + \tilde{L}_\beta f(\eta, J^n(x)), \tag{8.4}$$

for $x \in \Sigma_n \cup \{0\}$. To simplify notation, let $q^m_{x,x+1}(\eta) = a_{x,x+1}(\eta)(c^m_{x,x+1}(\eta) + n^{a-2})$. For each $x \in \Sigma_{n-1}$, we define the identity (8.4) into two parts, one corresponding to the jumps in the bulk, that is given by

$$\begin{aligned}\left(\tilde{L}_P^m + n^{a-2}\tilde{L}_S\right) f(\eta, J^n(x)) &= q^m_{x,x+1}(\eta)\left(f(\eta^{x,x+1}, J^n(x)+1) - f(\eta, J^n(x))\right)\\ &+\ q^m_{x+1,x}(\eta)\left(f(\eta^{x,x+1}, J^n(x)-1) - f(\eta, J^n(x))\right)\\ &+ \sum_{\substack{y\in\Sigma_{n-1}\\ y\neq x}} \left(q^m_{y,y+1}(\eta) + q^m_{y+1,y}(\eta)\right)\left(f(\eta^{y,y+1}, J^n(y)) - f(\eta, J^n(y))\right),\end{aligned} \tag{8.5}$$

and another one corresponding to the jumps in the boundary

$$\begin{aligned}\tilde{L}_\alpha f(\eta, J^n(0)) &= \frac{m}{n^\theta}\Big(\alpha(1-\eta(1))\left(f(\eta^1, J^n(0)+1) - f(\eta, J^n(0))\right)\Big)\\ &+ \frac{\kappa}{n^\theta}\Big((1-\alpha)\eta(1)\left(f(\eta^1, J^n(0)-1) - f(\eta, J^n(0))\right)\Big),\\ \tilde{L}_\beta f(\eta, J^n(n-1)) &= \frac{\kappa}{n^\theta}\Big(\beta(1-\eta(n-1))\left(f(\eta^{n-1}, J^n(n-1)-1) - f(\eta, J^n(n-1))\right)\\ &+ (1-\beta)\eta(n-1)\left(f(\eta^{n-1}, J^n(n-1)+1) - f(\eta, J^n(n-1))\right)\Big).\end{aligned} \tag{8.6}$$

Remark 8.1.1. *As in the previous sections, the process is sped up in the diffusive time scale* tn^2*.*

Remark 8.1.2. *It is worth noting that we recover the infinitesimal generator of* $\{\eta_t\}_{t\geq 0}$*, which is defined in* (2.1)*, if we take* $f(\eta, J) = f(\eta)$ *in* (8.5) *and* (8.6)*. Moreover, if we take* f *being the projection in the second variable, that is,* $f(\eta, J) = J$*, in* (8.5) *and* (8.6)*, we recover the instantaneous current through the bond* $(x, x+1)$ *as we can see below:*

For $x \in \Sigma_{n-1}$, we have

$$\begin{aligned}\left(\tilde{L}_P^m + n^{a-2}\tilde{L}_S\right) J^n(x) &= q^m_{x,x+1}(\eta)\left(J^n(x)+1-J^n(x)\right) + q^m_{x+1,x}(\eta)\left(J^n(x)-1-J^n(x)\right)\\ &= j^m_{x,x+1}(\eta).\end{aligned}$$

For the left (resp. right) boundary, we have

$$\begin{aligned}\tilde{L}_\alpha J^n(0) &= \frac{\kappa}{n^\theta}\Big(\alpha(1-\eta(1))\left(J^n(0)+1-J^n(0)\right)\Big)\\ &+ \frac{\kappa}{n^\theta}\Big(((1-\alpha)\eta(1))\left(J^n(0)-1-J^n(0)\right)\Big)\\ &= j^m_{0,1}(\eta),\end{aligned}$$

$$\begin{aligned}\tilde{L}_\beta J^n(n-1) &= \frac{\kappa}{n^\theta}\Big(\beta(1-\eta(n-1))\left(J^n(n-1)-1-J^n(n-1)\right)\Big)\\ &+ \frac{\kappa}{n^\theta}\Big(((1-\beta)\eta(n-1))\left(J^n(n-1)+1-J^n(n-1)\right)\Big)\\ &= j^m_{n-1,n}(\eta).\end{aligned}$$

Now, having defined the integrated currents, in the next section we will defined empirical measures associated with them.

8.2 Empirical measures

Recall the definition of the empirical measure in (2.19) and the conservative current in (8.2). Fix $t \in [0,T]$. For $\eta \in \Omega_n$, the empirical measure associated with the conservative current is defined as the signed measure on $[0,1]$

$$Q_t^n := \frac{1}{n^2}\sum_{x=1}^{n-2} Q_t^n(x)\delta_{x/n}, \tag{8.7}$$

where δ_u is the Dirac measure concentrated on $u \in [0,1]$. Note that the renormalization factor of order n^2 arises in (8.7) because we need to take into account the space renormalization and the diffusive scaling of the PMM and SSEP dynamics. Now, recall the definition (8.3). The empirical measure associated with this boundary current is defined as

$$K_t^n := \frac{1}{n}K_t^n(0)\delta_0 + \frac{1}{n}K_t^n(n-1)\delta_{n-1/n}. \tag{8.8}$$

Since (8.8) is related to the Glauber part of the dynamics, we only need to take into account the space renormalization factor of order n. Let $f \in C^1([0,1])$ be a test function. We define the empirical density of particles at time t, that is, the integral of f with respect to the empirical measure π_t^n, as

$$\langle \pi_t^n, f\rangle = \frac{1}{n}\sum_{x\in\Sigma_n} \eta_{tn^2}(x) f\left(\tfrac{x}{n}\right).$$

In the same manner, the current field J_t^n is defined as

$$J_t^n(f) := Q_t^n(f) + K_t^n(f),$$

where Q_t^n is the conservative current field

$$Q_t^n(f) := \frac{1}{n^2}\sum_{x=1}^{n-2} f\left(\tfrac{x}{n}\right) Q_t^n(x),$$

and K_t^n is the non-conservative current field

$$K_t^n(f) := \frac{1}{n}\left[f(0)K_t^n(0) - f\left(\tfrac{n-1}{n}\right) K_t^n(n-1)\right].$$

Theorem 8.2.1 (Fick's law)**.** *Under the same hypothesis of Theorem 2.3.1, for any $t \in [0,T]$ and any $\delta > 0$, we have*

$$\lim_{n\to\infty} \mathbb{P}_{\mu_n}\Big(\eta_\cdot \in \mathcal{D}([0,T],\Omega_n) : \Big|\frac{1}{n^2}\sum_{x=1}^{n-2} Q_t^n(x) H\left(\tfrac{x}{n}\right) - \int_0^1 H(u)\nabla\rho_t^m(u)\,du\Big| > \delta\Big) = 0,$$

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}_{\mu_n}\Big(\eta_\cdot \in \mathcal{D}([0,T],\Omega_n) : \Big|&\frac{1}{n}\left(H(0)K_t^n(0) - H\left(\tfrac{n-1}{n}\right) K_t^n(n-1)\right)\\ &- \mathbb{1}_{\{\theta=1\}}\kappa\int_0^t H(0)(\alpha-\rho_s(0)) + H(1)(\beta-\rho_s(1))ds\Big| > \delta\Big) = 0,\end{aligned}$$

where $\rho(\cdot)$ stands for a weak solution of the PME as Theorem 2.3.1, and $Q_t^n(x)$, $K_t^n(0)$ and $K_t^n(n-1)$ stand for the conserved and non-conserved integrated currents defined in (8.2) *and* (8.3)*, respectively.*

Remark 8.2.2. *From the previous theorem we have that J^n converges weakly to $J\,du$, where J is the weak solution of*

$$J = -D(\rho)\nabla\rho = -\nabla\rho^m.$$

8.3 Proof of Fick's law

In this section, we prove Theorem 8.2.1, that is, the validity of Fick's law: the current of particles that enter (resp. exit) from the system is at all times equal to the local density gradient at 0 (resp. 1). In order to prove it we will use the fact that we proved the validity of the hydrodynamic limit for the model, and we can then use the replacement lemmas stated in Chapter 6.

The result stated in Theorem 8.2.1, that we will prove here, is the Law of large numbers for the empirical measures defined in (8.7) and (8.8).

Proof. Let us prove the first identity of the theorem. Our proof starts with the observation that by Dynkin's formula (see Lemma A1.5.1 of [31]), for a fixed test function $H \in C^1([0,1])$, we have that

$$M_t^n(H) = Q_t^n(H) - Q_0^n(H) - \int_0^t n^2 \tilde{L}_n^m Q_s^n(H)\,ds, \tag{8.9}$$

is a martingale with respect to the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which vanishes as $n\to\infty$ in $L^2(\mathbb{P}_{\mu_n})$ (see Appendix A). Note that $Q_0^n(H)=0$. Hence, we can write (8.9) as

$$M_t^n(H)=Q_t^n(H)-\int_0^t\sum_{x=1}^{n-2}H\left(\tfrac{x}{n}\right)j_{x,x+1}^m(\eta_{sn^2})\,ds.$$

Since the PMM is a gradient model, performing a summation by parts in the previous expression, we can write (8.9) as

$$Q_t^n(H)-\int_0^t\frac{1}{n}\sum_{x=1}^{n-2}\nabla_n^-H\left(\tfrac{x}{n}\right)\tau_xh^m(\eta_{sn^2})+H\left(\tfrac{0}{n}\right)\tau_1h^m(\eta_{sn^2})-H\left(\tfrac{n-1}{n}\right)\tau_{n-1}h^m(\eta_{sn^2})\,ds,\tag{8.10}$$

where $\tau_xh^m(\eta_{sn^2})$ is defined in (2.10). Thus, we want to examine the convergence of (8.10) for each value of $\theta\geq 0$.

If $\theta<1$, the test function vanishes at the boundary. From the hydrodynamic limit and Theorem 6.1.1, we have that the integral term of (8.10) converges in $\mathbb{P}_{\mu_n}$, as $n\to\infty$ to

$$\begin{aligned}\int_0^t\int_0^1\nabla H(u)\rho_s^m(u)\,du\,ds=&\int_0^t H(1)\rho_s^m(1)-H(0)\rho_s^m(0)\,ds\\&-\int_0^t\int_0^1 H(u)\nabla\rho_s^m(u)\,du\,ds,\end{aligned}$$

which is equal to $-\int_0^t\int_0^1 H(u)\nabla\rho_s^m(u)\,du\,ds$.

If $\theta\geq 1$, the test function does not necessarily vanishes at the boundary. From the hydrodynamic limit, Theorems 6.1.1 and 6.2.3 it follows that the integral term of (8.10) converges in $\mathbb{P}_{\mu_n}$, as $n\to\infty$ to

$$\int_0^t\int_0^1\nabla H(u)\rho_s^m(u)\,du\,ds+\int_0^t H(0)\rho_s^m(0)-H(1)\rho_s^m(1)\,ds,$$

which is also equal to $-\int_0^t\int_0^1 H(u)\nabla\rho_s^m(u)\,du\,ds$.

Let us now prove the second identity of the theorem regarding the boundary current. In the same manner, for $H\in C^1([0,1])$ we have that

$$\tilde{M}_t^n(H)=K_t^n(H)+\kappa\int_0^t\frac{n}{n^\theta}\left[H(\tfrac{1}{n})(\alpha-\eta_{sn^2}(1))+H(\tfrac{n-1}{n})(\beta-\eta_{sn^2}(n-1))\right]\,ds,\tag{8.11}$$

is also a martingale that vanishes in $L^2(\mathbb{P}_{\mu_n})$ as $n\to\infty$, see Appendix A. Now, we want to examine the convergence of the integral term of (8.11), for each value of $\theta\geq 0$.
If $\theta<1$, the test function vanishes at the boundary, and by a Taylor expansion on H we get

$$\frac{\kappa}{n^\theta}\int_0^t-\nabla_n^+H(0)(\alpha-\eta_{sn^2}(1))-\nabla_n^-H\,(1)\,(\beta-\eta_{sn^2}(n-1))\,ds.$$

The previous expression is bounded from above by

$$\frac{\kappa}{n^\theta}\|\nabla H\|_\infty\int_0^t|\alpha-\eta_{sn^2}(1)|+|\beta-\eta_{sn^2}(n-1)|\,ds,$$

which vanishes as $n \to \infty$. If $\theta = 0$, the test function vanishes at the boundary, and by Lemma 6.2.1 we have that the integral term of (8.11) vanishes. If $\theta = 1$, the test function does not vanishes at the boundary, and from Theorem 6.2.2 we have that the integral term of (8.11) converges in $\mathbb{P}_{\mu_n}$, as $n \to \infty$ to

$$\kappa \int_0^t H(0)(\alpha - \rho(s,0)) + H(1)(\beta - \rho(s,1))\, ds.$$

Finally, if $\theta > 1$, we have that the integral term of (8.11) is bounded from above by

$$\frac{\kappa}{n^{\theta-1}} \|H\|_\infty \int_0^t |\alpha - \eta_{sn^2}(1)| + |\beta - \eta_{sn^2}(n-1)|\; ds,$$

which vanishes as $n \to \infty$, concluding the proof. $\square$

Chapter 9

Uniqueness of weak solutions

In this chapter, we prove the uniqueness of weak solutions of the hydrodynamic equations defined in Section 2.2. As we mentioned above, uniqueness is fundamental in the proof of the hydrodynamic limit using the Entropy Method. We start covering the Dirichlet case, in which we use Oleinik's trick, and we finish the chapter presenting the uniqueness for the Robin case. We remark that both methods presented below cover the Neumann case. We decided to include a brief description at the end of the proof for the Dirichlet case stating what would be the differences for the Neumann case.

Before presenting the proofs, suppose that $\rho_1(t,u)$ and $\rho_2(t,u)$ are weak solutions of the PME starting from the same initial condition $g(\cdot)$ and with suitable boundary conditions for each problem.

9.1 The Dirichlet and Neumann cases

Suppose that $\rho_1(t,u)$ and $\rho_2(t,u)$ are weak solutions of (2.15) starting from the same initial condition $g(\cdot)$. Performing an integration by parts in (2.15), we get

$$\langle \rho_1(T,\cdot)-\rho_2(T,\cdot), G_T\rangle + \int_0^T \langle \partial_u \rho_1^m(t,\cdot)-\partial_u\rho_2^m(t,\cdot), \partial_u G_t\rangle\, dt - \int_0^T \langle \rho_1(t,\cdot)-\rho_2(t,\cdot), \partial_t G_t\rangle\, dt = 0, \tag{9.1}$$

for all $G \in C_0^{1,2}([0,T]\times[0,1])$. Observe that the left-hand side of this identity is well defined even if we assume only that $G\in L^2(0,T;\mathcal{H}_0^1)$ and $\partial_t G \in L^2(0,T;L^2[0,1])$, where $\mathcal{H}_0^1([0,1])$ is the closure in $\mathcal{H}^1([0,1])$ of the space $C_c^\infty([0,1])$. In fact, by mollifying such G we can approximate it by smooth functions $G_k \in C_0^{1,2}([0,T]\times[0,1])$ and, using a limit argument, conclude that (9.1) holds for G, since it holds for G_k. We leave the details to the reader and we refer to [42] for more details. Now we consider the function $\zeta\in L^2(0,T;\mathcal{H}_0^1)$ such that $\partial_t\zeta \in L^2(0,T;L^2[0,1])$ given by

$$\zeta(t,u) = \begin{cases} \int_t^T \rho_1^m(s,u)-\rho_2^m(s,u)\, ds\,, & \text{if } 0<t<T\,,\\ 0\,, & \text{if } t\geq T\,,\end{cases}$$

where $T > 0$. Note that $\zeta(t,0) = \zeta(t,1) = 0$ for all $t \in [0,T]$, comes from the fact that $\rho_1(t,u)$ and $\rho_2(t,u)$ satisfy item (3) of Definition 4. Observe that

$$\begin{aligned} \partial_t \zeta(t,u) &= -\left(\rho_1^m(t,u) - \rho_2^m(t,u)\right) \in L^2([0,T]\times[0,1]), \\ \partial_u \zeta(t,u) &= \int_t^T \left(\partial_u \rho_1^m(s,u) - \partial_u \rho_2^m(s,u)\right) ds \in L^2([0,T]\times[0,1]). \end{aligned} \tag{9.2}$$

Replacing G by ζ in (9.1), we have

$$\int_0^T \langle \partial_u \rho_1^m(t,\cdot) - \partial_u \rho_2^m(t,\cdot), \partial_u \zeta_t \rangle \, dt - \int_0^T \langle \rho_1(t,\cdot) - \rho_2(t,\cdot), \partial_t \zeta_t \rangle \, dt = 0.$$

Using (9.2) it follows that

$$\begin{aligned} \int_0^1 \int_0^T \Bigg\{ &(\rho_1(t,u) - \rho_2(t,u))(\rho_1^m(t,u) - \rho_2^m(t,u)) \\ &+ \left(\partial_u \rho_1^m(t,u) - \partial_u \rho_2^m(t,u)\right) \left(\int_t^T (\partial_u \rho_1^m(s,u) - \partial_u \rho_2^m(s,u)\, ds \right) \Bigg\} \, dt\, du = 0, \end{aligned}$$

that is

$$\int_0^1 \left\{ \int_0^T (\rho_1(t,u) - \rho_2(t,u))(\rho_1^m(t,u) - \rho_2^m(t,u))\, dt + \frac{1}{2} \left(\int_0^T (\partial_u \rho_1^m(t,u) - \partial_u \rho_2^m(t,u))\, dt \right)^2 \right\} du = 0.$$

From last identity, we conclude that $\rho_1(t,u) = \rho_2(t,u)$ a.s. in $[0,T]\times[0,1]$. Now, we remark that the previous proof also shows uniqueness in the Neumann case. The only difference with respect to the proof above is that we do not need to require the profile $\rho(\cdot)$ to have a fixed value at the boundary. We now give a sketch of the proof in this case. Suppose that $\rho_1(t,u)$ and $\rho_2(t,u)$ are now weak solutions of (2.17) (with $\kappa = 0$), starting from the same initial condition $g(\cdot)$. Performing an integration by parts in (2.17) (with $\kappa = 0$), we get

$$\langle \rho_1(T,\cdot) - \rho_2(T,\cdot), G_T \rangle + \int_0^T \langle \partial_u \rho_1^m(t,\cdot) - \partial_u \rho_2^m(t,\cdot), \partial_u G_t \rangle \, dt - \int_0^T \langle \rho_1(t,\cdot) - \rho_2(t,\cdot), \partial_t G_t \rangle \, dt = 0,$$

for all $G \in C^{1,2}([0,T]\times[0,1])$. Note that the last equation is exactly the same as in (9.1). Now, by the same arguments used in the Dirichlet case, we can reach the same conclusion for the Neumann case.

9.2 The Robin case

We adapt Filo's proof to our equation (see [19], Theorem 3), and we present it in details below. Although the proof there holds for any spatial dimension, we consider only the one-dimensional case. Before starting the proof, we need some technical results. The following result is concerning the final value problem with Robin boundary conditions:

Lemma 9.2.1. *Suppose that $a = a(t,u)$ is a positive $C^{2,2}([0,T]\times[0,1])$ function, $b = b(t,u)$ is a positive*

$C^2([0,T])$ function, for $u=0$ and $u=1$, $h=h(u)\in C_0^2([0,1])$, and $\lambda\geq 0$. Then, for $t\in(0,T]$, the problem with Robin conditions

$$\begin{cases} \partial_s\varphi + a\Delta\,\varphi = \lambda\varphi, \quad \text{for} \quad (s,u)\in[0,t)\times(0,1), \\ \partial_u\varphi(s,0) = b(s,0)\,\varphi(s,0), \quad \text{for} \quad s\in[0,t), \\ \partial_u\varphi(s,1) = -b(s,1)\,\varphi(s,1), \quad \text{for} \quad s\in[0,t), \\ \varphi(t,u) = h(u), \quad \text{for} \quad u\in(0,1), \end{cases} \tag{9.3}$$

has a unique solution φ_0 in $C^{1,2}([0,t]\times[0,1])$. Moreover, if $0\leq h\leq 1$, then

$$0\leq\varphi_0(s,u)\leq e^{-\lambda(t-s)}, \quad \text{for} \quad (s,u)\in[0,t]\times[0,1]. \tag{9.4}$$

Proof. First, observe that by setting $\tau=t-s$ and $\zeta(\tau,u)=e^{-\lambda(t-\tau)}\varphi(t-\tau,u)$, (9.3) is equivalent to

$$\begin{cases} \partial_\tau\zeta - a\Delta\,\zeta = 0, \quad \text{for} \quad (\tau,u)\in(0,t]\times(0,1), \\ \partial_u\zeta(\tau,0) = b(t-\tau,0)\,\zeta(\tau,0), \quad \text{for} \quad \tau\in(0,t], \\ \partial_u\zeta(\tau,1) = -b(t-\tau,1)\,\zeta(\tau,1), \quad \text{for} \quad \tau\in(0,t], \\ \zeta(0,u) = e^{-\lambda t}h(u), \quad \text{for} \quad u\in(0,1), \end{cases} \tag{9.5}$$

which has a unique $C^{1,2}([0,t]\times[0,1])$ solution $\zeta_0(\tau,u)$ according to [33] (see Theorem 5.3) or [34] (see Theorem 4). Now, we need to show that $0\leq\zeta_0\leq e^{-\lambda t}$ in $[0,t]\times[0,1]$, under the assumption that $0\leq h\leq 1$. Suppose that

$$\max_{[0,t]\times[0,1]}\zeta_0 > e^{-\lambda t}.$$

From the maximum principle for parabolic equations,

$$\max_{[0,t]\times[0,1]}\zeta_0 = \max_{\Sigma_t\cup(\{0\}\times[0,1])}\zeta_0,$$

where $\Sigma_t=([0,t]\times\{0\})\cup([0,t]\times\{1\})$. Since $\zeta_0(0,u)=e^{-\lambda t}h(u)\leq e^{-\lambda t}$, for $0\leq u\leq 1$, there exists some $(\tau_1,u_1)\in\Sigma_t$ that realizes the maximum of ζ_0. Suppose, without loss of generality, that $u_1=0$. Observe that $\tau_1>0$, due to the fact that ζ_0 is continuous in $[0,t]\times[0,1]$ and $\zeta_0(0,0)=e^{-\lambda t}h(0)=0$. Since $\zeta_0(\tau_1,u_1)>e^{-\lambda t}$ and b is positive, it follows that

$$\partial_u\zeta_0(\tau_1,0) = b(t-\tau_1,0)\,\zeta_0(\tau_1,0) > 0.$$

Hence, for $u>0$ sufficiently close to 0, we have

$$\zeta_0(\tau_1,u) > \zeta_0(\tau_1,0),$$

contradicting the fact that $(\tau_1, 0)$ is a point of maximum of ζ_0. Therefore, $\zeta_0 \le e^{-\lambda t}$. By an analogous argument, we can prove that $\zeta_0 \ge 0$, concluding that $0 \le \zeta_0 \le e^{-\lambda t}$.

Now, let $\varphi_0(s,u) = e^{\lambda s}\zeta_0(t-s,u)$. As we have already mentioned, since ζ_0 is the solution of (9.5), then φ_0 is the solution of (9.3). Furthermore, since $0 \le \zeta_0 \le e^{-\lambda t}$, we have that $0 \le \varphi_0(s,u) \le e^{-\lambda(t-s)}$, which proves the lemma. □

Lemma 9.2.2. *Let φ_0 be the solution of the parabolic problem* (9.3). *There exists a positive constant $C = C(b,h)$ such that*

$$\int_0^t \int_0^1 a(s,u)(\Delta\varphi_0(s,u))^2\, duds \le C(b,h).$$

Proof. Multiplying the first line of (9.3) by $\Delta\varphi_0(s,u)$, and integrating it in space and time, we obtain

$$\int_0^t \int_0^1 \partial_s\varphi_0\, \Delta\varphi_0\, du\, ds + \int_0^t \int_0^1 a(\Delta\,\varphi_0)^2\, du\, ds - \int_0^t \int_0^1 \lambda\varphi_0\, \Delta\varphi_0\, duds = 0.$$

Integrating last equation by parts, we have

$$\begin{aligned}
&\int_0^t \partial_s\varphi_0(s,1)\, \partial_u\varphi_0(s,1)\, ds - \int_0^t \partial_s\varphi_0(s,0)\, \partial_u\varphi_0(s,0)\, ds \\
&- \frac{1}{2}\int_0^t \int_0^1 \partial_s|\partial_u\varphi_0|^2\, duds + \int_0^t \int_0^1 a(\Delta\,\varphi_0)^2\, du\, ds \\
&- \int_0^t \lambda\varphi_0(s,1)\, \partial_u\varphi_0(s,1)\, ds + \int_0^t \lambda\varphi_0(s,0)\, \partial_u\varphi_0(s,0)\, ds + \int_0^t \int_0^1 \lambda\, |\partial_u\varphi_0|^2\, duds = 0\,.
\end{aligned}$$

Integrating the third term in the last equation and using the boundary conditions, it follows that

$$\begin{aligned}
&\int_0^t \int_0^1 \big(a(\Delta\,\varphi_0)^2 + \lambda\, |\partial_u\varphi_0|^2\big)\, du\, ds + \int_0^t \lambda b(s,1)(\varphi_0(s,1))^2\, ds + \int_0^t \lambda b(s,0)(\varphi_0(s,0))^2\, ds \\
&- \int_0^t \partial_s\varphi_0(s,1)\, b(s,1)\varphi_0(s,1)\, ds - \int_0^t \partial_s\varphi_0(s,0)\, b(s,0)\varphi_0(s,0)\, ds \\
&- \frac{1}{2}\int_0^1 |\partial_u\varphi_0|^2(t,u) - |\partial_u\varphi_0|^2(0,u)\, du = 0\,.
\end{aligned}$$

Now, doing an integration by parts on the fourth and fifth terms in the above display, and using the initial condition, we obtain:

$$\begin{aligned}
&\int_0^t \int_0^1 \big(a(\Delta\,\varphi_0)^2 + \lambda\, |\partial_u\varphi_0|^2\big)\, du\, ds + \int_0^t \lambda b(s,1)(\varphi_0(s,1))^2\, ds + \int_0^t \lambda b(s,0)(\varphi_0(s,0))^2\, ds \\
&- \frac{1}{2}b(t,1)(\varphi_0(t,1))^2 + \frac{1}{2}b(0,1)(\varphi_0(0,1))^2 + \frac{1}{2}\int_0^t \partial_s b(s,1)(\varphi_0(s,1))^2\, ds \\
&- \frac{1}{2}b(t,0)(\varphi_0(t,0))^2 + \frac{1}{2}b(0,0)(\varphi_0(0,0))^2 + \frac{1}{2}\int_0^t \partial_s b(s,0)(\varphi_0(s,0))^2\, ds \\
&- \frac{1}{2}\int_0^1 |h'(u)|^2\, du + \frac{1}{2}\int_0^1 |\partial_u\varphi_0|^2(0,u)\, du = 0\,.
\end{aligned}$$

Therefore,

$$\int_0^t\int_0^1 a(\Delta\,\varphi_0)^2\,du\,ds \leq \frac{1}{2}\int_0^1 |h'(u)|^2\,du$$
$$+\frac{1}{2}b(t,1)(\varphi_0(t,1))^2-\frac{1}{2}b(0,1)(\varphi_0(0,1))^2-\frac{1}{2}\int_0^t \partial_s b(s,1)(\varphi_0(s,1))^2\,ds$$
$$+\frac{1}{2}b(t,0)(\varphi_0(t,0))^2-\frac{1}{2}b(0,0)(\varphi_0(0,0))^2-\frac{1}{2}\int_0^t \partial_s b(s,0)(\varphi_0(s,0))^2\,ds\,.$$

Since φ_0 is bounded, according to Lemma 9.2.1, the right-hand side of last inequality is bounded from above by some constant C, that depends only on h and b. □

Before presenting the uniqueness of weak solutions of the hydrodynamic equation with Robin boundary conditions, we need two more technical results:

Lemma 9.2.3. *Let b be a nonnegative and bounded measurable function in $[0,T]$ and $1\leq p<\infty$. There exists a sequence $\{b_k\}_{k\in\mathbb{N}}$ of positive functions in $C^\infty([0,T])$, such that b_k converges to b in $L^p([0,T])$ and*

$$\left\|\frac{b}{b_k}-1\right\|_{L^p(A)}\to 0,$$

where $A=\{t\in(0,T]\,:\,b(t)>0\}$.

Proof. Let $\varepsilon_k=1/k>0$. Consider a sequence of positive numbers $\{\delta_j\}_{j\in\mathbb{N}}$, such that $\delta_j\to 0$. Since $b>0$ in A, we have

$$\frac{b(t)}{b(t)+\delta_j}-1\to 0\quad\text{for any } t\in A\;\text{ as } j\to\infty,\quad\text{ and }\quad\left|\frac{b(t)}{b(t)+\delta_j}-1\right|<2.$$

From the dominated convergence theorem, $b/(b+\delta_j)-1$ converges to 0 in $L^p(A)$. Hence, for a large j_0, we have

$$\left\|\frac{b}{b+\delta_{j_0}}-1\right\|_{L^p(A)}<\frac{\varepsilon_k}{2}. \tag{9.6}$$

Let $\{c_m\}_{m\in\mathbb{N}}$ be a sequence in $C^\infty([0,T])$, such that $c_m\to b+\delta_{j_0}$ in $L^p([0,T])$. Since $b+\delta_{j_0}\geq\delta_{j_0}$, we can assume that $c_m\geq\delta_{j_0}$. Then

$$\left\|\frac{b}{c_m}-\frac{b}{b+\delta_{j_0}}\right\|_{L^p(0,T)}=\left\|\frac{b(b+\delta_{j_0}-c_m)}{c_m(b+\delta_{j_0})}\right\|_{L^p(0,T)}\leq\frac{\|b\|_{L^\infty([0,T])}\|b+\delta_{j_0}-c_m\|_{L^p([0,T])}}{\delta_{j_0}^2}.$$

Hence, using that $c_m\to b+\delta_{j_0}$ in $L^p([0,T])$, for a large m_0, we have that

$$\left\|\frac{b}{c_{m_0}}-\frac{b}{b+\delta_{j_0}}\right\|_{L^p(0,T)}<\frac{\varepsilon_k}{2}. \tag{9.7}$$

Defining $b_k=c_{m_0}$, (9.6) and (9.7) imply that

$$\left\|\frac{b}{b_k}-1\right\|_{L^p(A)}<\varepsilon_k,$$

proving the result. $\square$

Remark 9.2.4. *Using the same argument above, we can prove the following result that is used in [19]: if a is a nonnegative bounded measurable function in $[0,T]\times[0,1]$, then there exists a sequence $\{a_k\}_{k\in\mathbb{N}}$ of positive C^∞ functions in time and space, such that*

$$\frac{1}{k}\leq a_k\leq \|a\|_{L^\infty}+\frac{1}{k}\quad\text{and}\quad \left\|\frac{a-a_k}{\sqrt{a_k}}\right\|_{L^2([0,T]\times[0,1])}\to 0.$$

Proof of uniqueness for the Robin case ([19]): Although the proof that we will present is true for $\kappa\geq 0$, we will only consider the case $\kappa>0$. But the interested reader can check that for $k=0$, the proof also holds. Suppose that $\rho_1(t,u)$ and $\rho_2(t,u)$ are weak solutions of (2.17). We stress that throughout this proof we will use the notation

$$w(t,u)=\rho_1(t,u)-\rho_2(t,u)\ \text{ and }\ v(t,u)=\sum_{i=0}^{m-1}\rho_1^{m-1-i}(t,u)\rho_2^i(t,u), \tag{9.8}$$

for $(t,u)\in[0,T]\times[0,1]$, and we note that $w_t(u)v_t(u)=\rho_1^m(t,u)-\rho_2^m(t,u)$. Since $\rho_1(t,u)$ and $\rho_2(t,u)$ satisfy (2.18), we get

$$\begin{aligned}\langle w_t,G_t\rangle-\int_0^t\langle w_s,\partial_sG_s\rangle\,ds-\int_0^t\langle w_s,v_s\Delta G_s\rangle\,ds+\int_0^t w_s(1)v_s(1)\partial_uG_s(1)-w_s(0)v_s(0)\partial_uG_s(0)\,ds\\ +\kappa\int_0^t w_s(0)G_s(0)+w_s(1)G_s(1)ds=0,\end{aligned}$$

for all $G\in C^{1,2}([0,T]\times[0,1])$. Thus, the previous equation can be rewritten as

$$\begin{aligned}\langle w_t,G_t\rangle\ =&\ \int_0^t\langle w_s,\partial_sG_s+v_s\Delta G_s\rangle\,ds-\int_0^t w_s(1)\left(\kappa G_s(1)+v_s(1)\partial_uG_s(1)\right)ds\\ &+\int_0^t w_s(0)\left(v_s(0)\partial_uG_s(0)-\kappa G_s(0)\right)ds\,.\end{aligned} \tag{9.9}$$

To estimate the integrals above we need to use a suitable test function, which is the solution of the parabolic equation (9.3). Unfortunately, the function v above does not have regularity enough. To avoid this difficulty, observe that $0\leq v(t,u)\leq m$, since $0\leq\rho_1(t,u),\rho_2(t,u)\leq 1$. Then, according to Lemma 9.2.3, taking b equal to v, and $p=1$, for $\varepsilon>0$ there exists a positive function $b_\varepsilon\in C^2([0,T]\times\{0,1\})$ such that

$$\left\|\frac{v(t,u_i)}{b_\varepsilon(t,u_i)}-1\right\|_{L^1(A_i)}<\varepsilon\quad\text{for}\quad i\in\{0,1\}, \tag{9.10}$$

where $u_0=0$, $u_1=1$ and $A_i=\{t\in(0,T]\ :\ v(t,u_i)>0\}$. Moreover, from Remark 9.2.4 with $a=v$, there exists a sequence of functions $\{a_n\}_{n\in\mathbb{N}}$ in $C^\infty([0,T]\times[0,1])$, such that

$$\frac{1}{n}\leq a_n\leq m+\frac{1}{n}\quad\text{and}\quad\frac{a_n-v}{\sqrt{a_n}}\to 0\quad\text{in}\quad L^2([0,T]\times[0,1])\quad\text{as}\quad n\to\infty. \tag{9.11}$$

For fixed $\lambda=0$ and $h\in C_0^2([0,1])$, consider the parabolic problem (9.3) with a and b replaced by a_n and

κ/b_ε, respectively. Observe that κ/b_ε is a positive C^2 function. Then, from Lemma 9.2.1 there exists a unique solution $\varphi_n(s,u)$ to this problem associated to a_n and κ/b_ε.

Now, for $G(s,u) = \varphi_n(s,u)$, we estimate each integral of the right-hand side of (9.9). For the first integral, using the fact that φ_n is a solution of (9.3) (with $\lambda = 0$), and the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&\int_0^t \langle w_s, \partial_s\varphi_n(s,\cdot) + v_s\Delta\varphi_n(s,\cdot)\rangle \, ds \\
&= \int_0^t \langle w_s, \partial_s\varphi_n(s,\cdot) + a_n(s,\cdot)\Delta\varphi_n(s,\cdot)\rangle \, ds + \int_0^t \langle w_s, (v_s - a_n(s,\cdot))\Delta\varphi_n(s,\cdot))\rangle \, ds \\
&\leq \int_0^t \left\| w_s \, \frac{(v-a_n)}{\sqrt{a_n}} \right\|_{L^2([0,1])} \|\sqrt{a_n}\Delta\varphi_n\|_{L^2([0,1])} \, ds \,.
\end{aligned}
$$

Hence, from Cauchy-Schwarz inequality, (9.4), Lemma 9.2.2, and $|w_s| = |\rho_1 - \rho_2| \leq 2$, we have

$$
\int_0^t \langle w_s, \partial_s\varphi_n(s,\cdot) + v_s\Delta\varphi_n(s,\cdot)\rangle \, ds \;\leq\; 2 \left\| \frac{(v-a_n)}{\sqrt{a_n}} \right\|_{L^2([0,T]\times[0,1])} \sqrt{C(\kappa/b_\varepsilon, h)} \,. \tag{9.12}
$$

For the boundary integrals of (9.9) we use the Robin condition satisfied by φ_n. For the right-hand side of the boundary ($u_1 = 1$), we have

$$
\partial_u\varphi_n(s,1) = -\frac{\kappa}{b_\varepsilon(s,1)} \, \varphi_n(s,1).
$$

Then, for $G(s,u) = \varphi_n(s,u)$, the second integral on the right-hand side of (9.9) becomes

$$
\int_0^t w_s(1)(\kappa\varphi_n(s,1) + v_s(1)\partial_u\varphi_n(s,1)) \, ds \;=\; \int_0^t w_s(1) \left(\kappa\varphi_n(s,1) - v_s(1)\frac{\kappa}{b_\varepsilon(s,1)}\varphi_n(s,1) \right) \, ds \,.
$$

Note that if $s_0 \notin A_1^t := \{s \in [0,t] \;:\; v_s(1) > 0\}$, then $\rho_1(s_0,1) = \rho_2(s_0,1) = 0$ and, therefore, $w(s_0,1) = 0$. Hence, from the fact that $|w| \leq 2$, and (9.4) together with the choice $\lambda = 0$, we get

$$
\begin{aligned}
\left| \int_0^t w_s(1)(\kappa\varphi_n(s,1) + v_s(1)\partial_u\varphi_n(s,1)) \, ds \right| &= \left| \int_{A_1^t} w_s(1) \left(\kappa\varphi_n(s,1) - v_s(1)\frac{\kappa}{b_\varepsilon(s,1)}\varphi_n(s,1) \right) \, ds \right| \\
&\leq 2\kappa \left\| 1 - \frac{v_s(1)}{b_\varepsilon(s,1)} \right\|_{L^1(A_1^t)} \,.
\end{aligned}
$$

Then, using (9.10) and that $A_1^t \subset A_1$, we have

$$
\left| \int_0^t w_s(1)(\kappa\varphi_n(s,1) + v_s(1)\partial_u\varphi_n(s,1)) \, ds \right| \leq 2\kappa\varepsilon. \tag{9.13}
$$

By an analogous argument, we also have

$$
\left| \int_0^t w_s(0)(v_s(0)\partial_u\varphi_n(s,0) - \kappa\varphi_n(s,0)) \, ds \right| \leq 2\kappa\varepsilon. \tag{9.14}
$$

Therefore, from the fact that $\varphi_n(t,u) = h(u)$, and from (9.9), (9.12), (9.13), and (9.14), we conclude that

$$\langle w_t, h\rangle \;\leq\; 2\left\|\frac{(v-a_n)}{\sqrt{a_n}}\right\|_{L^2([0,T]\times[0,1])}\sqrt{C(\kappa/b_\varepsilon, h)} \;+\; 4\kappa\varepsilon\,.$$

Taking $n\to\infty$ and using (9.11), it follows that

$$\langle w_t, h\rangle \;\leq\; 4\kappa\varepsilon.$$

Since $\varepsilon>0$ is arbitrary,

$$\langle w_t, h\rangle \;\leq\; 0,$$

for any $h\in C_0^2([0,1])$. Now, consider a sequence $h_n\in C_0^2([0,1])$ such that $h_n(\cdot)\to \mathbb{1}_{\{u\in[0,1]\,:\,w_t(u)>0\}}(t,\cdot)$ in $L^2([0,1])$. Then, from the last inequality, we obtain

$$\int_0^1 w^+(t,u)\,du \;\leq\; 0,$$

where $w^+=\max\{w,0\}$. Therefore, for any $t\in[0,T]$, $\rho_1(t,u)\leq\rho_2(t,u)$ for almost every $u\in[0,1]$. That is, $\rho_1\leq\rho_2$ for almost every $(t,u)\in[0,T]\times[0,1]$. In the same way, $\rho_2\leq\rho_1$ a.e., completing the proof. □

Chapter 10

Future work

There are a couple of questions that still have no answer and are left for future work. We highlight one that is concerned with the *hydrostatic limit*. In our result on the hydrodynamic limit, we need to impose the starting measure to be associated with a profile, see (2.20). We note that when the boundary rates α and β coincide with ρ, the Bernoulli product measure with constant parameter ρ is a reversible measure for this model and, in particular, it is invariant. Nevertheless, when $\alpha \neq \beta$, this measure is no longer invariant, and we have no information on the invariant measure of the system. The matrix method of Derrida [11] can not be straightforwardly applied to this model due to the complicated action of the bulk dynamics. One way to prove that the invariant measure of the model is associated with a profile, namely the stationary profile of the respective hydrodynamic equation, is to prove that its space correlations decay to 0 when $n \to \infty$. For this model, it is not easy to obtain information on the correlations since the generator of the process does not preserve the degree of functions of η. Another interesting problem is to derive the hydrodynamic limit for the PMM with slow reservoirs without the perturbation with the SSEP jumps. We will face difficulty in studying how long it takes for the Glauber dynamics to create a mobile cluster in the system. Due to the recent advances in the study of nonequilibrium fluctuations of interacting particle systems, see [30], another interesting problem is to study the nonequilibrium fluctuations for the PMM. Another interesting problem is studying the hydrodynamic limit and the large deviation principle of the PMM with long jumps. The first is a work in progress, and we hope to finish it soon.

Bibliography

[1] Andersen, H., Kob, W.: Kinetic lattice-gas model of cage effects in high-density liquids and a test of mode-coupling theory of the ideal-glass transition. Phys. Rev. E 48, 4359–4363, (1993).

[2] Baldasso, R., Menezes, O., Neumann, A., Souza, R.: Exclusion Process with Slow Boundary. Journal of Statistical Physics, Volume 167, Issue 5, 1112–1142, (2017).

[3] Barenblatt, G.I.: On some unsteady motions of a liquid or a gas in a porous medium. Prikl. Mat. Mekh., 16(1), 67–78 (in Russian), (1952).

[4] Bernardin, C., Gonçalves, P., Jimenez, B.: Slow to fast infinitely extended reservoirs for the symmetric exclusion process with long jumps, Markov Processes and Related Fields, no. 25, 217-274, (2019).

[5] Bodinou T., Lagouge M.: Large deviations of the empirical currents for a boundary driven reaction diffusion model. The Annals of Applied Probability Vol. 22, No. 6, 2282–2319, (2012).

[6] Bonorino, L., de Paula, R., Gonçalves, P. et al. Hydrodynamics of Porous Medium Model with Slow Reservoirs. J Stat Phys 179, 748–788, (2020). https://doi.org/10.1007/s10955-020-02550-y

[7] Cancrini, N., Martinelli, F., Roberto, C., Toninelli, C.: Kinetically constrained lattice gases. Communications in Mathematical Physics 297(2): 299–344, (2010).

[8] De Masi, A., Presutti, E.: Mathematical methods for hydrodynamic limits, volume1501 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, (1991).

[9] de Paula, R., Gonçalves, P., Neumann, A.: Energy estimates and convergence of weak solutions of the porous medium equation, (2021). https://arxiv.org/abs/2101.08900

[10] de Paula, R., Gonçalves, P., Neumann, A.: Porous medium model in contact with slow reservoirs. From Particle Systems to Partial Differential Equations, Springer International Publishing, 123–147, (2018).

[11] Derrida, B., Evans, M., Hakim, V., Pasquier, V.: Exact solution of a 1D asymmetric exclusion model using a matrix formulation. Journal of Physics A: Mathematical and General 26 no. 7, 1493, (1993).

[12] Dobrušin, R. L.: Markov processes with a large number of locally interacting components: Existence of a limit process and its ergodicity. Problemy Peredači Informacii, 7(2):70–87, (1971).

[13] Dobrušin, R. L.: Markov processes with a large number of locally interacting components: The invertible case and certain generalizations. Problemy Peredači Informacii, 7(3):57–66, (1971).

[14] Ekhaus, M., Seppalainen, T.: Stochastic Dynamics Macroscopically Governed by the Porous Medium Equation. Annales- Academia e Scientiarum Fennicae Mathematica 21(2): 309–352, (1996).

[15] Evans, L.: *Partial Differential Equations*. Graduate Studies in Mathematics, American Mathematical Society, (1998).

[16] Feller, W.: An introduction to a probability theory and its applications, Vol. 2, Wiley series in probability and mathematical statistics, (1970).

[17] Feng, S., Iscoe, I., Seppalainen, T.: A microscopic mechanism for the porous medium equation. Stochastic Processes and their Applications 66 147–182, (1997).

[18] Fick A.: Ueber Diffusion. Annalen der Physik. 94 (1): 5986, (1855).

[19] Filo, J.: A nonlinear diffusion equation with nonlinear boundary conditions: methods of lines. Mathematica Slovaca, Volume 38, Issue 3, 273–296, (1988).

[20] Fourier, J.: Théorie analytique de la Chaleur; reprint of the 1822 original: Éditions Jacques Gabay, Paris, 1988. English version: The Analytical Theory of Heat, Dover, New York, (1955).

[21] Franco, T., Gonçalves, P., Neumann, A.: Hydrodynamical behavior of symmetric exclusion with slow bonds, Annales de l'Institut Henri Poincaré: Probability and Statistics, Volume 49, No. 2, 402–427, (2013).

[22] Franco, T., Gonçalves, P., Neumann, A.: Phase transition of a heat equation with Robin's boundary conditions and exclusion process. Trans. Amer. Math. Soc. 367, 6131-6158, (2015).

[23] Franco, T., Tavares, M.: Hydrodynamic Limit for the SSEP with a Slow Membrane. J Stat Phys 175, 233–268, (2019).

[24] Gonçalves, P.: Equilibrium Fluctuations for Totally Asymmetric Particle Systems: exclusion and zero-range processes, VDM Verlag Dr. Müller, (2010).

[25] Gonçalves, P.: Hydrodynamics for symmetric exclusion in contact with reservoirs, Stochastic Dynamics Out of Equilibrium, Institut Henri Poincaré, Paris, France, 2017, Springer Proceedings in Mathematics and Statistics book series, 137–205, (2019).

[26] Gonçalves, P., Landim, C., Toninelli, C.: Hydrodynamic Limit for a Particle System with degenerate rates. Ann. Inst. H. Poincaré: Probab. Statist, Volume 45, Issue 4, 887–909, (2009).

[27] Guo, M.Z., Papanicolaou, G.C., Varadhan, S.R.S.: Nonlinear diffusion limit for a system with nearest neighbor interactions. Commun. Math. Phys. 118, 31–59, (1988).

[28] Gurney, W., Nisbet, R.: The regulation of inhomogeneous populations. J. Theoret. Biol. 52 441–457, (1975).

[29] Gurtin, M., MacCarny, R.: On the diffusion of biological populations. Math. Biosci. 33 35–49, (1977).

[30] Jara, M., Menezes, O.: Non-equilibrium Fluctuations of Interacting Particle Systems, (2018). https://arxiv.org/abs/1810.09526

[31] Kipnis, C., Landim, C.: *Scaling Limits of Interacting Particle Systems*. Grundlehrender Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 320. Springer-Verlag, Berlin, (1999).

[32] Kompaneets, A. S., Zel'dovich, Ya. B: Towards a theory of heat conduction with thermal conductivity depending on the temperature. In Collection of Papers Dedicated to 70th Anniversary of A. F. Ioffe. Izd. Akad. Nauk SSSR, Moscow, pp. 61–72, (1950).

[33] Ladyženskaja, A., Solonnikov, A., Ural'ceva, N.: Linear and Quasi-linear Equations of Parabolic Type. Amer. Math. Soc., Providence, RI, (1968).

[34] Lieberman, M.: Mixed Boundary Value Problems for Elliptic and Parabolic Differential Equations of Second Order. Journal of Mathematical Analysis and Applications, Volume 113, 422–440, (1986).

[35] Liggett, T. M.: *Interacting particle systems*, volume 276 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, (1985).

[36] Liggett, T. M.: *Stochastic interacting systems: contact, voter, and exclusion processes*, volume 324 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, (1999).

[37] Muskat, M.: *The Flow of Homegeneous Fluids Through Porous Media*. McGrawHill, New York, (1937).

[38] Ritort, F., Sollich, P.: Glassy dynamics of kinetically constrained models. Advances in Physics 52, no. 4, 219–342, (2003).

[39] Shapira, A.: Hydrodynamic limit of the Kob-Andersen model, (2020). https://arxiv.org/abs/2003.08495

[40] Spitzer, F.: *Interaction of Markov processes*. Adv. Math., 5:246–290, (1970).

[41] Spohn, H.: *Large Scale Dynamics of Interacting Particles*. Berlin: Springer, (1991).

[42] Vazquez, J.: *The Porous Medium Equation* - Mathematical Theory. Claredon Press, Oxford, (2007).

[43] Yau, H.: Relative entropy and hydrodynamics of Ginzburg-Landau models. Lett. Math. Phys. 22(1) 63–80, (1991).

[44] Zel'dovich, B., Raizer, P.: Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena II. Academic Press, New York, (1966).

Appendix A

Some results

This appendix is dedicated to the proof of some results that we used along the thesis. We start by presenting a result from analysis that were used in Chapters 5 and 7. Then, we present microscopic results used in Chapters 5, 6, 7, and 8.

Lemma A.0.1. *Let $m \in \mathbb{N}\backslash\{1\}$ such that $\rho^m \in L^2(0,T;\mathcal{H}^1)$. For all $\varepsilon > 0$ it holds that*

$$|\rho_s(0) - \langle \pi_s, \overrightarrow{\iota_\varepsilon}^{i\varepsilon} \rangle| \leq 2\varepsilon^{1/4(m-1)} + \varepsilon^{1/4} \|\partial_u \rho_s^m\|_2 4m^{3/2},$$

for any $i \in \{0,1,\ldots,m-1\}$ and for almost every $s \in [0,T]$. Besides that the same inequality above holds for $\overleftarrow{\iota_\varepsilon}^{1-i\varepsilon}$ in place of $\overrightarrow{\iota_\varepsilon}^{i\varepsilon}$ and $\rho_s(1)$ in place of $\rho_s(0)$.

Proof. For $A > 0$ and $s \in [0,T]$, let $E_{s,A} = \{v \in [0,1] : \rho_s(0) \leq A$ and $\rho_s(v) \leq A\}$. We start the proof by noticing that

$$\begin{aligned}
\rho_s(0) - \langle \pi_s, \overrightarrow{\iota}_\varepsilon^{i\varepsilon} \rangle &= \frac{1}{\varepsilon} \int_{i\varepsilon}^{i\varepsilon+\varepsilon} \rho_s(0) - \rho_s(v)\, dv \\
&= \frac{1}{\varepsilon} \int_{i\varepsilon}^{i\varepsilon+\varepsilon} (\rho_s(0) - \rho_s(v)) \left(\mathbb{1}_{E_{s,A}}(v) + \mathbb{1}_{\bar{E}_{s,A}}(v) \right) dv \qquad \text{(A.1)} \\
&\leq 2A + \frac{1}{\varepsilon} \int_{i\varepsilon}^{i\varepsilon+\varepsilon} (\rho_s(0) - \rho_s(v)) \mathbb{1}_{\bar{E}_{s,A}}(v)\, dv,
\end{aligned}$$

for any $i \in \{0,1,\ldots,m-1\}$ and for almost every $s \in [0,T]$. Let

$$P_m^{\rho_s(0)}(\rho_s(v)) = \sum_{i=0}^{m-1} (\rho_s)^{m-1-i}(0)(\rho_s)^i(v),$$

and note that $(\rho_s)^m(0) - (\rho_s)^m(v) = (\rho_s(0) - \rho_s(v))\, P_m^{\rho_s(0)}(\rho_s(v))$. Thus, for every $v \in \bar{E}_{s,A}$, it holds that

$P_m^{\rho_s(0)}(\rho_s(v)) \geq A^{m-1}$ and that

$$\begin{aligned} |\rho_s(0) - \rho_s(v)| &= \left| \frac{(\rho_s)^m(0) - (\rho_s)^m(v)}{P_m^{\rho_s(0)}(\rho_s(v))} \right| \\ &\leq \frac{|(\rho_s)^m(0) - (\rho_s)^m(v)|}{A^{m-1}} \\ &= \frac{1}{A^{m-1}} \left| \int_0^v \partial_u (\rho_s)^m(u)\, du \right|. \end{aligned}$$

From Cauchy-Schwarz's inequality we can bound the previous expression from above by

$$\frac{1}{A^{m-1}} \left(\int_0^1 \left(\partial_u (\rho_s)^m(u)\right)^2 \, du \right)^{1/2} \left(\int_0^1 \mathbb{1}^2_{(0,v)}(u)\, du \right)^{1/2},$$

which is equal to $\frac{1}{A^{m-1}} \|\partial_u \rho_s^m\|_2 \sqrt{v}$. Hence, from the previous computations we get

$$|\rho_s(0) - \rho_s(v)| \mathbb{1}_{\bar{E}_{s,A}}(v) \leq \frac{1}{A^{m-1}} \|\partial_u \rho_s^m\|_2 \sqrt{v}. \tag{A.2}$$

Combining (A.1) with (A.2), we have that

$$\begin{aligned} |\rho_s(0) - \langle \pi_s, \overrightarrow{\iota}_\varepsilon^{i\varepsilon} \rangle| &\leq 2A + \frac{1}{\varepsilon A^{m-1}} \|\partial_u \rho_s^m\|_2 \int_{i\varepsilon}^{i\varepsilon+\varepsilon} \sqrt{v}\, dv \\ &\leq 2A + \frac{2}{3\varepsilon A^{m-1}} \|\partial_u \rho_s^m\|_2 \left((i\varepsilon + \varepsilon)^{3/2} - (i\varepsilon)^{3/2} \right) \\ &\leq 2A + \frac{2\varepsilon^{1/2}}{3A^{m-1}} \|\partial_u \rho_s^m\|_2 m^{3/2}. \end{aligned}$$

Taking $A = \varepsilon^{1/4(m-1)}$, we have therefore that

$$|\rho_s(0) - \langle \pi_s, \overrightarrow{\iota}_\varepsilon^{i\varepsilon} \rangle| \leq 2\varepsilon^{1/4(m-1)} + \frac{2\varepsilon^{1/4}}{3} \|\partial_u \rho_s^m\|_2 m^{3/2},$$

for every $i = \{0, 1, \ldots, m-1\}$ and almost every $s \in [0, T]$. □

Microscopic computations: Now, we will establish some microscopic technical results that were used in the proof of Proposition 7.0.6 and in the proof of the Fick's law. Let μ_n be a probability measure on Ω_n and $f : \Omega_n \to \mathbb{R}$ be a density with respect to $\nu^n_{\rho(\cdot)}$. We denote by $H\left(\mu_n | \nu^n_{\rho(\cdot)}\right)$ the relative entropy of μ_n with respect to $\nu^n_{\rho(\cdot)}$, which is defined by

$$H\left(\mu_n | \nu^n_{\rho(\cdot)}\right) := \sup_f \left\{ \int f(\eta) d\mu_n - \log \int e^{f(\eta)} d\nu^n_{\rho(\cdot)} \right\},$$

where the supremum is carried over all continuous functions.

Lemma A.0.2. *There exists a constant $C = C(\alpha, \beta)$, such that*

$$H\left(\mu_n | \nu^n_{\rho(\cdot)}\right) \leq nC.$$

Proof. Since Ω_n is a countable state space, the entropy of μ_n with respect to $\nu^n_{\rho(\cdot)}$ (see [31] for more details) can be computed using the explicit formula

$$H\left(\mu_n|\nu^n_{\rho(\cdot)}\right)=\sum_{\eta\in\Omega_n}\mu_n(\eta)\log\left(\frac{\mu_n(\eta)}{\nu^n_{\rho(\cdot)}(\eta)}\right).$$

Moreover, since μ_n is a probability measure and $\log(\cdot)$ is an increasing function, we can bound the previous expression by

$$\sum_{\eta\in\Omega_n}\mu_n(\eta)\log\left(\frac{1}{\nu^n_{\rho(\cdot)}(\eta)}\right).$$

Using the definition of the Bernoulli product measure $\nu^n_{\rho(\cdot)}$ and the fact that $\rho(\cdot)$ satisfies the hypothesis of Lemma 4.2.1, last expression can be written as

$$\sum_{\eta\in\Omega_n}\mu_n(\eta)\log\left(\frac{1}{\prod_{x=1}^{n-1}\rho\left(\frac{x}{n}\right)^{\eta(x)}\left(1-\rho\left(\frac{x}{n}\right)\right)^{1-\eta(x)}}\right)\leq\sum_{\eta\in\Omega_n}\mu_n(\eta)\log\left(\frac{1}{\left(\min\{\rho\left(\frac{x}{n}\right),1-\rho\left(\frac{x}{n}\right)\}\right)^n}\right).$$

Now, using properties of logarithmic functions and the fact that μ_n is a probability measure, last expression can be bounded by

$$n\left(-\log\left(\min\left\{\rho\left(\tfrac{x}{n}\right),1-\rho\left(\tfrac{x}{n}\right)\right\}\right)\right).$$

Therefore, using again the fact that $\rho(\cdot)$ satisfies the hypothesis of Lemma 4.2.1 we conclude that

$$H\left(\mu_n|\nu^n_{\rho(\cdot)}\right)\leq nC.$$

□

Proposition A.0.3. *Let ρ be a function that satisfies the hypothesis of Lemma 4.2.1 and $\eta\in\Omega_n$.*

$i)$ *There exists a constant $C(\alpha,\beta)$ such that*

$$\frac{d\nu^n_{\rho(\cdot)}(\eta^{x,x+1})}{d\nu^n_{\rho(\cdot)}(\eta)}\leq C(\alpha,\beta). \tag{A.3}$$

$ii)$ *It is true that*

$$I_1^\alpha(\eta^1)\frac{d\nu^n_{\rho(\cdot)}(\eta^1)}{d\nu^n_{\rho(\cdot)}(\eta)}=I_1^\alpha(\eta). \tag{A.4}$$

The same is true for β (resp. 1) in place of α (resp. $n-1$).

Proof. Let us begin by proving item $i)$ above. Note that

$$\begin{aligned}\frac{d\nu^n_{\rho(\cdot)}(\eta^{x,x+1})}{d\nu^n_{\rho(\cdot)}(\eta)}&=\mathbb{1}_{\eta(x)=1,\eta(x+1)=0}\frac{(1-\rho(\frac{x}{n}))\rho(\frac{x+1}{n})}{\rho(\frac{x}{n})(1-\rho(\frac{x+1}{n}))}\\&+\mathbb{1}_{\eta(x)=0,\eta(x+1)=1}\frac{\rho(\frac{x}{n})(1-\rho(\frac{x+1}{n}))}{(1-\rho(\frac{x}{n}))\rho(\frac{x+1}{n})}+\mathbb{1}_{\eta(x)=\eta(x+1)}.\end{aligned}$$

Since for $u\in(0,1)$, it holds that $1-\beta\leq1-\rho(u)\leq1-\alpha$, we have that last expression is bounded from above by

$$\mathbb{1}_{\eta(x)=1,\eta(x+1)=0}\frac{(1-\alpha)\beta}{\alpha(1-\beta)}+\mathbb{1}_{\eta(x)=0,\eta(x+1)=1}\frac{\beta(1-\alpha)}{(1-\beta)\alpha}+\mathbb{1}_{\eta(x)=\eta(x+1)}\leq C(\alpha,\beta)=\max\left\{\frac{\beta(1-\alpha)}{(1-\beta)\alpha},1\right\}.$$

Now, let us prove item $ii)$.

$$\begin{aligned}
I_1^\alpha(\eta^1)\frac{d\nu^n_{\rho(\cdot)}(\eta^1)}{d\nu^n_{\rho(\cdot)}(\eta)} &= \mathbb{1}_{\eta(1)=0}(1-\alpha)\frac{\rho(\frac{1}{n})}{1-\rho(\frac{1}{n})}+\mathbb{1}_{\eta(1)=1}(\alpha)\frac{1-\rho(\frac{1}{n})}{\rho(\frac{1}{n})}\\
&=(1-\eta(1))(1-\alpha)\frac{\alpha}{1-\alpha}+\eta(1)\alpha\frac{1-\alpha}{\alpha}\\
&=\alpha(1-\eta(1))+(1-\alpha)\eta(1)\\
&=I_1^\alpha(\eta).
\end{aligned}$$

□

Now, we will show that the quadratic variation vanishes in $L^2(\mathbb{P}_{\mu_n})$, as n goes to infinity. This is a necessary result in order to prove the Fick's law in Chapter 8.

Quadratic variation: Our goal now is to prove that the quadratic variation of (8.9) vanishes in $L^2(\mathbb{P}_{\mu_n})$, as n goes to infinity.

Fix $f \in C^1(0,1)$. From Dynkin's formula (see Lemma A1.5.1 of [31]) we have that

$$M_t^n(f)=J_t^n(f)-J_0^n(f)-\int_0^t n^2\tilde{L}_n^m J_s^n(f)\,ds,$$

is a martingale with respect to the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The quadratic variation of M_t^n is given by $\langle M^n(f)\rangle_t=\int_0^t B_s^n(f)\,ds$, where

$$B_s^n(f):=n^2\left(\tilde{L}_n^m J_s^n(f)^2-2J_s^n(f)\tilde{L}_n^m J_s^n(f)\right).$$

Recalling the definition of $\tilde{L}_n^m$ in (8.4), we can write $B_s^n(f)$ in the following form

$$B_s^n(f)=B_{s,\alpha}^n(f)+B_{s,P}^n(f)+n^{a-2}B_{s,S}^n(f)+B_{s,\beta}^n(f). \tag{A.5}$$

Let us examine the conservative part of (A.5). Note that

$$\left(B_{s,P}^n+n^{a-2}B_{s,S}^n\right)(f)=n^2\left(\left(\tilde{L}_P^m+n^{a-2}\tilde{L}_S\right)Q_s^n(f)^2-2Q_s^n(f)\left(\tilde{L}_P^m+n^{a-2}\tilde{L}_S\right)Q_s^n(f)\right). \tag{A.6}$$

To simplify notation, take $Q_s^n(f)=F(\eta_{sn^2},Q_s^n(x))$. Now, we can write (A.6) as

$$\left(B_{s,P}^n+n^{a-2}B_{s,S}^n\right)(f)=n^2\sum_{x=1}^{n-1}\left(B_{s,P}^n+n^{a-2}B_{s,S}^n\right)(x),$$

where

$$\left(B_{s,P}^{n}+n^{a-2}B_{s,S}^{n}\right)(x)=\left(\tilde{L}_{P}^{m}+n^{a-2}\tilde{L}_{S}\right)F(\eta_{sn^2},Q_s^n(x))^2$$
$$-2F(\eta_{sn^2},Q_s^n(x))\left(\tilde{L}_{P}^{m}+n^{a-2}\tilde{L}_{S}\right)F(\eta_{sn^2},Q_s^n(x)).$$

The previous expression is equal to

$$a_{x,x+1}(\eta_{sn^2})(c_{x,x+1}^m(\eta_{sn^2})+n^{a-2})\left(F\left(\eta_{sn^2}^{x,x+1},Q_s^n(x)+1\right)-F\left(\eta_{sn^2},Q_s^n(x)\right)\right)^2$$
$$+a_{x+1,x}(\eta_{sn^2})(c_{x,x+1}^m(\eta_{sn^2})+n^{a-2})\left(F\left(\eta_{sn^2}^{x,x+1},Q_s^n(x)-1\right)-F\left(\eta_{sn^2},Q_s^n(x)\right)\right)^2$$
$$+\sum_{\substack{y=1\\y\neq x}}^{n-2}\left(\eta_{sn^2}(x)-\eta_{sn^2}(x+1)\right)^2(c_{x,x+1}^m(\eta_{sn^2})+n^{a-2})\left(F\left(\eta_{sn^2}^{y,y+1},Q_s^n(y)\right)-F(\eta_{sn^2},Q_s^n(y))\right)^2.$$

Thus, since $Q_s^n(f)=F(\eta_{sn^2},Q_s^n(x))$, we get

$$a_{x,x+1}(\eta_{sn^2})(c_{x,x+1}^m(\eta_{sn^2})+n^{a-2})\left(\frac{1}{n^2}\sum_{y=1}^{n-2}f\left(\tfrac{y}{n}\right)Q_s^{n,x+1}(y)-\frac{1}{n^2}\sum_{y=1}^{n-2}f\left(\tfrac{y}{n}\right)Q_s^n(y)\right)^2$$
$$+a_{x+1,x}(\eta_{sn^2})(c_{x,x+1}^m(\eta_{sn^2})+n^{a-2})\left(\frac{1}{n^2}\sum_{y=1}^{n-2}f\left(\tfrac{y}{n}\right)Q_s^{n,x-1}(y)-\frac{1}{n^2}\sum_{y=1}^{n-2}f\left(\tfrac{y}{n}\right)Q_s^n(y)\right)^2,$$

which is equal to

$$a_{x,x+1}(\eta_{sn^2})(c_{x,x+1}^m(\eta_{sn^2})+n^{a-2})\left(\frac{1}{n^2}f\left(\tfrac{x}{n}\right)(Q_s^n(x)+1)-\frac{1}{n^2}f\left(\tfrac{x}{n}\right)Q_s^n(x)\right)^2$$
$$+a_{x+1,x}(\eta_{sn^2})(c_{x,x+1}^m(\eta_{sn^2})+n^{a-2})\left(\frac{1}{n^2}f\left(\tfrac{x}{n}\right)(Q_s^n(x)-1)-\frac{1}{n^2}f\left(\tfrac{x}{n}\right)Q_s^n(x)\right)^2.$$

Hence,

$$\left(B_{s,P}^{n}+n^{a-2}B_{s,S}^{n}\right)(x)=\frac{1}{n^4}f\left(\tfrac{x}{n}\right)^2(a_{x,x+1}(\eta_{sn^2})+a_{x+1,x}(\eta_{sn^2}))(c_{x,x+1}^m(\eta_{sn^2})+n^{a-2}).$$

Therefore,

$$\left(B_{s,P}^{n}+n^{a-2}B_{s,S}^{n}\right)(f)=\frac{1}{n^2}\sum_{x=1}^{n-2}f\left(\tfrac{x}{n}\right)^2(a_{x,x+1}(\eta_{sn^2})+a_{x+1,x}(\eta_{sn^2}))(c_{x,x+1}^m(\eta_{sn^2})+n^{a-2})$$
$$\leq 2\frac{\|f^2\|_\infty}{n}+\frac{\|f^2\|_\infty}{n^{3-a}},$$

which vanishes when n goes to infinity since $1<a<2$.

Let us now examine the non-conservative part of the quadratic variation. Note that

$$\left(B_{s,\alpha}^n+B_{s,\beta}^n\right)(f)=n^2\left(\left(\tilde{L}_\alpha+\tilde{L}_\beta\right)K_s^n(f)^2-2K_s^n(f)\left(\tilde{L}_\alpha+\tilde{L}_\beta\right)K_s^n(f)\right).$$

We will examine only $B_{s,\alpha}^n(f)$ since the computations for $B_{s,\beta}^n(f)$ are the same. Take $K_s^n(f)=F(\eta_{sn^2},K_s^n(0))$.

Repeating the same arguments used above, we have that

$$B^n_{s,\alpha}(f) = n^2\frac{\kappa}{n^\theta}\Bigg\{\alpha(1-\eta_{sn^2}(1))\left(F\left((\eta^1_{sn^2}, K^n_s(0)+1\right) - F(\eta_{sn^2}, K^n_s(0))\right)^2$$

$$+ (1-\alpha)(\eta_{sn^2}(1))\left(F\left((\eta^1_{sn^2}, K^n_s(0)-1\right) - F(\eta_{sn^2}, K^n_s(0))\right)^2\Bigg\}.$$

Since $K^n_s(f) = F(\eta_{sn^2}, K^n_s(0))$, we get

$$n^2\frac{\kappa}{n^\theta}\Bigg\{\alpha(1-\eta_{sn^2}(1))\left(\frac{1}{n}f(0)(K^n_s(0)+1) - \frac{1}{n}f(0)K^n_s(0)\right)^2$$

$$+(1-\alpha)(\eta_{sn^2}(1))\left(\frac{1}{n}f(0)(K^n_s(0)-1) - \frac{1}{n}f(0)K^n_s(0)\right)^2\Bigg\}.$$

Hence,

$$B^n_{s,\alpha}(f) = \frac{\kappa}{n^\theta}f(0)^2(\alpha-\eta_{sn^2}(1))^2.$$

In the same manner, we also have

$$B^n_{s,\beta}(f) = \frac{\kappa}{n^\theta}f\left(\tfrac{n-1}{n}\right)^2(\beta-\eta_{sn^2}(n-1))^2.$$

Therefore,

$$B^n_{s,\alpha}(f) + B^n_{s,\beta}(f) \leq C(\alpha,\beta)\frac{\kappa}{n^\theta}\|f^2\|_\infty, \tag{A.7}$$

which vanishes as n goes to infinity for any $\theta > 0$. In order to conclude the proof we need to show that (A.7) vanishes for $\theta = 0$. The proof of this case follows the same ideas used in the proof of Proposition 4.1.2 in the case $\theta \in [0,1)$.