New toric polarizations in $\mathbb{CP}^1$

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Thesis to obtain the Master of Science Degree in

Master in Applied Mathematics and Computation

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6 November 2023

”I’m not great at advice. But can I interest you in a sarcastic comment?”

Chandler Bing

Declaration

I declare that this document is an original work of my own authorship and that it fulfills all the requirements of the Code of Conduct and Good Practices of the Universidade de Lisboa.

Acknowledgments

Antes de mais, quero agradecer ao meu orientador Professor João Pimentel Nunes por me ter aturado durante este ano e por toda a ajuda, motivação e paciência que teve comigo durante a realização deste projeto.

Quero ainda agradecer a todos os meus Professores pela sua dedicação ao ensino e por terem contagiado com o “bichinho” da matemática. Em particular, gostava de deixar um grande obrigado aos Professores Ana Paula Jardim, Cristina Sernadas, Leonor Godinho, Rosa Sena Dias, Patrícia Gonçalves, Pedro Resende, João Pimentel Nunes e José Natário.

Um agradecimento especial vai para todos os meus amigos, em particular ao Luis Maia pelas ótimas conversas e memes matemáticos; ao Wormy por todos os cabelos brancos que ganhei; ao João pela companhia nas cadeiras; ao Vasco, Gonçalo, Rita, Água, João, Miaw, Alves e Gui pelos jantares e secções online de discord; ao Simão pelos jogos de civ, memes e fotocópias; à Maria Madrugo pelo gossip; ao Caria pelas longas conversas filosóficas; à Marta pela tua felicidade e caos; à Filipa pelos berros logo de manhã; à Mariana por deixares-me chatear-te, Huzzah; ao Zé pelos teus batidos; à Catarina pelos longos áudios; ao Pedro Leite pela tua serenidade; à Ana Santos pelas longas conversas ao telemóvel; ao Ruben pelos gelados, almoços e idas ao cinema; à Mena, Carolina e Mariana pelas idas ao forum; à Mafalda, Carolina e Laura Maria pelos jantares de sushi; ao António, Barbara Rivas e Nóbrega pela companhia na ilha.

Quero também agradecer à Fernanda por ter aguardado a todo o momento.

Quero ainda agradecer à minha família de Lisboa por terem cuidado de mim quando vim para cá estudar e, em particular, por me terem alimentado.

Por fim, quero agradecer aos meus pais por tudo. Sem vocês, isto não seria possível.

Resumo

Em [1], [2] e [3] foi mostrado, usando diferentes técnicas, como a escolha de uma função estritamente convexa no politopo de momento de uma variedade Kähler tórica permite a degeneração das polarizações Kähler na polarização real. Com esta degeneração, foi ainda mostrado que as secções holomorfas convergem para as secções delta de Dirac com suporte nos pontos intergais do politopo de momento.

Esta tese explora o caso especial de $S^2 \cong \mathbb{CP}^1$ e generaliza os resultados prévios, considerando funções com uma "*bump function*" como sua segunda derivada. Iremos abordar dois dos métodos: secções normalizadas L^1 e abordagem de fluxo hamiltoniano em tempo complexo para secções corrigidas meia forma. Seguindo essas abordagens, as polarizações Kähler convergem para uma nova polarização mista. Assim, somos então capazes de dividir o politopo do momento em três partes, que correspondem a uma decomposição do espaço de Hilbert da quantização mista em três partes. Fora do suporte da "*bump function*", as seções holomorfa convergem para sua restrição normalizada na respectiva parte. Se houver pontos inteiros no suporte de nossa função, as secções correspondentes convergem para secções distribucionais. Além disso, generalizamos estes resultados para o caso quando temos mais "*bump functions*".

Estes novos resultados são interessantes porque, em geral, não há como "decompor" um espaço de fase em subconjuntos, de modo que a quantização da variedade simplética também "decomponha" como uma soma das quantizações desses subconjuntos.

Palavras-chave: Quantização Geométrica, Geometria Tórica, fluxos Hamiltonianos em tempo imaginário, Polarizações mistas, secções distribucionais.

Abstract

In [1], [2], and [3] it was shown, using different techniques, how the choice of a strictly convex function on the moment polytope of a toric Kähler manifold allows for the degeneration of the Kähler polarization into the real polarization. With this degeneration, it was further proved that the holomorphic sections converge to the Dirac delta distributional sections supported on the integral points of the moment polytope.

This thesis explores the special case of $S^2 \cong \mathbb{CP}^1$ and generalizes the previous results, considering functions ψ, which have bump functions as their second derivative. We will do this using two methods: L^1-normalized sections and Complex time Hamiltonian flow approach for half-form corrected sections. Following these approaches, the Kähler polarizations converge to a new mixed polarization. The moment polytope becomes divided into three parts, corresponding to the splitting of the Hilbert space of the mixed quantization into three parts. Outside the support of the bump function, the holomorphic sections converge to their normalized restriction on the respective part. If there are integral points in the support of our function, the corresponding sections converge to distributional sections. Moreover, we then generalize this for the case when we have more bump functions.

These new results are interesting as, in general, there is no way of ”decomposing” a phase space into subsets, such that the quantization of the symplectic manifold also ”decomposes” as a sum of the quantizations of those subsets.

Keywords: Geometric Quantization, Toric geometry, Imaginary Time Hamiltonian Flows, Mixed polarization, Distributional sections.

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Chapter 1

Symplectic Forms

In this chapter we will first define what is a symplectic structure on a vector space and then explore some basic properties. Afterwards we will generalize this idea to manifolds.

1.1 Skew-Symmetric bilinear maps

From now on, let V be an m-dimensional real vector space. Let $\Omega : V \times V \to \mathbb{R}$ be a bilinear skew-symmetric map.

Theorem 1.1.1. *Let Ω be a skew-symmetric bilinear map over V. Then there is a basis $u_1, ..., u_k$, $e_1, ..., e_n$, $f_1, ..., f_n$ such that:*

$$\Omega(u_i, v) = 0,\ \forall i \in \{1, ..., k\}\ \forall v \in V,$$
$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0,\ \forall i, j \in \{1, ..., n\},$$
$$\Omega(e_i, f_j) = \delta_{ij},\ \forall i, j \in \{1, ..., k\}.$$

Proof. The proof can be found, for instance, [4] on page 3. ■

The basis given in Theorem 1.1.1 is not unique, despite being called the **canonical basis**. Let $U = \{u \in V : \Omega(u, v) = 0,\ \forall v \in V\}$. Consider the following map

$$\tilde{\Omega} : V \to V^*$$
$$v \mapsto \tilde{\Omega}(v)(u) := \Omega(v, u).$$

It is clear that the kernel of $\tilde{\Omega}$ is U.

Definition 1.1.1. *We say that Ω is **symplectic** (or **non-degenerate**) if $\tilde{\Omega}$ is bijective (equivalently, if $U = \{0\}$). Therefore, we call the tuple (V, Ω) a **symplectic vector space**.*

It is immediate to see by theorem 1.1.1 that if Ω is symplectic than $dim\, V = 2n$, therefore we have the following corollary:

Corollary 1.1.1. *A symplectic vector space must have even dimension.*

Example 1.1.1. *In $\mathbb{R}^{2n}$ we have the prototype of a symplectic vector space $(\mathbb{R}^{2n}, \Omega_0)$, where Ω_0 is such that the basis:*

$$e_1 = (1, ..., 0), ..., e_n = (0, ..., \underbrace{1}_{n}, 0, ...0)$$
$$f_1 = (0, ..., 0, \underbrace{1}_{n+1}, 0, ..., 0), ..., f_n = (0, ..., 1),$$

is a symplectic basis. In particular, in this basis, the symplectic map is of the following form:

$$\begin{bmatrix} 0_n & Id_n \\ -Id_n & 0_n \end{bmatrix}$$

As usual, it is useful to consider transformations that "preserve" the symplectic structure.

Definition 1.1.2. *A **symplectomorphism** ϕ between symplectic spaces (V, Ω) and (V', Ω') is linear isomorphism $\phi : V \to V'$ such that $\phi^*\Omega' = \Omega$, where $(\phi^*\Omega')(u, v) = \Omega'(\phi(u), \phi(v))$.*

Similarly to what happens with the inner product, there is a way to find, given a subspace of a symplectic vector space, the "complement" of this subspace with respect to symplectic structure.

Definition 1.1.3. *Let (V, Ω) be a symplectic vector space and Y a subspace of it. Then its **symplectic orthogonal** Y^Ω is the linear subspace defined by*

$$Y^\Omega := \{v \in V : \Omega(v, y) = 0,\ \forall y \in Y\}$$

Definition 1.1.4. *We say that Y is **isotropic** when $Y \subset Y^\Omega$ and that Y is **coisotropic** when $Y^\Omega \subset Y$. If Y is both isotropic and coisotropic then we say that Y is **lagrangian** (i.e.$Y = Y^\Omega$), which implies that $\dim Y = \frac{1}{2} \dim V$.*

1.2 Symplectic Manifolds

Let ω be a 2-form on a manifold M such that for all $p \in M$, the map $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is skew-symmetric bilinear and ω_p varies smoothly with p (i.e. ω is de Rham 2-form).

Definition 1.2.1. *The 2-form ω is **symplectic** if ω_p is symplectic for all $p \in M$ and if it is closed, i.e. $d\omega = 0$. In this case we say that (M, ω) is **symplectic manifold**.*

We therefore have, as a consequence of corollary 1.1.1, the following corollary:

Corollary 1.2.1. *A symplectic manifold is even dimensional.*

Example 1.2.1. *Consider de 2-sphere S^2. Then we can consider its volume form in spherical coordinates*

$$\omega = \sin(\phi) d\phi \wedge d\theta,$$

which tell us that S^2 is a symplectic manifold. (This extends smoothly to the whole of S^2.)

Definition 1.2.2. *Let (M, ω) and (M', ω') be symplectic manifolds and let $\phi : M \to M'$ be a diffeomorphism. Then ϕ is a **symplectomorphism** if $\phi^* \omega' = \omega$, where $(\phi^* \omega')(u, v) = \omega'(d\phi_p(u), d\phi_p(v))$ is the pullback.*

Theorem 1.2.1 (Darboux)**.** *Let (M, ω) be a 2n-dimensional symplectic manifold, and $p \in M$. Then there is a chart U with local coordinates $x_1, ..., x_n, y_1, ..., y_n$ centered at p such that on U*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

*These coordinates are known as **Darboux coordinates**.*

Proof. The proof can be found in any book that deals with symplectic geometry. In particular, it can be found in [4] on page 55. ■

This theorem tell us that any symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. In fact, symplectic manifolds are locally indistinguishable. This a clearly very different from Riemannian geometry, where different metrics can be distinguished locally by curvature. In particular, this result tells us that in symplectic geometry we are interested in looking at global properties.

Definition 1.2.3. *A **submanifold** of M is a manifold X with a proper injective immersion (also known as a closed embedding) $i : X \hookrightarrow M$.*

We usually regard the embedding $i : X \hookrightarrow M$ as being an inclusion(i.e. $i(p) = p$).

Definition 1.2.4. *Given a symplectic manifold (M, ω), we say that a submanifold Y of M is a **lagrangian submanifold** if $\forall p \in Y$, T_pY is a lagrangian subspace of T_pM, that is $\omega|_{T_pY} \equiv 0$ (using the inclusion map, this is equivalent to $i^*\omega = 0$) and $dim\, T_pY = \frac{1}{2} dim\, T_pM$.*

Let X be any n-dimensional manifold and $M = T^*X$ its cotangent bundle. Then, considering the usual cotangent coordinates given coordinates $(T^*U, x_1, ..., x_n, \xi_1, ..., \xi_n)$, we can thus define a 2-form ω in T^*U by

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i, \tag{1.1}$$

which is symplectic. And we can define the following 1-form on T^*U

$$\alpha = \sum_{i=1}^{n} \xi_i dx_i, \tag{1.2}$$

Such that $\omega = -d\alpha$. α is known as the **tautological form** or the **Liouville 1-form** and ω is the **canonical symplectic form**. The Tautological form is coordinate independent. Consider the following definition:

Definition 1.2.5. *Let $\pi : T^*X = M \to X$ be the natural projection (i.e. $\pi(x, \xi) = x$). Then the **tautological 1-form** α is defined pointwise by*

$$\alpha_p = (d\pi_p)^* \xi \in T_p^* M$$

Where $(d\pi_p)^*$ represents the transpose of $d\pi_p$.

Consider $v = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i}$ then:

$$\alpha_p(v) = \xi(d\pi_p \cdot v) = \xi\left(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\right) = \sum_{i=1}^{n} \xi_i a_i = \sum_{i=1}^{n} \xi_i dx_i \left(\underbrace{\sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j}}_{=v}\right),$$

which shows that the tautological form is well defined.

Definition 1.2.6. *The canonical symplectic form ω on $M = T^*X$ is defined as*

$$\omega = -d\alpha$$

And thus it is given in local coordinates by 1.1

By a simple induction argument, it is easy to see that $\omega^n = \omega \wedge ... \wedge \omega$ does not vanish. Thus, it defines a volume form. In particular, the form

$$\frac{\omega^n}{n!}$$

is called the **symplectic volume** or the **Liouville volume** of (M, ω).

Therefore, we have found a way to construct symplectic manifolds, by simply considering the cotangent bundle of an existing manifold. However, it is not the case that all symplectic manifolds are a cotangent bundle of another manifold, as we have seen in the case of S^2.

Corollary 1.2.2. *The ω^n of any symplectic form ω on a 2n-dimensional manifold M is a volume form. This can easily be seen by the above proposition and noting that a symplectic form on M is* *a 2-form, and therefore ω^n is top degree.*

Corollary 1.2.3. *A symplectic manifold is orientable.*

Proposition 1.2.1. *If (M,ω) is a compact symplectic manifold of dimension n, then $[\omega^n] \in H^{2n}_{dR}(M) \neq 0$.*

Proof. This result follows form a simple application of Stokes theorem. ■

Proposition 1.2.2. *If (M,ω) is a compact symplectic manifold of dimension n, then $[\omega] \neq 0$.*

Proof. We know by the above proposition that $[\omega^n] \in H^{2n}_{dR}(M) \neq 0$. Then, by the cup product on the cohomology, we have $[\omega^n] = [\omega]^n$, which allows us to conclude that $[\omega] \neq 0$. ■

Corollary 1.2.4. *For $n > 1$ S^{2n} is not symplectic. This is easily seen because $H^2_{dR}(S^{2n}) = 0$ for $n > 1$.*

Definition 1.2.7. *Let M be a manifold and $\rho : M \times \mathbb{R} \to M$ (we will write $\rho_t(p) := \rho(t,p)$). Then ρ is said to be an **isotopy** if $\rho_t : M \to M$ is a diffeomorphism for every t and $\rho_0 = id_M$.*

Definition 1.2.8. *Given an isotopy ρ, we have a **time-dependent vector field**, that is, a family of vector fields $X_t,\ t \in \mathbb{R}$ such that :*

$$\frac{d\rho_t}{dt} = X_t(\rho_t)$$

Conversely, assuming that either M is compact or that X_t have compact support for all of $t \in \mathbb{R}$, then there is an isotopy associated to the time-dependent vector field. If it happens that X_t is independent of t, then the isotopy associated is the flow or the exponential map of X.

Proposition 1.2.3. *Let $\omega_t, t \in \mathbb{R}$ be a family of forms. Then*

$$\frac{d}{dt}(\rho_t^*\omega_t) = \rho_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt}\right)$$

Proof. See [4] on page 42-43. ■

A natural question that may be asked is that given two symplectic forms in the same cohomology class if there exists a diffeomorphism (homotopic to the identity of our manifold) such that it behaves like a symplectomorphism? Moser answered this question in the positive in what is now known as Moser theorem.

Theorem 1.2.2 (Moser)**.** *Let M be a compact manifold with two symplectic forms ω_1, ω_2 such that $[\omega_1] = [\omega_2]$ and that the 2-form $\omega_t = (1-t)\omega_1 + t\omega_2$, is symplectic $\forall t \in [0,1]$. Then there is an isotopy $\rho : M \times \mathbb{R} \to M$ such that $\rho_t^*\omega_t = \omega_1,\ \forall t \in [0,1]$.*

Proof. See, for instance, page 50 in [4]. ■

Chapter 2

Compatible Almost Complex Structures

In this chapter we will dwell into almost complex structures on manifolds. Our main goal is to define what is almost complex structures and to show that any symplectic and riemmanian manifold has a almost complex structure which is "compatible" in some sense that we will also define. We will also look into some consequences of this as well as define the Dolbeaut theory. This will be the basis for the next chapter where we will deal with complex structures on manifolds.

2.1 Almost Complex structures

Example 2.1.1. $\mathbb{R}^{2n}$ *with the standard coordinates* $(x_1, ..., x_n, y_1, ..., y_n)$ *has the standard symplectic form :*

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

making it into a symplectic manifold. On the other hand we also have the standard Riemannian metric, given by the standard inner product:

$$g_0 = \langle \cdot, \cdot \rangle$$

Finally we can think of $\mathbb{R}^{2n}$ *being isomorphic to* $\mathbb{C}^n$ *with coordinates* $z_j = x_j + iy_j$*. In turn the multiplication by* i *induces a linear map* J_0 *on the tangent space of* $\mathbb{R}^{2n}$ *as follows. Let* $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}$ *be the standard basis for tangent space of* $\mathbb{R}^{2n}$ *then:*

$$J_0\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \qquad J_0\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}$$

Notice that $J_0^2 = -1$. Using the above coordinates we can write all of this maps in matrix form:

$$J_0(u) = \begin{bmatrix} 0 & -Id \\ Id & 0 \end{bmatrix} u, \qquad \omega_0(u,v) = v^t \begin{bmatrix} 0 & -Id \\ Id & 0 \end{bmatrix} u, \qquad g_0(u,v) = v^t u$$

It is also worth to point out that we can define the symplectic form in terms of the metric and the complex form and vice versa:

$$\omega_0(u,v) = g_0(J_0(u), v)$$
$$g_0(u,v) = \omega_0(u, J_0(v))$$

This is not a coincidence, as we will soon see.

Definition 2.1.1. *Let V be a vector space. A* ***complex structure*** *on V is a linear map: $J : V \to V$, such that $J^2 = -Id$. The pair (V, J) is called a* ***complex vector space****.*

Definition 2.1.2. *Let (V, Ω) be a symplectic vector space. A complex structure J on V is said to be* ***compatible*** *(with Ω) if*

$$G_J(u,v) = \Omega(u, J(v)),\ \forall u, v \in V \text{ is a positive inner product on } V.$$

That is,

$$\Omega(Ju, Jv) = \Omega(u,v) \ [\textit{symplectomorphism}]$$
$$\Omega(u, Ju) > 0 \ [\textit{taming condition}]$$

Proposition 2.1.1. *Let (V, Ω) be a symplectic vector space. Then there is a compatible complex structure J on V.*

Proof. The proof can be found in [4] on page 84. ∎

2.2 Almost Complex Manifold

Borrowing the idea from vector spaces, we are then able to extend this concept into manifolds, in the following way

Definition 2.2.1. *An* ***almost complex structure*** *on a manifold M is a smooth field of complex structures on its tangent space: $x \mapsto J_x : T_xM \to T_xM$ linear, and $J_x^2 = -Id$. The pair (M, J) is called a* ***almost complex manifold****.*

Definition 2.2.2. *Let (M, ω) be a symplectic manifold. An almost complex structure J on M is* *called **compatible** (with ω) if the map:*

$$x \mapsto g_x : T_xM \times T_xM \to \mathbb{R}$$
$$g_x(u, v) := \omega_x(u, J_x v)$$

*is a riemmanian metric on M. The triple (ω, g, J) is called the **compatible triple** when $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$.*

Proposition 2.2.1. *Let (M, ω) be a symplectic vector space and g a riemmanian metric on M. Then there is a canonical compatible almost complex structure J on M.*

Proof. The proof follows immediately from proposition 2.1.1 and by noting that on its proof, the J structure is canonical after the choice of the inner product. ■

In particular, if (ω, J, g) is a compatible triple, then any one of these maps can be written in terms of the other two:

$$g(u, v) = \omega(u, Jv), \qquad \omega(u, v) = g(Ju, v), \qquad J(u) = \tilde{g}^{-1}(\tilde{\omega}(u)),$$

where

$$\tilde{\omega} : TM \to T^*M \qquad \tilde{g} : TM \to T^*M$$
$$u \mapsto \omega(u, \cdot) \qquad u \mapsto g(u, \cdot)$$

As in the other areas of geometry, we may then be interested to check when g is flat and ω is closed. For the almost complex structure, the corresponding property we may dwell into is when is J **integrable**, that is when is J induced by a structure of a complex manifold, i.e. the coordinates maps establish a homeomorphism with $\mathbb{C}^n$ and the transition maps are biholomorphic.

Now, we will present the example of $\mathbb{R}^2$, which will be useful later on.

Example 2.2.1. *Take $M = \mathbb{R}^2 \cong \mathbb{C}$. Then we can take the coordinates to be $z = p + iq$, where (p, q) are the usual coordinates in $\mathbb{R}^2$. Thus*

$$p = \frac{1}{2}(z + \overline{z}) \qquad q = \frac{1}{2i}(z - \overline{z})$$

Then

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial p} - i\frac{\partial}{\partial q}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right).$$

Then if $f = u + iv : \mathbb{C} \to \mathbb{C}$, where u, v are real functions, satisfy the Cauchy-Riemman equations:

$$\begin{cases} \frac{\partial u}{\partial p} & = \frac{\partial v}{\partial q} \\ \frac{\partial u}{\partial q} & = -\frac{\partial v}{\partial p} \end{cases}$$

is equivalent to

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} = 0 \iff & \frac{1}{2}\left(\frac{\partial f}{\partial p} + i\frac{\partial f}{\partial q}\right) = 0 \\ \iff & \frac{\partial u}{\partial p} + i\frac{\partial v}{\partial p} + i\frac{\partial u}{\partial q} - \frac{\partial v}{\partial q} = 0 \\ \iff & \frac{\partial u}{\partial p} = \frac{\partial v}{\partial q}, \quad \frac{\partial u}{\partial q} = -\frac{\partial v}{\partial p} \end{aligned}$$

Thus f is holomorphic iff $\frac{\partial f}{\partial \bar{z}} = 0$. Thus it is natural to define

$$J\left(\frac{\partial}{\partial p}\right) = \frac{\partial}{\partial q}, \qquad J\left(\frac{\partial}{\partial q}\right) = -\frac{\partial}{\partial p}.$$

Therefore

$$J\left(\frac{\partial}{\partial z}\right) = J\left(\frac{1}{2}\left(\frac{\partial}{\partial p} - i\frac{\partial}{\partial q}\right)\right) = \frac{1}{2}\left(J\left(\frac{\partial}{\partial p}\right) - iJ\left(\frac{\partial}{\partial q}\right)\right) = \frac{1}{2}\left(\frac{\partial}{\partial q} + i\frac{\partial}{\partial p}\right) = i\frac{\partial}{\partial z},$$

and in the same way

$$J\left(\frac{\partial}{\partial \bar{z}}\right) = -i\frac{\partial}{\partial \bar{z}}.$$

We will now check that this complex structure is compatible with $\omega = dx \wedge dy$.

$$\begin{aligned} u &= a\frac{\partial}{\partial p} + b\frac{\partial}{\partial q} & \qquad Ju &= -b\frac{\partial}{\partial p} + a\frac{\partial}{\partial q} \\ v &= c\frac{\partial}{\partial p} + d\frac{\partial}{\partial q} & \qquad Jv &= -d\frac{\partial}{\partial p} + c\frac{\partial}{\partial q} \end{aligned}$$

Then, $\omega(u, Ju) = a^2 + b^2 \geq 0$. So if $u \neq 0$, $\omega(u, Ju) > 0$. On the other hand, $\omega(Ju, Jv) = ad - bc$ and $\omega(u, v) = ad - bc$, hence, it is compatible. The associated Riemannian metric is then $\omega(u, Jv) = ac + bd = \langle u, v\rangle$ which is the usual one in $\mathbb{R}^2$.

Definition 2.2.3. *A submanifold X of an almost complex manifold (M, J) is an* ***almost complex submanifold*** *when $J(TX) \subset TX$.*

Proposition 2.2.2. *Let (M, ω) be a sympletic manifold equipped with a compatible almost complex structure J. Then any almost complex submanifold X of (M, J) is a symplectic submanifold of (M, ω).*

Proof. The Proof can be found in [4] on page 91. ■

2.3 Dolbeault Theory

Let (M, J) be an 2n-dimensional almost complex manifold. As stated before, J has eigenvalues $\pm i$. Therefore we can not decompose TM (and T^*M) with respect to the eigenvalues, because TM (T^*M) is real. But we may "complexify" it, using extension by scalars. So consider the **complexified tangent bundle** of M to be $TM \otimes \mathbb{C}$ such that $p \in M,\ \ (TM \otimes \mathbb{C})_p = T_pM \otimes \mathbb{C}$. Notice that now $T_pM \otimes \mathbb{C}$ is 2n-dimensional complex vector space.
We extend linearly J to $TM \otimes \mathbb{C}$ as

$$J(v \otimes c) = Jv \otimes c, \quad \forall v \in TM, \quad \forall c \in \mathbb{C}$$

Thus we may now define:

$$\begin{aligned} T^{1,0} &= \{v \in TM \otimes \mathbb{C} : Jv = iv\} = \{v \otimes 1 - Jv \otimes i : v \in TM\} \\ T^{0,1} &= \{v \in TM \otimes \mathbb{C} : Jv = -iv\} = \{v \otimes 1 + Jv \otimes i : v \in TM\} \end{aligned}$$

$T^{1,0}$ is known as the $(J-)$**holomorphic tangent vectors** and $T^{0,1}$ is known as the $(J-)$**anti-holomorphic tangent vectors** . We also have the natural projections:

$$\begin{aligned} \pi^{1,0} :& TM \to T^{1,0} \\ & v \mapsto \frac{1}{2}(v \otimes 1 - Jv \otimes i) \end{aligned} \qquad \begin{aligned} \pi^{0,1} :& TM \to T^{0,1} \\ & v \mapsto \frac{1}{2}(v \otimes 1 + Jv \otimes i) \end{aligned}$$

Note that:

$$(\pi^{1,0} \circ J)(v) = \frac{1}{2}(Jv \otimes 1 - JJv \otimes i) = \frac{1}{2}(Jv \otimes 1 + v \otimes i) = i\pi^{1,0}(v)$$

And similarly

$$(\pi^{0,1} \circ J)(v) = -i\pi^{0,1}(v)$$

Thus, these projections are isomorphisms (of complex vector bundles) and hence $T^{1,0} \cong \overline{T^{0,1}}$. Thus extending the above projections to $TM \otimes \mathbb{C}$ we get that following decomposition:

$$TM \otimes \mathbb{C} \cong T^{1,0} \oplus T^{0,1}$$

We can repeat the process above to the cotangent bundle:

$$\begin{aligned} T_{1,0} &= \{\xi \in T^*M \otimes \mathbb{C} : \xi(Jv) = i\xi(v),\ \forall v \in TM \otimes \mathbb{C}\} = \{\xi \otimes 1 - (\xi \circ J) \otimes i : \xi \in T^*M\} \\ T_{0,1} &= \{\xi \in T^*M \otimes \mathbb{C} : \xi(Jv) = -i\xi(v),\ \forall v \in TM \otimes \mathbb{C}\} = \{\xi \otimes 1 + (\xi \circ J) \otimes i : \xi \in T^*M\} \end{aligned}$$

$T_{1,0}$ is known as the $(J-)$**holomorphic cotangent vectors** and $T-0,1$ is known as the $(J-)$**anti-holomorphic cotangent vectors** with projections:

$$\pi_{1,0} : T^*M \otimes \mathbb{C} \to T_{1,0} \qquad \pi_{0,1} : T^*M \otimes \mathbb{C} \to T_{0,1}$$
$$\xi \mapsto \xi_{1,0} := \frac{1}{2}(\xi - i\xi \circ J) \qquad v \mapsto \xi_{0,1} := \frac{1}{2}(\xi + i\xi \circ J)$$

Thus extending the above projections to $T^*M \otimes \mathbb{C}$ we get that following decomposition:

$$T^*M \otimes \mathbb{C} \cong T_{1,0} \oplus T_{0,1}$$

Let

$$\Omega^k(M;\mathbb{C}) := \text{sections of } \Lambda^k(T^*M \otimes \mathbb{C}),$$

be the set of **complex-valued k-forms** on M. Using the above decomposition we get

$$\Lambda^k(T^*M \otimes \mathbb{C}) = \Lambda^k(T_{1,0} \oplus T_{0,1}) = \bigoplus_{l+m=k} \underbrace{\Lambda^l(T_{1,0}) \wedge \Lambda^m(T_{0,1})}_{:=\Lambda^{l,m}} = \bigoplus_{l+m=k} \Lambda^{l,m}$$

Definition 2.3.1. *The **differential forms of type** (l,m) on (M,J) are the sections of $\Lambda^{l,m}$, and let $\Omega^{l,m}$ denote the set consisting of them. Then $\Omega^k(M;\mathbb{C}) = \bigoplus_{l+m=k} \Omega^{l,m}$.*

We may then define $\pi_{l,m} : \Lambda^k(T^*M \otimes \mathbb{C}) \to \Lambda^{l,m}$, with $l+m=k$. We may then define two analogous differentials operators on forms of type (l,m), using these projections, as follows:

$$\partial : \Omega^{l,m}(M) \to \Omega^{l+1,m}, \ \partial = \pi_{l+1,m} \circ d$$
$$\overline{\partial} : \Omega^{l,m}(M) \to \Omega^{l,m+1}, \ \overline{\partial} = \pi_{l,m+1} \circ d$$

Let $f : M \to \mathbb{C}$ be any smooth function. Then we extend the exterior derivative to f by setting $df = d(Re\, f) + id(Im\, f)$. We then say that f is $(J-)$**holomorphic at** $p \in M$ if df_p is linear complex, that is $df_p \circ J = idf_p$. We then say that f is $(J-)$**holomorphic** if it is $(J-)$holomorphic at all $p \in M$. In the same way, we say that f is $(J-)$**anti-holomorphic at** $p \in M$ if df_p is complex antilinear, that is $df_p \circ J = -idf_p$. We then say that f is $(J-)$**anti-holomorphic** if it is $(J-)$anti-holomorphic at all $p \in M$.

On functions, $d = \partial + \overline{\partial}$, thus we say that f is **holomorphic** if $\overline{\partial} f = 0$ and **anti-holomorphic** if $\partial f = 0$. However, in general, this does not hold for general k-forms. Nevertheless, will see next chapter some extra conditions on the manifold for which it holds for general k-forms.

For now, suppose that $d = \partial + \overline{\partial}$. Then for any $\beta \in \Omega^{l,m}$ we have that

$$0 = d^2\beta = d(\partial\beta + \overline{\partial}\beta) = \underbrace{\partial^2\beta}_{\in\Omega^{l+2,m}} + \underbrace{\partial\overline{\partial}\beta + \overline{\partial}\partial\beta}_{\in\Omega^{l+1,m+1}} + \underbrace{\overline{\partial}^2\beta}_{\in\Omega^{l,m+2}}$$

Hence $\partial^2 = \partial\overline{\partial} + \overline{\partial}\partial = \overline{\partial}^2 = 0$. Then this allows one to define a cohomology theory in the following way. The following long sequence is exact

$$0 \longrightarrow \Omega^{l,0} \xrightarrow{\overline{\partial}} \Omega^{l,1} \xrightarrow{\overline{\partial}} \Omega^{l,2} \xrightarrow{\overline{\partial}} ...$$

Thus we may define the **Dolbeault cohomology** groups:

$$H^{l,m}_{Db}(M) := \frac{\ker \overline{\partial} : \Omega^{l,m} \to \Omega^{l,m+1}}{Im\ \overline{\partial} : \Omega^{l,m-1} \to \Omega^{l,m}}$$

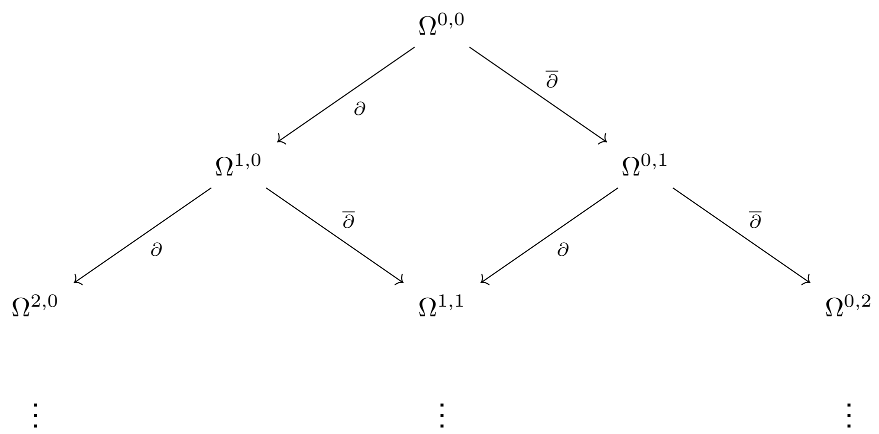

Figure 2.1: relation between $\Omega^{l,k}$.

The Dolbeault cohomology is a very important tool in complex geometry. We will dwell a little bit into it in the next chapter. Now, we will define the Nijenhuis tensor, which allows one to analyse if the almost complex structure is integrable.

Definition 2.3.2. *Let (M, J) be an almost complex manifold. Its **Nijenhuis tensor** $\mathcal{N}$ is:*

$$\mathcal{N}(v, w) := [Jv, Jw] - J[v, Jw] - J[Jv, w] - [v, w]$$

Where $v, w \in \mathcal{X}(M)$ and $[\cdot, \cdot]$ is the Lie bracket.

Chapter 3

Kähler Manifolds

In this chapter our main goal is to define a Kähler manifold. These manifolds are particularly unique as they are complex, symplectic and riemmanian manifolds. In the first section, we will define what is a complex manifold. Then we will deal with Kähler forms, which is going to introduce a restriction on the sympletic form. Finally we will dwell a bit into Hodge theory.

3.1 Complex manifolds

Definition 3.1.1. *A n-dimensional **complex manifold** M is a manifold with an atlas of charts to open sets of $\mathbb{C}^n$, such that the transitions maps are biholomorphic, that is bijective, holomorphic and with holomorphic inverse.*

Proposition 3.1.1. *Any complex manifold has a canonical almost complex structure.*

Proof. See, for instance, [4] page 101. ∎

We would like now to study what $\Omega^k(M;\mathbb{C})$ looks like. Let $U \subset M$ be a coordinate neighborhood with coordinates $z_j = x_j + iy_j,\ \forall i \in \{1, ..., n\}$, then at $p \in U$:

$$T_pM = span_{\mathbb{R}} \left\{ \frac{\partial}{\partial x_j}|_p, \frac{\partial}{\partial y_j}|_p \right\}$$

$$\begin{aligned} T_pM \otimes \mathbb{C} &= span_{\mathbb{C}} \left\{ \frac{\partial}{\partial x_j}|_p, \frac{\partial}{\partial y_j}|_p \right\} \\ &= span_{\mathbb{C}} \left\{ \frac{1}{2} \left(\frac{\partial}{\partial x_j}|_p - i \frac{\partial}{\partial y_j}|_p \right) \right\} \oplus span_{\mathbb{C}} \left\{ \frac{1}{2} \left(\frac{\partial}{\partial x_j}|_p + i \frac{\partial}{\partial y_j}|_p \right) \right\} \\ &= span_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_j}|_p \right\} \oplus span_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_j}|_p \right\} \end{aligned}$$

Where we have decomposed the space according to the eigenvalues of J and used the result from the example 2.2.1. Similarly

$$T^*M \otimes \mathbb{C} = span_{\mathbb{C}}\{dx_j, dy_j\} = span_{\mathbb{C}}\{dz_j\} \oplus span_{\mathbb{C}}\{d\overline{z}_j\}$$

Thus

$$\Omega^{l,m} = \left\{ \sum_{|J|=l, |K|=m} b_{J,K} dz_J \wedge d\overline{z}_K : \; b_{J,K} \in C^\infty(U; \mathbb{C}) \right\}$$

As we have seen, on almost complex manifolds only for function we had that $d = \partial + \overline{\partial}$. What about complex manifolds?

Theorem 3.1.1. *Let $\beta \in \Omega^k(M; \mathbb{C})$, where M is any complex manifold. Then*

$$d\beta = \partial\beta + \overline{\partial}\beta.$$

Proof. See [4] on page 104-105. ■

What this tell us is that in complex manifolds, we have a counterpart to de Rham cohomology, the Dolbeaut cohomology, which we have defined in 2.3. We will analyse the relationship between the two in section 3.3.

Example 3.1.1. *If f is a function on M then in local coordinates:*

$$\begin{aligned} df &= \sum_j \left(\frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j \right) \\ &= \sum_{j,k} \left(\left(\frac{\partial z_k}{\partial x_j} \frac{\partial f}{\partial z_k} + \frac{\partial \overline{z}_k}{\partial x_j} \frac{\partial f}{\partial \overline{z}_k} \right) \frac{1}{2}(dz_j + d\overline{z}_j) \right) + \sum_{j,k} \left(\left(\frac{\partial z_k}{\partial y_j} \frac{\partial f}{\partial z_k} + \frac{\partial \overline{z}_k}{\partial y_j} \frac{\partial f}{\partial \overline{z}_k} \right) \frac{1}{2i}(dz_j - d\overline{z}_j) \right) \\ &= \sum_j \left(\left(\frac{\partial f}{\partial z_j} + \frac{\partial f}{\partial \overline{z}_j} \right) \frac{1}{2}(dz_j + d\overline{z}_j) + \left(\frac{\partial f}{\partial z_j} - \frac{\partial f}{\partial \overline{z}_j} \right) \frac{1}{2}(dz_j - d\overline{z}_j) \right) \\ &= \sum_j \left(\frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j \right). \end{aligned}$$

Theorem 3.1.2 (Newlander-Nirenberg, 1957)**.** *Let (M, J) be an almost complex manifold. Let $\mathcal{N}$ be the Nijenhuis tensor. Then:*

$$\begin{aligned} M \text{ is a complex manifold} &\iff J \text{ is integrable} \\ &\iff \mathcal{N} \equiv 0 \\ &\iff d = \partial + \overline{\partial} \\ &\iff \overline{\partial}^2 = 0 \\ &\iff \pi_{2,0} d|_{\Omega^{0,1}} = 0 \end{aligned}$$

Proof. The proof can be found on the original paper [5] and also [4]. ∎

3.2 Kähler forms

Definition 3.2.1. *A **Kähler manifold** is a symplectic manifold (M, ω) equipped with an integrable compatible almost complex structure. The symplectic form is then called a **Kähler form**.*

A natural question that one might ask is what restrictions does this add to the symplectic form. As it turns out, quite a lot:

Proposition 3.2.1. *Locally, the Kähler form is given by*

$$\omega = \sum_{j,k=1}^{n} \frac{i}{2} h_{jk} dz_j \wedge d\overline{z}_k,$$

where, at each point of the chart, h_{jk} is a positive-definite hermitian matrix.

Proof. See [4] on page 110-111. ∎

Definition 3.2.2. *Let M be a complex manifold. A function $\rho \in C^{\infty}(M;\mathbb{R})$ is **strictly plurisubharmonic** (s.p.s.h) if on each local coordinates $U, z_1, ..., z_n$ the matrix $\left(\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p)\right)$ is positive definite at all $p \in U$.*

Proposition 3.2.2. *Let M be a complex manifold and $\rho \in C^{\infty}(M;\mathbb{R})$ be s.p.s.h. Then*

$$\omega = \frac{i}{2} \partial \overline{\partial} \rho$$

*is Kähler. ρ is then called a (global) **Kähler potential**.*

Proof. Because M is complex, ω being closed comes trivially. It is also immediate to check that ω is real, as it is equal to its conjugate.

$$J^*\omega(v,u) = \frac{i}{2} \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} dz_j \wedge d\overline{z}_k(Jv, Ju) = \frac{i}{2} \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} i(-i) dz_j \wedge d\overline{z}_k(v,u) = \omega(v,u)$$

Now because ρ is s.p.s.h we have that $h_{j,k} = \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}$ which is positive definite. ∎

Example 3.2.1. *Take $M = \mathbb{C}^n$ with the usual coordinates $z_j = x_j + iy_j$. Let*

$$\rho(z_1, ..., z_n) = \sum z_j \overline{z}_j = |\boldsymbol{z}|^2$$

Then it is easy to see that $h_{j,k} = \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} = \delta_{jk}$, thus it is s.p.s.h and the Kähler form associated to *it is:*

$$\omega = \frac{i}{2}\partial\overline{\partial}\rho = \frac{i}{2}\sum_{j,k}\delta_{jk}dz_j \wedge d\overline{z}_k = \sum_j dx_j \wedge dy_j$$

which is the standard symplectic form.

Theorem 3.2.1. *Let ω be a closed real-valued $(1,1)$-form on a complex manifold M and let $p \in M$. Then there exist a neighborhood U of p and $\rho \in C^\infty(U;\mathbb{R})$ such that on U.*

$$\omega = \frac{i}{2}\partial\overline{\partial}\rho$$

Proof. The proof can be found on [6]. ■

The function ρ is then called a (local) **Kähler potential**.

Proposition 3.2.3. *Let M be a complex manifold, $\rho \in C^\infty(M;\mathbb{R})$ s.p.s.h., X a complex submanifold, and $i : X \hookrightarrow M$ the inclusion map. Then $i^*\rho$ is s.p.s.h..*

Proof. See, for instances, page 113-114 in [4]. ■

Corollary 3.2.1. *Any complex submanifold of a Kähler manifold is Kähler.*

Definition 3.2.3. *Let (M,ω) be a Kähler manifold and X a complex submanifold, with the inclusion map $i : X \hookrightarrow M$. Then $(X, i^*\omega)$ is called a **Kähler submanifold**.*

3.3 Hodge theory

Now, we have theorem 3.1.1, what may we say more about Dolbeault cohomology?

Theorem 3.3.1 (Hodge)**.** *On a compact Kähler manifold (M,ω) the Dolbeault cohomology groups satisfy*

$$H^k_{dR}(M;\mathbb{C}) \cong \bigoplus_{l+m=k} H^{l,m}_{Db}(M) \tag{3.1}$$

with $H^{l,m} \cong \overline{H^{m,l}}$. In particular, the spaces $H^{l,m}_{Db}(M)$ are finite-dimensional.

The decomposition in 3.1 is known as the **Hodge decomposition**. In order to do this decomposition, Hodge identified the spaces of cohomology classes of forms with the space of actual forms, by choosing the representative in each class that solves the Laplace equation, which is known as the harmonic representative.
As such, we will need to define what is the Laplacian of a form. For that we will need to use the **Hodge $\star$-operator**.

Definition 3.3.1. *Consider a vector space V with inner product $\langle\cdot,\cdot\rangle$. Let $e_1, ..., e_n$ be a positively oriented orthonormal basis for V and $\omega = e_1 \wedge ... \wedge e_n$. Then the star operator is the unique linear operator*

$$\star : \Lambda^k(V) \to \Lambda^{n-k}(V)$$

such that for all $\alpha, \beta \in \Lambda^k(V)$

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega$$

It also follows that $\star\star = (-1)^{k(n-k)}$.

Example 3.3.1. *If $V = \mathbb{R}^2$ then*

$$\star(1) = dx \wedge dy \qquad \star\, dx = dy$$
$$\star(dx \wedge dy) = 1 \qquad \star\, dy = -dx.$$

Now consider a Riemannian manifold M. Then we may take $V = T_pM,\ p \in M$ and $\langle\cdot,\cdot\rangle$ the riemannian metric. Then, assuming that the manifold is compact, one can define the following inner product on the forms $\langle\cdot,\cdot\rangle : \Omega^k \times \Omega^k \to \mathbb{R}$:

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star\beta$$

Definition 3.3.2.

$$\delta = (-1)^{n(k+1)+1} \star d \star \ : \Omega^k(M) \to \Omega^{k-1}(M)$$
$$\Delta = d\delta + \delta d \ : \Omega^k(M) \to \Omega^k(M)$$

The operator δ is knows as the **codifferential**, and Δ the **Laplacian** (or sometimes de **Laplacian-Beltrami**) operator.

Example 3.3.2. *We will now check that the Laplacian defined above is the usual Laplacian for function in $\mathbb{R}^n$.*
Let $f : \mathbb{R}^n \to \mathbb{R}$. Then:

$$\delta f = (-1)^{n+1} \star d \star f = (-1)^{n+1} \star d(f dx_1 \wedge ... \wedge dx_n) = 0$$

On the other hand

$$\begin{aligned}
\delta df &= (-1)^{2n+1} \star d \star df \\
&= \sum_{i=1}^{n} (-1) \star d \star \frac{\partial f}{\partial x_i} dx_i \\
&= \sum_{i=1}^{n} (-1) \star d \left(\frac{\partial f}{\partial x_i} \star dx_i \right) \\
&= \sum_{i=1}^{n} (-1) \star d \left(\frac{\partial f}{\partial x_i} (-1)^{i-1} dx_1 \wedge ... \wedge d\hat{x}_i \wedge ... \wedge dx_n \right) \\
&= \sum_{i,j=1}^{n} (-1) \star \frac{\partial^2 f}{\partial x_i \partial x_j} (-1)^{i-1} \underbrace{dx_j \wedge dx_1 \wedge ... \wedge d\hat{x}_i \wedge ... \wedge dx_n}_{=(-1)^{j-1}\delta_{i,j} dx_1 \wedge ... \wedge dx_n} \\
&= \sum_{i}^{n} (-1) \frac{\partial^2 f}{\partial x_i^2} \\
&= \Delta f.
\end{aligned}$$

Proposition 3.3.1. *δ is the adjoint of d with respect to the above inner product.*

Proof.

$$\langle \alpha, \delta\beta \rangle = \int_M \alpha \wedge \star\delta\beta = \int_M \alpha \wedge (-1)^k d \star \beta = \int_M -d(\alpha \wedge \beta) + d\alpha \wedge \star\beta = \int_M d\alpha \wedge \star\beta = \langle d\alpha, \beta \rangle.$$

■

In a similar fashion, we may define the adjoint of ∂ and $\overline{\partial}$, $\partial^\star$ and $\overline{\partial}^\star$, respectively, which are defined by the following proposition:

Proposition 3.3.2.

$$\langle \overline{\partial}\alpha, \beta \rangle = \langle \alpha, - \star \partial \star \beta \rangle$$

$$\langle \partial\alpha, \beta \rangle = \langle \alpha, - \star \overline{\partial} \star \beta \rangle$$

Proof. See, for instances, page 82-83 in [6]. ■

Proposition 3.3.3. *The Laplacian is self-adjoint and* $\langle \Delta\alpha, \alpha \rangle = |d\alpha|^2 + |\delta\alpha|^2$

Proof.

$$\langle \alpha, \Delta\beta \rangle = \langle \alpha, d\delta\beta \rangle + \langle \alpha, \delta d\beta \rangle = \langle \delta\alpha, \delta\beta \rangle + \langle d\alpha, d\beta \rangle = \langle d\delta\alpha, \beta \rangle + \langle \delta d\alpha, \beta \rangle = \langle \Delta\alpha, \beta \rangle$$

It follows form the above computations that $\langle \Delta\alpha, \alpha \rangle = |d\alpha|^2 + |\delta\alpha|^2$. ■

The **Harmonic k-forms** are the elements of $H^k := \{\alpha \in \Omega^k : \Delta\alpha = 0\}$. Notice that

$$\Delta\alpha = 0 \iff d\alpha = \delta\alpha = 0$$

Thus they define a de Rham cohomology class. The case when M is Kähler, it can be shown that $\Delta = 2(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})$ (for instances, see pag 106 in [6] or page 103 in [7]) and $\Delta : \Omega^{l,m} \to \Omega^{l,m}$. Hence

$$H^k = \bigoplus_{l+m=k} H^{l,m}$$

Theorem 3.3.2 (Hodge)**.** *Every Dolbeault cohomology class on a compact Kähler Manifold (M, ω) possesses a unique harmonic representative*

$$H^{l,m} \cong H^{l,m}_{Db}(M)$$

Thus $H^{l,m}$ are finite dimensional. Thus, we have the following isomorphisms:

$$H^k_{dR}(M) \cong H^k \cong \bigoplus_{l+m=k} H^{l,m} \cong \bigoplus_{l+m=k} H^{l,m}_{Db}(M)$$

Proof. See [6] on page 116 for the proof. ■

There is also have the following useful decomposition.

Theorem 3.3.3 (Hodge-Dolbeault decomposition)**.** *Let M be a compact kähler manifold. Then*

$$\Omega^{l,m}(M) = H^{l,m}(H) \oplus \overline{\partial}\Omega^{l,m-1}(M) \oplus \overline{\partial}^*\Omega^{l,m+1}(M)$$

Proof. See, for instances, page 108 in [7]. ■

We also have a version of the Poincaré lemma

Lemma 3.3.4 ($\overline{\partial}$ lemma)**.** *Let M be a complex manifold and let $\omega \in \Omega^{0,1}(M)$ such that $\overline{\partial}\omega = 0$. Then, for all $p \in M$ there is a open neighborhood U of p and $\phi \in C^\infty(U;\mathbb{C})$ such that $\omega|_U = \overline{\partial}\phi$ (i.e. ω is locally $\overline{\partial}$-exact).*

Proof. See, for instances, [6] on page 25-27. ■

Lemma 3.3.5 (Global $i\partial\overline{\partial}$ lemma)**.** *Let M be a complex manifold and let ω be an exact, real, type $(1,1)$ form on M. Then, there is $\phi \in C^\infty(M)$ such that $\omega = i\partial\overline{\partial}\phi$.*

Proof. The proof can be found in [8] on page 9. ■

Chapter 4

Hamiltonian Mechanics

In this chapter we will explorer an application of symplectic geometry. In particular we will study classical mechanics.

4.1 Hamiltonian Vector fields

Let (M, ω) be a symplectic manifold and $H \in C^\infty(M; \mathbb{R})$. Then its exterior derivative dH is a 1-form. Because ω is symplectic and therefore nondegenerate, there is a unique vector field X_H such that

$$\iota_{X_H}(\omega) = dH.$$

We then call X_H the **hamiltonian vector field** and we call H an **hamiltonian function**. In particular, we may say that X_H is a hamiltonian vector field if $\iota_{X_H}\omega$ is exact.
If X_H is complete, then we may define the usual flow of X_H as usual.

$$\phi^t_{X_H} : M \to M,\ t \in \mathbb{R}$$
$$\begin{cases} \phi^0_{X_H} = \text{id}_M \\ \frac{d\phi^t_{X_H}}{dt} = X_H(\phi^t_{X_H}) \end{cases}$$

Proposition 4.1.1. *The flow is a symplectomorphism* $\forall t \in \mathbb{R}$.

Proof. Notice that for $t = 0$ then this is trivially true. Therefore we will show that $(\phi^t_{X_H})^*\omega$ is constant $\forall t \in \mathbb{R}$, which will then imply the result.

$$\begin{aligned}\frac{d}{dt}(\phi^t_{X_H})^*\omega = \frac{d}{ds}_{|s=0}(\phi^{t+s}_{X_H})^*\omega = (\phi^t_{X_H})^*\frac{d}{ds}_{|s=0}(\phi^s_{X_H})^*\omega &= (\phi^t_{X_H})^*\mathcal{L}_{X_H}\omega \\ &= (\phi^t_{X_H})^*(d\iota_{X_H}\omega + \iota_{X_H}\underbrace{d\omega}_{=0}) \\ &= (\phi^t_{X_H})^*(ddH) \\ &= 0.\end{aligned}$$

■

An important thing to note is that due to the fact that ω is a symplectic form, we have that:

$$\mathcal{L}_{X_H}H = \iota_{X_H}dH = \iota_{X_H}\iota_{X_H}\omega = \omega(X_H, X_H) = 0$$

Which shows that hamiltonian vector fields preserve their hamiltonian functions. Hence:

$$(\phi^t_{X_H})^*H = H,\ \forall t \in \mathbb{R}$$

In the same way, we say that X is a **symplectic vector field** if $\iota_X\omega$ is closed. Note that because $d^2 = 0$, every hamiltonian vector field is symplectic. Locally on a contractible open set every symplectic vector field is also hamiltonian. As a consequence, if $H^1_{dR}(M) = 0$ we have that every symplectic vector field is hamiltonian.

Notice that proposition 4.1.1 is still valid for symplectic vector fields and its proof is essentially the same.

Proposition 4.1.2. *For any form α,*

$$\iota_{[X,Y]}\alpha = \mathcal{L}_X\iota_Y\alpha - \iota_Y\mathcal{L}_X\alpha.$$

Proof. Notice that we only have to check for functions and 1-forms. Let $f \in C^\infty$. Then $\iota_{[X,Y]}f = \mathcal{L}_X\iota_Y f = \iota_Y\mathcal{L}_X f = 0$.
Let now α be a one form. Then

$$\mathcal{L}_X\iota_Y\alpha = X \cdot \alpha(Y)$$

$$\begin{aligned}\iota_Y\mathcal{L}_X\alpha &= \iota_Y d\alpha(X) + \iota_Y\iota_X d\alpha \\ &= (d\alpha(X))(Y) + d\alpha(X,Y) \\ &= Y\cdot\alpha(X) + X\cdot\alpha(Y) - Y\cdot\alpha(X) - \alpha[X,Y] \\ &= X\cdot\alpha(Y) - \alpha[X,Y]\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{L}_X \iota_Y \alpha - \iota_Y \mathcal{L}_X \alpha &= X \cdot \alpha(Y) - X \cdot \alpha(Y) + \alpha[X, Y] \\ &= \alpha[X, Y] \\ &= \iota_{[X,Y]}\alpha\end{aligned}$$

■

Proposition 4.1.3. *If X and Y are symplectic vector fields on (M, ω), then $[X, Y]$ is hamiltonian with hamiltonian function $\omega(Y, X)$.*

Proof. See page 130 in [4]. ■

Definition 4.1.1. *Let (M, ω) be a symplectic manifold. Then we define the* ***Poisson bracket*** *of two functions $f, g \in C^\infty(M; \mathbb{R})$ to be*

$$\{f, g\} = \omega(X_f, X_G).$$

Proposition 4.1.4.

$$X_{\{f,g\}} = -[X_f, X_g].$$

Proof. Notice that X_f and X_g are hamiltonian vector fields. Therefore by proposition 4.1.3 we have that

$$\iota_{[X_f, X_g]}\omega = d\omega(X_g, X_f).$$

And

$$\iota_{X_{\omega(X_f, X_G)}}\omega = d\omega(X_f, X_g) = -d\omega(X_g, X_f) = -\iota_{[X_f, X_g]}\omega$$

Thus $X_{\{f,g\}} = X_{\omega(X_f, X_G)} = -[X_f, X_g]$. ■

Proposition 4.1.5. *The Poisson bracket satisfies the Jacobi identity:*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Proof. See page 579 in [9]. ■

4.2 Actions

Let M be a manifold and let X be a complete vector field in M. Then we define $\rho_t : M \to M,\ t \in \mathbb{R}$ the flow of X. We then call $\{\rho_t; t \in \mathbb{R}\}$ the **one-parameter group of diffeomosphisms of** M and denote $\rho_t = \text{exp}(tX)$.

Let G be a Lie group. Then a representation of G on a vector space V is a group homomorphism $\phi : G \to GL(V)$. We will denote the **left action** of a Lie group G on M by

$$\psi : G \to \text{Diff}(M), \qquad g \mapsto \psi_g$$

where $\psi_g : M \to M$ is a bijection such that $\psi_g(p) = g \cdot p$. Similarly, the evaluation map associated to ψ will be represented as

$$\text{ev}_\psi : M \times G \to M, \qquad (g, p) \mapsto \psi_g(p).$$

The action ψ is smooth if the evaluation map is smooth.

Note: We will only consider left actions, although right-actions are defined in the exact same way.

Definition 4.2.1. *An action ψ is a **symplectic action** if*

$$\psi : G \to \textit{Sympl}(M, \omega) \subset \textit{Diff}(M), \qquad g \mapsto \phi_g.$$

That is, G acts by symplectomorphisms.

Definition 4.2.2. *Let ψ be a symplectic action of S^1 or $\mathbb{R}$ on a symplectic manifold (M, ω). Then we say that ψ is an **hamiltonian action** if the vector field generated by ψ is hamiltonian.*

Note: in the case of $G = \mathbb{T}^n = S^1 \times ... \times S^1$ the action is hamiltonian when the restriction to each component is hamiltonian and the hamiltonian function is preserved by the action of “the rest of G”. A similar reasoning may be done when G is a product of S^1 and $\mathbb{R}$.

Consider now the action of a Lie group G on itself by **conjugation**, that is,

$$\psi : G \to \text{Diff}(G)$$
$$g \mapsto \psi_g, \qquad \psi_g(\tilde{g}) = g \cdot \tilde{g} \cdot g^{-1}.$$

We then take the derivative of ψ_g at the identity to be the map $\text{Ad}_g : \mathfrak{g} \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra associated to G. Letting g vary, we thus obtain the **adjoint representation** (or the **adjoint action**) of G on $\mathfrak{g}$:

$$\text{Ad} : G \to GL(\mathfrak{g}) \qquad g \mapsto \text{Ad}_g$$

Let $\langle \cdot, \cdot \rangle$ be the natural pairing of $\mathfrak{g}^*$ and $\mathfrak{g}$

$$\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$$
$$(\xi, g) \mapsto \xi(g)$$

Thus we may naturally define the $\mathrm{Ad}^*_g \xi$ to be such that $\langle \mathrm{Ad}^*_g \xi, \tilde{g} \rangle$=$\langle \xi, \mathrm{Ad}_{g^{-1}} \tilde{g} \rangle$. In the same way we define the **coadjoint representation** (or the **coadjoint action**) of G on $\mathfrak{g}$:

$$\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*)$$
$$g \mapsto \mathrm{Ad}^*_g$$

Note: the inverse on the definition of $\mathrm{Ad}^*_g \xi$ is such that we obtain a left representation, the following proposition may show why this makes sense.

Proposition 4.2.1. *$Ad^*_g \circ Ad^*_h = Ad^*_{gh}$*

Proof.

$$\begin{aligned}
\langle \mathrm{Ad}^*(g \times h)(\xi), \tilde{g} \rangle &= \langle \mathrm{Ad}^*_{g \times h}(\xi), \tilde{g} \rangle \\
&= \langle \xi, \mathrm{Ad}^*_{h^{-1} \times g^{-1}} \tilde{g} \rangle \\
&= \langle \xi, \mathrm{Ad}^*_{h^{-1}} g^{-1} \tilde{g} g \rangle \\
&= \langle \mathrm{Ad}^*_h \xi, \mathrm{Ad}^*_{g^{-1}} \tilde{g} \rangle \\
&= \langle \mathrm{Ad}^*_{gh} \xi, \tilde{g} \rangle \\
&= \langle \mathrm{Ad}^*(gh) \xi, \tilde{g} \rangle.
\end{aligned}$$

■

We will now define what it means for an action of a general group to be hamiltonian. For that we have to use the “moment map”.

Definition 4.2.3. *Let (M, ω) be a symplectic manifold and G a Lie group with Lie algebra $\mathfrak{g}$. Then the action ψ is hamiltonian if there is a map*

$$\mu : M \to \mathfrak{g}^*$$

such that:

1. *For each $X \in \mathfrak{g}$, let:*
   - *$\mu^X : M \to \mathbb{R}, \mu^X(p) := \langle \mu(p), X \rangle$, be the component of μ along X,*

- *$X^{\#}$ be the vector field on M generated by the one-parameter subgroup $\{exp(tX); t \in \mathbb{R}\} \subset G$.*

Then μ^X is a hamiltonian function for the vector field $X^{\#}$, i.e.

$$\iota_{X^{\#}}\omega = d\mu^X.$$

2. *μ is equivariant with respect to the given action ψ of G on M and the coadjoint action Ad* of G on $\mathfrak{g}^*$:*

$$\mu \circ \psi_g = Ad^*_g \circ \mu, \qquad \forall g \in G.$$

Then, (M, ω, G, μ) is then called the ***hamiltonian G-space*** *and μ is a* ***moment map****.*

Example 4.2.1. *If we take $G = S^1$ then the Lie algebra is $\mathfrak{g} \cong \mathbb{R}$ and thus $\mathfrak{g}^* \cong \mathbb{R}$. Then the moment map μ must satisfy the following:*

1. *The generator of $\mathfrak{g}$ is 1 thus we take $X = 1$ and as such $\mu^X = \mu$ and $X^{\#}$ is the usual vector field associated to the action of S^1 on M. Hence $d\mu = \iota_{X^{\#}}\omega$.*

2. *μ is invariant because $\mathcal{L}_{X^{\#}}\mu = \iota_{X^{\#}}d\mu = 0$.*

The moment map may be used for "symplectic reduction". Borrowing from physics, we may realize a system of n particles as a symplectic manifold. Thus if there is a k dimensional symmetry group free action on the mechanical system, then the degrees of freedom for the position and momenta of particle may be reduced by k. This is the spirit of the symplectic redution. One of the most well known theorem about reduction is the following

Theorem 4.2.1 (Marsden-Weinstein-Meyer)**.** *Let (M, ω, G, μ) be a hamiltonian G-space for a compact Lie group G. Let $\iota : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that $\mu^{-1}(0)$ is smooth and that G acts freely on $\mu^{-1}(0)$. Then*

1. *the orbit space $M_{red} = \mu^{-1}(0)/G$ is a manifold,*

2. *$\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle, and*

3. *there is a symplectic form ω_{red} on M_{red} satisfying $\iota^*\omega = \pi^*\omega_{red}$.*

The pair $(M_{\text{red}}, \omega_{\text{red}})$ is called the **reduction** of (M, ω) with respect to G, μ.

Proof. See [4] on pag 171. ■

Chapter 5

Imaginary time flows

In this chapter we will study flows of vector fields with "imaginary time". We will first provide a motivating example and then will develop the general case .

Example 5.0.1. *Recall that in example 2.2.1 we showed that $\mathbb{R}^2$ has a Kähler structure. Consider now the hamiltonian function $h = \frac{y^2}{2}$. Then, it follows that the associated hamiltonian vector field is*

$$X_h = y\frac{\partial}{\partial x}$$

Let $\phi^t_{X_h}$ be the flow of X_h, then it must satisfy the following equation:

$$\dot{\phi}^t_{X_h} = X_h(\phi^t_{X_h})$$

It then follows that the flow is given by

$$\phi^t_{X_h}(x, y) = (yt + x, y)$$

Recall that, given real-analytic conditions, the flow of a vector field may also be denoted by e^{tX_h}. Consider now the following family of coordinates in $\mathbb{R}^n$:

$$z_t = e^{tX_h} \cdot z$$

Where z is the usual complex coordinates. Then we see that:

$$z_t = e^{tX_h} \cdot z = z(yt + x, y) = yt + x + iy.$$

Then if we take $t = is$, for some $s \in \mathbb{R}$, we get that

$$z_{is} = x + i(s+1)y.$$

Then in these new coordinates we get that

$$x = \frac{z_{is} + \overline{z_{is}}}{2}, \qquad y = \frac{z_{is} - \overline{z_{is}}}{2i(s+1)}.$$

Thus we also obtain coordinates on the tangent space given by

$$\frac{\partial}{\partial z_{is}} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i(s+1)}\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial \overline{z}_{is}} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i(s+1)}\frac{\partial}{\partial y}$$

In order to find the associated complex structure J_s we recall that on section 2.3 we saw that the $\frac{\partial}{\partial z}$ is the eigenvector associated to the eigenvalue i and that $\frac{\partial}{\partial \overline{z}}$ is the eigenvector associated to the eigenvalue $-i$. Hence we get that :

$$J_s\left(\frac{\partial}{\partial z_{is}}\right) = i\frac{\partial}{\partial z_{is}} \iff \begin{cases} J_s\left(\frac{\partial}{\partial x}\right) & = \frac{1}{(s+1)}\frac{\partial}{\partial y} \\ J_s\left(\frac{\partial}{\partial y}\right) & = -(s+1)\frac{\partial}{\partial x} \end{cases}.$$

Thus

$$J_s = \begin{bmatrix} 0 & -(s+1) \\ \frac{1}{(s+1)} & 0 \end{bmatrix}$$

And it is straightforward to check that $J_s^2 = -Id$. Now we check that J_s is compatible with ω. Let $u, v \in T_p\mathbb{R}^2$ be given by:

$$u = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \qquad J_s(u) = -b(s+1)\frac{\partial}{\partial x} + \frac{a}{s+1}\frac{\partial}{\partial y}$$
$$v = c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y} \qquad J_s(v) = -d(s+1)\frac{\partial}{\partial x} + \frac{c}{s+1}\frac{\partial}{\partial y}$$

1.

$$\omega(u, J_s(u)) = \frac{a^2}{s+1} + b^2(s+1)$$

Which is positive if $s > -1$ and is only 0 if $u = 0$.

2.

$$\omega(J_s(u), J_s(u)) = ad - cb = \omega(u, v)$$

Thus it is compatible and the associated Riemannian metric is

$$g_s = \omega(u, J_s(v)) = \frac{ac}{s+1} + bd(s+1) = \frac{1}{s+1}dx^2 + (s+1)dy^2.$$

Thus we see that (ω, J_s, g_s) *is Kähler, for all* $s > -1$. *In particular if we take* $s \to \infty$ *we see that the metric collapses in the* x*-axis while it diverges in the* y*-axis, that is, there is metric collapse of* $\mathbb{R}^2$ *into the vertical axis.*
We also see that $(\mathbb{R}^2, J_s)$ *and* $(\mathbb{R}^2, J_0)$ *are biholomorphic. Indeed consider:*

$$\begin{aligned} \varphi_s : (\mathbb{R}^2, J_s) &\to (\mathbb{R}^2, J_0) \\ (x, y) &\mapsto (x, (s+1)y) \end{aligned}$$

Then

$$d\varphi_s = \begin{bmatrix} 1 & 0 \\ 0 & (1+s) \end{bmatrix}$$

Hence:

$$d\varphi_s \circ J_s = J_0 \circ d\varphi_s$$

i.e. the map is a holomorphism.
Alternatively, we can see that when we changed the complex structure what we are doing is changing which functions are holomorphic. In particular, if f *is an holomorphism with respect to the usual complex structure, then* $f(\varphi_s)$ *is an holomorphism with respect to* J_s*:*

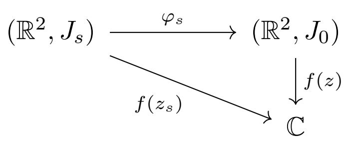

Figure 5.1: Diagram of the relationship between $(\mathbb{R}^2, J_s)$ and $(\mathbb{R}^2, J_0)$

This example is the motivation for this all chapter, as we will generalise this concept to manifolds.

5.1 Lie Series

Definition 5.1.1. *Let* M *be a compact complex manifold,* S *be a real analytic tensor field and* X *be a real analytic vector field on* M. *Then we define the exponential of* $\tau\mathcal{L}_X$ *to be the lie Series:*

$$e^{\tau\mathcal{L}_X}S = \sum_{k=0}^{\infty} \frac{\tau^k}{k!}\mathcal{L}_X^k S, \qquad \tau \in \mathbb{C}.$$

Theorem 5.1.1. *For all* S *real analytic tensor field and* X *real analytic vector field, there exists a*

T such that if $t \in \mathbb{R}$ and $|t| < T$ then $e^{t\mathcal{L}_X} S$ converges and

$$e^{t\mathcal{L}_X} S = (\phi_X^t)^* S.$$

Proof. See lemma 2.1 and theorem 3.1 in [10], or page 15-16 in [8]. ∎

Theorem 5.1.2. *For all S real analytic tensor field and X real analytic vector field, there exists a T such that if $|\tau| < T,\ \tau \in \mathbb{C}$ then $e^{\tau\mathcal{L}_X} S$ converges.*

Proof. Notice that first $e^{\tau\mathcal{L}_X} S$ converges iff $\sum_{k=0}^{\infty} \frac{\tau^k}{k!} \mathcal{L}_X^k S(X_1, ..., X_m, \omega_1, ..., \omega_n)$ converges $\forall p \in M$ and $X_i \in T_pM,\ i \in \{1, ..., m\}$ and $\omega_j \in T_p^*M\ j \in \{1, ..., n\}$. In particular, by theorem 5.1.1 we know that if t is real then there is T such that it converges. Let R be the radius of convergence of the series. Then we must have $R \geq T$ and as such $\sum_{k=0}^{\infty} \frac{\tau^k}{k!} \mathcal{L}_X^k S(X_1, ..., X_m, \omega_1, ..., \omega_n)$ converges for all $|\tau| < T$. ∎

Proposition 5.1.1. *Suppose that all the series below converges, then for $\tau \in \mathbb{C}$*

- *If S, R are tensor fields then:*

$$e^{\tau\mathcal{L}_X}(S \otimes R) = e^{\tau\mathcal{L}_X}(S) \otimes e^{\tau\mathcal{L}_X}(R)$$

- *If S is tensor field type (m, n) then:*

$$e^{\tau\mathcal{L}_X}(S(X_1, ..., X_m, \omega_1, ..., \omega_n)) = e^{\tau\mathcal{L}_X} S(e^{\tau\mathcal{L}_X} X_1, ..., e^{\tau\mathcal{L}_X} X_m, e^{\tau\mathcal{L}_X} \omega_1, ..., e^{\tau\mathcal{L}_X} \omega_n)$$

- *if $Y, Z \in \mathfrak{X}(M)$ then:*

$$e^{\tau\mathcal{L}_X}[Y, Z] = [e^{\tau\mathcal{L}_X} Y, e^{\tau\mathcal{L}_X} Z].$$

Proof. Notice that if we take τ to be real, then $e^{\tau\mathcal{L}_X}$ is simply $(\phi_X^t)^*$, i.e. the pullback, and the above properties hold for the pullback. Then using analytic continuation on both sides of the equations yields the desired result. Alternatively, one could easily use the definition to prove the above. ∎

Theorem 5.1.3 (Mourão and Nunes). *Let (M, J_0) be a compact complex manifold and X be a real analytic vector field on M. Then there is a $T > 0$ such that $\forall \tau \in B(0, T)$, there exists an integrable almost complex structure J_τ such that:*

1. *For $p \in M$ and $(U_\alpha, z_0^1, ..., z_0^n)$ J_0 holomorphic coordinates of p then there exists an open neighborhood $V_{\alpha,p}$ of p such that:*
    - $p \in V_{\alpha,p} \subset \overline{V_{\alpha,p}} \subset U_\alpha$;

- *$\overline{V_{\alpha,p}}$ is compact;*
- *the series $z_\tau^j := e^{\tau X} \cdot z_0^j$ are uniformly convergent on $V_{\alpha,p}$;*
- *$(V_{\alpha,p}, z_\tau^1, ..., z_\tau^n)$ is J_τ holomorphic coordinates of p.*

2. *There exists a unique biholomorphism $\phi_\tau : (M, J_\tau) \to (M, J_0)$ such that, using the same sets as in (1), $\phi_\tau(V_{\alpha,p}) \subset U\alpha$ and $z_\tau^j = z_o^j \circ \phi_\tau$.*

Proof. See [10] for a proof. ■

Note that by conjugation we have that $e^{\overline{\tau}X} \cdot \overline{z}_0^j = \overline{z_\tau}^j = \overline{z_0}^j \circ \phi_\tau$. Notice also that ϕ_τ depends also on the original complex structure J_0, although for sake of simplicity we will omit the complex structure , unless it is not obvious from the context. In fact, this implies that in general ϕ_τ is not a flow as $\phi_{\tau+\sigma} \neq \phi_\tau + \phi_\sigma$, unless $\tau, \sigma \in \mathbb{R}$. In general, we have the following commutative diagram:

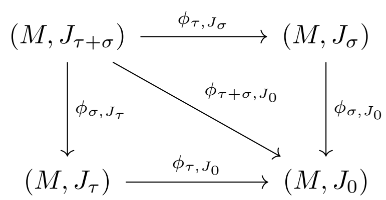

Figure 5.2: Commutative diagram of the relationships induced by the complex time flow.

Proposition 5.1.2. *Let (M, ω_0, J_0, g_0) be a compact Kähler manifold, with all the structures analytic and let h an analytic function on M and X_h the hamiltonian vector field associated to it. Let f be an analytic function on M. If $e^{\tau X_h} \cdot f$ is well-defined, then its hamiltonian function is given by $e^{\tau \mathcal{L}_{X_h}} \cdot X_f$.*

Proof. See on page 19 in [8]. ■

Corollary 5.1.1. *Let (M, ω_0, J_0, g_0), h and X_h as above. Let $(U, z^1, ..., z^n)$ be a J_0 complex coordinate chart on M and let $(V, z_\tau^1, ..., z_\tau^n)$ be a J_τ complex coordinate chart defined by $(U, z^1, ..., z^n)$ as in theorem. Then on V we have:*

$$e^{\tau \mathcal{L}_{X_h}} X_{z^j} = X_{z_\tau^j}, \qquad e^{\overline{\tau} \mathcal{L}_{X_h}} X_{z^j} = X_{\overline{z}_\tau^j}.$$

In summary, given a complex structure (M, ω_0, J_0) we obtain a new complex structure (M, ω_0, J_τ). One might ask if this new structure is a Kähler. The answer follows from this theorem:

Theorem 5.1.4 (Mourão and Nunes)**.** *Let (M, J_0, ω_0, g_0) be a compact Kähler manifold. with J_0, ω_0 and g_0 be real analytic. Let $h \in C^{an}(M)$, then there is a $T > 0$ such that:*

1. *For all* $t \in B_0(T)$ $(M, J_\tau, \omega_0, g_\tau)$ *is a Kähler manifold, where* J_τ *is the complex structure obtained from theorem 5.1.3 to* (M, J_0, ω_0, g_0) *with the vector field* X_h *and setting* $g_\tau(\cdot,\cdot) := \omega_0(\cdot, J_\tau\cdot)$.
2. *For all* $p \in M$ *there exists:*
   - $(U_\alpha, z_0^1, ..., z_0^n)$ J_0*- holomorphic coordinates neighborhood of* p*;*
   - $k_0 : U_\alpha \to \mathbb{R}$ *a local Kähler potential for* (M, ω_0, J_τ)*:*
   - $V_{\alpha,p}$ *open set such that:*
     - $p \in V_{\alpha,p} \subset \overline{V_{\alpha,p}} \subset U_\alpha$;
     - $\overline{V_{\alpha,p}}$ *is compact;*
     - *for all* $\tau \in B_0(T),\ \varphi_\tau(V_{\alpha,p}) \subset U_\alpha$;
     - *for all* $\tau \in B_0(T)$*,* k_τ *is defined by:*

$$\begin{aligned}
\theta &:= \frac{i}{2}(\partial_0 - \overline{\partial}_0)k_0\\
\alpha_t &:= \int_0^t e^{s\,X_h}(\theta(X_h))ds\\
\alpha_\tau &:= \textit{unique complex analytic continuation of } \alpha_t\\
\psi_\tau &:= \frac{-i}{2}e^{\tau\,X_h}\cdot k_0 + \tau h - \alpha_\tau\\
k_\tau &:= -2\mathit{Im}\psi_\tau
\end{aligned}$$

is well defined on $V_{\alpha,p}$ *and is a local Kähler potential for* (M, ω_0, J_τ)*.*

Proof. The proof can be found in [10] theorem 4.1 ■

5.2 The space of Kähler metrics

Definition 5.2.1. *The* ***space of Kähler metrics on*** M *in the cohomology class of* $[\omega_0]$ *is*

$$\mathcal{H}(\omega_0, J_0) := \{\varphi^*\omega_0;\ \varphi \in \textit{Diff}(M), [\varphi^*\omega_0] = [\omega_0],\ (M, J_0, \varphi^*\omega_0, g_0) \textit{ is Kähler }\}$$

Definition 5.2.2. *The* ***space of Kähler potentials on*** M *with base point* $[\omega_0]$ *is*

$$\mathcal{K}(\omega_0, J_0) := \{\phi \in C^\infty(M);\ \tilde{g}(\cdot,\cdot) := (\omega_0 + i\partial_0\overline{\partial}_0\phi)(\cdot, J_0\cdot) \textit{ is positive definite }\}$$

One important remark about this spaces comes from an application of the $\partial\overline{\partial}$-lemma 3.3.5 . By this lemma, given any other Kähler metric that is in the same cohomology class of $[\omega_0]$ can be written using a global Kähler potential. Due to this we can regard $\mathcal{H}(\omega_0, J_0)$ as being the quotient of $\mathcal{K}(\omega_0, J_0)$ by constants:

$$\mathcal{H}(\omega_0, J_0) \cong \mathcal{K}(\omega_0, J_0)/\mathbb{R} \cong \left\{\phi \in C^\infty(M);\ \tilde{g}(\cdot,\cdot) := (\omega_0 + i\partial_0\overline{\partial}_0\phi)(\cdot, J_0\cdot) \succ 0, \int_M \phi\omega_0^n = 0\right\}$$

Moreover, $\mathcal{H}(\omega_0, J_0)$ can be regarded as an infinite dimensional manifold. In particular, its tangent vector at φ_0, denoted by $\delta\varphi_0$ is a function on M. Indeed, consider the following curve in $\mathcal{H}(\omega_0, J_0)$:

$$c : I \subset \mathbb{R} \to \mathcal{H}(\omega_0, J_0)$$
$$t \mapsto \phi_t \in C^\infty(M)$$

Where ϕ_t is a family of representatives of the classes such that $\int_M \phi_t\omega_0^n = 0$ so that c is smooth map. Then we define the tangent space

$$\delta\varphi_0 := \frac{d}{dt}|_{t=0}\phi_t \in C^\infty(M).$$

Additionally, $\mathcal{H}(\omega_0, J_0)$ can be equipped with a Riemannian metric called the **Mabuchi metric** defined as:

$$\langle\delta_1\phi, \delta_2\phi\rangle = \int_M \frac{1}{n!}(\delta_1\phi \cdot \delta_2\phi)\omega_\phi \wedge ... \wedge \omega_\phi$$

where $\omega_\phi = \omega_0 + i\partial_0\overline{\partial}_0\phi$. It can also be shown that it admits a unique Levi-Civita connection. As such, a curve $\{\phi_t\}_{t\in\mathbb{R}}$ is a geodesic iff

$$\ddot{\phi}_t = \frac{1}{2}||\nabla_{\tilde{g}_t}\dot{\phi}_t||^2_{\tilde{g}_t} \tag{5.1}$$

Where $\tilde{g}(\cdot,\cdot) := (\omega_0 + i\partial_0\overline{\partial}_0\phi_t)$, $||\cdot||_{\tilde{g}_t}$ is its norm and $\nabla_{\tilde{g}_t}$ the gradient with respect to this norm.

We have now seen two different ways one may change the Kähler of a manifold: fixing ω_0 and change J_0 to J_τ; or fixing J_0 and change ω_0 to ω_τ. How different are these two approaches? As we will see next, they are equivalent. Consider (M, J, ω) a Kähler manifold and let $\phi : M \to M$ be a diffeomorphism. Then it can be seen that $(M, \phi^*J, \phi^*\omega)$ is also a Kähler manifold. Then consider the following, starting with a Kähler manifold (M, J_0, ω_0), we can obtain Kähler structure (J_τ, ω_0), just like before. We also obtain a biholomorphism $\phi_\tau : (M, J_\tau) \to (M, J_0)$. As such, we define $\omega_\tau = (\phi_\tau^{-1})^*\omega_0$. Therefore, we can regard $\phi_\tau : (M, J_\tau, \omega_0) \to (M, J_0, \omega_\tau)$ as a Kähler isomorphism. Thus (J_τ, ω_0) and (J_0, ω_τ) are isomorphic Kähler structures.

We are now ready to see an example of geodesic in $\mathcal{H}(\omega_0, J_0)$. Consider a Kähler manifold (M, J_0, ω_0) and choose $h \in C^{an}(M)$ and let X_h be the associated hamiltonian vector field. Choose T as in theorem 5.1.4, and take $\tau = it \in B_0(T)$, where $t \in \mathbb{R}$. Then, we obtain new Kähler structures (ω_0, J_{it}) for each $t \in (-T, T)$. As such, we obtain a path $\{(\omega_0, J_{it})\}_{t\in(-T,T)}$ which has an isomorphic path $\{(\omega_{it}, J_0)\}_{t\in(-T,T)}$. Our goal for the remainder of this section is to show that the path $\{(\omega_{it}, J_0)\}_{t\in(-T,T)}$ is a geodesic.

First, we fix our symplectic structure ω_0 and show that $\omega_{it} \in \mathcal{H}(\omega_0, J_0)$ for all $t \in (-T, T)$.

Proposition 5.2.1. *$\omega_{it} \in \mathcal{H}(\omega_0, J_0)$ for all $t \in (-T, T)$.*

Proof. By definition, $\phi_{it}^*\omega_{it} = \omega_0$. Now using the fact that ϕ_{it} is homotopic to the identity we obtain:

$$[\omega_0] = [\phi_{it}^*\omega_{it}] = \phi_{it}^*[\omega_{it}] = [\omega_{it}].$$

■

This result allows us to find and φ_t such that $\omega_{it} = \omega_0 + i\partial_0\overline{\partial}_0\varphi_t$, for each $t \in (-T, T)$. Now we want to get a better grip on what these φ_{it} are. Writing ω_0 is terms of the Kähler potentials we get:

$$\omega_0 = i\partial_0\overline{\partial}_0 k_0 \qquad\qquad \omega_0 = i\partial_{it}\overline{\partial}_{it} k_{it}$$

From which the following equation follows (using the notation from the above proof):

$$i\partial_{it}\overline{\partial}_{it}k_{it} = \omega_0 = \Phi_t^*\omega_{it} = i\Phi_t^*(\partial_0\overline{\partial}_0(k_0 + \varphi_t)) = i\partial_0\overline{\partial}_0((k_0 + \varphi_t) \circ \Phi_t)$$

Therefore, we are tempted to define $\varphi_t = k_{it} \circ \Phi_t^{-1} - k_0$, and thus, we need to show that is independent of the choice of k_0.

Proposition 5.2.2. *Let $p \in M$ and let U and V be neighborhoods of $\phi_\tau^{-1}(p)$ is M just like in theorem 5.1.4 with the associated Kähler potentials $k_0 : U \to \mathbb{R}$ and $k_\tau : V \to \mathbb{R}$. Then, in a neighborhood $W = \phi_\tau^{-1}(V)$ of p we define:*

$$\varphi_t|_W = k_{it} \circ \Phi_t^{-1} - k_0.$$

Then, φ_t is well defined and $\omega_{it} = \omega_0 + i\partial_0\overline{\partial}_0\varphi_t$.

Proof. See [8] on page 27. ■

Theorem 5.2.1 (Mourão and Nunes)**.** *Let φ_t be defined as above. Then φ_t is a geodesic, i.e.*

$$\ddot{\varphi}_t = \frac{1}{2}||\nabla_{\tilde{g}_{it}}\dot{\varphi}_t||^2_{\tilde{g}_{it}}$$

As such $\{(\omega_{it}, J_0)\}_{t\in(-T,T)}$ is a geodesic.

Proof. See [10] proposition 9.1. ■

Chapter 6

Prequantization

In this Chapter we will explore the first step into quantization, called prequantization. In the first section we will develop some basic concepts of line bundles, and in particular we will arrive at the integrality condition, which will be fundamental for quantization. Then in the next section we will define what an hermitian and holomorphic line bundles are. After that we will present the concept of prequantization.

6.1 Integrality condition

Let M be a smooth manifold and be $L \xrightarrow{\pi} M$ be a line bundle with connection ∇. Let α be the connection form of ∇, Ω be the curvature form of (L, ∇).

Locally, in a trivialization chart (U, ψ), the connection is of the form:

$$\nabla_X s = (X \cdot f - i\alpha(X) f) s_1,$$

where $s = f s_1$, $f \in C^\infty(U)$, $p \in U \subset M$ $s_1(p) = \psi^{-1}(p, 1)$, and $X \in \mathfrak{X}(U)$. Moreover, let F_∇ be the curvature operator defined as follows:

$$\begin{aligned} F_\nabla : \mathfrak{X}(U) \times \mathfrak{X}(U) &\to \text{End}(\Gamma(U, L)) \\ (X, Y) &\mapsto i([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}) \end{aligned}$$

where U is an open set of M. Consider the curve $\gamma : I := [a, b] \to M$. Naturally, one may want to lift this curve to the line bundle. This will ultimately lead us to a very important result.

We say that $\Gamma : I \to L$ is **parallel** (or **horizontal**), if there is a section s and a vector field $X \in \mathcal{X}(M)$ such that $\Gamma \subset s(M)$

1. $X = d\pi(\xi)$ on $\pi(\Gamma(I))$ and ξ is a tangent vector to Γ,

2. $\nabla_\xi s = 0.$

It turns out that any smooth curve γ has a unique parallel curve $\tilde{\gamma}$, when we fix a base point, that is, for each $x_a \in L_{\gamma(a)}$, satisfying $\pi \circ \tilde{\gamma} = \gamma$. This can be seen by noting that γ can be covered by a finite $\{U_j\}$ trivializations. In each of these trivilizations, the above conditions amount to find $\tilde{\gamma}$ given by the following formula

$$\tilde{\gamma}(t) = \psi_j(\gamma(t), z(t)),$$

where $z : I \to \mathbb{C}^\times$ is the unique solution of

$$z' = i\alpha_j(\xi)z, \qquad z(0) = z_0. \tag{6.1}$$

The uniqueness is guaranteed by Picard-Lindelöf theorem.

This in turn connects to the usual parallel transport along a curve γ:

$$\mathbb{P}^\gamma : L_{\gamma(a)} \to L_{\gamma(t)},$$

which associates each $v \in L_{\gamma(a)}$ to $\tilde{\gamma}(t) \in L_{\gamma(t)}$ starting at v.

Now observe the following, let $S \subset M$ be an oriented compact surface. We may assume that this surface is contained in some trivialization U (otherwise we would have to “cut” the surface into finitely many surfaces until this happens) . Choose γ a curve such that it divides S into two compact oriented surfaces S^+ and S^-. Then $\partial S^+ = \partial S^- = \gamma$. It then follows from equation 6.1 and Stokes theorem (and choosing and orientation) that the parallel transport is given by

$$\mathbb{P}^\gamma = \exp\left(i\int_{S^+}\Omega\right) = \exp\left(-i\int_{S^-}\Omega\right).$$

This then implies that

$$1 = \exp\left(i\int_S \Omega\right),$$

hence we obtain that

$$\frac{1}{2\pi}\int_S \Omega \in \mathbb{Z}.$$

Theorem 6.1.1. *Let (L, ∇) be a line bundle with connection. Then the curvature Ω satisfies the following* ***Integrality condition****:*

$$\frac{1}{2\pi}\int_S \Omega \in \mathbb{Z}$$

for every oriented closed compact surface $S \subset M$ in M.

6.2 Hermitian and Holomorphic Line Bundles

Definition 6.2.1. *Let M be a manifold, $L \xrightarrow{\pi} M$ be a complex line bundle. We say that L is a **Hermitian Line bundle** if for all fibers L_p have a Hermitian metric that smoothly depends on the base point. As such the Hermitian metric will be given by the map:*

$$H : \bigcup_{p \in M} L_p \times L_p \to \mathbb{C}.$$

We will denote this map by

$$H(p, \tilde{p}) = (p, \tilde{p}).$$

Example 6.2.1. *Let L be the trivial line bundle, i.e. $L = M \times \mathbb{C}$. Then it has a natural Hermitian metric H_0 defined as follows:*

$$H_0((a, z_1), (a, z_2)) := z_1 \overline{z_2}.$$

*This Hermitian metric is called the **constant Hermitian metric** and it follows if H is any other Hermitian metric on L is given by*

$$H((a, z_1), (a, z_2)) = H(a, a) H_0((a, z_1), (a, z_2)) = H(a, a) z_1 \overline{z_2}.$$

It is also immediate to see that given a Hermitian line bundle such that $L = M \times \mathbb{C}$ is isomorphic to the trivial line bundle with constant Hermitian metric H_0.

What about for general lines bundles? Does a hermitian metric always exists? The answer is yes. Indeed by the above observation we see that locally this metric always exists. Then using partitions of unity (in the exact same way that one proves that every manifold admits Riemannian metric) the result follows.

Definition 6.2.2. *Let M be a manifold, $L \xrightarrow{\pi} M$ be a Hermitian line bundle. A connection ∇ on L is said to be compatible with H if for all sections $s, t \in \Gamma(U, L)$ and all vector fields $X \in \mathfrak{X}(U)$, $U \subset M$ open, we have:*

$$\mathcal{L}_X(s, t) = (\nabla_X s, t) + (s, \nabla_X t).$$

*In particular, such a connection is said to be **Hermitian connection**.*

The above condition is an analogue to the condition for a connection to be compatible with the metric. From now on, every time we refer to "A Hermitian Line bundle with connection", we assume that the connection is Hermitian. One important remark is

Suppose now that M is also a complex manifold. Naturally, we may want to consider now **holomorphic line bundles**, which are simply complex line bundles whose trivialization maps are holomorphic. As a consequence, the transitions function are also going to be holomorphic. A section $s \in \Gamma(U, L)$ is said to be a holomorphic section if s is a holomorphic map. Let $\Gamma_{hol}(U, L)$ be the space of such sections.

Definition 6.2.3. *Let M be a manifold, $L \xrightarrow{\pi} M$ be a Holomorphic line bundle. A connection ∇ on L is said to be a* ***holomorphic connection*** *if on all trivializing holomorphic frames over U and $s \in \Gamma_{hol}(U, L)$, the map*

$$X \mapsto \nabla_X s/s, \ X \in \mathfrak{X}(U) \ X \text{ holomorphic}$$

is a holomorphic one-form.

Just like before, a connection ∇ on L is said to be ***compatible with holomorphic structure*** *on L if on all trivializing holomorphic frame U and $s \in \Gamma_{hol}(U, L)$ the one form*

$$X \mapsto \nabla_X s/s$$

is a (1,0)-form, that is, in local holomorphic coordinates

$$\nabla s = ds + \sum_j f_j dz_j s,$$

where $f_j : U \to \mathbb{C}$ are holomorphic.

It follows, using basically the same proof for the connection, that every holomorphic line bundle admits a holomorphic connection compatible with the holomorphic structure on L.

6.3 Prequantization

The concept of quantization comes from physics. The main idea is to obtain a “quantum system” based on a mechanical systems. These mechanical systems are mathematically described as a tuple (M, ω, H), where H is a scalar function. This tuple is known as a Hamiltonian system. Dirac was the first to try to describe this idea, and according to him, the quantization is a $\mathbb{C}$-linear map from the space of smooth functions on the classical phase space to the space of linear operators on some Hilbert space of “quantum states”, denoted by $\mathbb{H}$,

$$q : C^\infty(M) \to \text{Op}(\mathbb{H}),$$

such that the following conditions are satisfied:

1. $q(1) = \text{id}_{\mathbb{H}}$,
2. $q(f)$ is self-adjoint,
3. $[q(f), q(g)] = iq(\{f, g\}), \ \forall f, g \in C^\infty(M)$

4. If the set $\{f_1, ..., f_n\}$ is complete, in the sense of if $\{g, f_i\} = 0, \forall i \in \{1, ..., n\}$ then g is constant, then the set $\{q(f_1), ..., q(f_n)\}$ is also complete, in the sense that if $[A, q(f_i)] = 0\ \forall i \in \{i, ..., n\}$ then $A = a\text{id}_{\mathbb{H}}$ for some $a \in \mathbb{H}$. This condition says that the representation of $\mathbb{H}$ is irreducible.

As it turns out, this is too much to ask for, and even in the most elementary examples, such as $M = \mathbb{R}^2$, there is no solution. So in general, one weakens the above requirements. However, this general idea still leads to rich and interesting Hilbert spaces $\mathbb{H}$. The main goal of this chapter is to begin to see how one may obtain these spaces and to start to analise its structure.

Definition 6.3.1. *A symplectic manifold (M, ω) is said to be **quantizable** if there exists a complex line bundle $L \xrightarrow{\pi} M$ with connection ∇ such that $Curv(L, \nabla) = -i\omega$.*
*A **prequantum Line Bundle** (L, ∇, H) on a symplectic manifold (M, ω) is a Hermitian line bundle (L, H) together with a compatible connection ∇ such that $Curv(L, \nabla) = -i\omega$.*

Recall that we have seen that the condition for a manifold to be quantizable is a topological one, given by the integrality condition. Indeed, now is a good time to see an example.

Example 6.3.1. *Let $M = T^*Q$ for some $Q \subset R^n$ open, and consider the usual symplectic form $\omega = \sum_i dp_i \wedge dq_i$ with tautological form $\alpha = \sum_i q_i dp_i$. Consider L to be the trivial line bundle with connection form α. Then it follows that the curvature is $-i\omega$.*
Moreover, consider any closed, compact and oriented surface $S \subset M$. Then by Stokes, it follows that:

$$\int_S \omega = 0.$$

So (T^*Q, ω) is quantizable. In fact, by the exact same reasoning, any sympletic manifold (M, ω) such that ω is exact, is quantizable.

Theorem 6.3.1. *Let (M, ω) be a symplectic manifold and consider (L, ∇, H) a prequantum line bundle over M. Then the following operator*

$$q : C^\infty(M, \mathbb{C}) \to \textit{End}_{\mathbb{C}}(\Gamma(M, L))$$
$$f \mapsto -i\nabla_{X_f} + f$$

is $\mathbb{C}$-linear and satisfies the following:

- $q(1) = \textit{id}_{\Gamma(M,L)}$,
- $[q(f), q(g)] = iq(\{f, g\})$.

*This operator is known as the **prequantum operator**.*

Proof. It trivially follows that q is $\mathbb{C}$-linear and that $q(1) = \text{id}_{\Gamma(M,L)}$. So we only have to prove the second condition. Now notice that given $X, Y \in \mathfrak{X}(M)$ we have by definition of the curvature tensor:

$$i\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) = F_\nabla(X,Y) = \omega(X,Y)$$

Taking $X = X_f$ and $Y = X_g$ and by proposition 4.1.4, $[X_f, X_g] = -X_{\{f,g\}}$ we obtain

$$[\nabla_{X_f}, \nabla_{X_g}] = -i\{f,g\} - \nabla_{X_{\{f,g\}}}.$$

Therefore

$$\begin{aligned}
[q(f), q(g)] &= \left[-i\nabla_{X_f} + f, -i\nabla_{X_g} + g\right] \\
&= (-i)^2 \left[\nabla_{X_f} \nabla_{X_g}\right] - if\nabla_{X_g} + i \underbrace{\nabla_{X_g} \circ f}_{\mathcal{L}_{X_g} f + f\nabla_{X_g}} + ig\nabla_{X_f} - i \underbrace{\nabla_{X_f} \circ g}_{\mathcal{L}_{X_f} g + g\nabla_{X_f}} \\
&= (-i)^2 \left(-i\{f,g\} - \nabla_{X_{\{f,g\}}}\right) - i\left(-\underbrace{\mathcal{L}_{X_g} f}_{\{f,g\}} + \underbrace{\mathcal{L}_{X_f} g}_{-\{f,g\}}\right) \\
&= i\left(-\{f,g\} - i\nabla_{X_{\{f,g\}}} + 2\{f,g\}\right) \\
&= i\left(-i\nabla_{X_{\{f,g\}}} + \{f,g\}\right) \\
&= iq(\{f,g\}).
\end{aligned}$$

■

In spirit of quantum mechanics, we want to represent our observables as operators on a Hilbert space (the so called representation space of a model). As such, we replace $\Gamma(M, L)$, the space in which our operator acts, with a natural Hilbert space of sections.

Recall now that if (M, ω) is simplectic manifold, there is a natural volume form (called the Liouville volume) given by

$$\text{vol} := \frac{\omega^n}{n!}.$$

Definition 6.3.2. *Let (M, ω) be a simplectic manifold and (L, H) an Hermitian line bunlde over it. We define the* ***space of square integrable smooth sections*** *to be*

$$H_{pre} := \left\{ s \in \Gamma(M, L) : \int_M |s|^2 d\textit{vol} < \infty \right\},$$

where $|s|^2 = (s, s)$. This space is a pre-hilbert space with respect to the inner-product

$$\langle s, t \rangle := \int_M (s, t) d\textit{vol},$$

and its completion with respect to the norm

$$||s|| := \left(\int_M |s|^2 d\textit{vol}\right)^{1/2}$$

is the Hilbert space $\mathbb{H}(M, L)$.

We will write $\mathbb{H}$ for $\mathbb{H}(M, L)$ when (M, L) is clear from the context.
It is obvious that given $f \in C^\infty(M, \mathbb{C})$ that $q(f)$ is defined on the space of compact supported sections $(\Gamma_0(M, L))$, which in turn is a subspace of $\mathbb{H}$, and in turn $q(f)(\Gamma_0(M, L)) \subset \Gamma_0(M, L)$, thus $q(f)$ induces an operator whose domain contains $\Gamma_0(M, L)$.

Theorem 6.3.2. *Whenever $f \in C^\infty(M)$ is such that X_f is complete, then $q(f)$ is an essentially self-adjoint operator in $\mathbb{H}$ (that is, the closure of $q(f)$ is a self-adjoint operator).*

Proof. See proposition 7.16 in [11]. ■

Example 6.3.2. *Let $M = T^*Q$, where $Q \subset \mathbb{R}^n$ open and consider the usual symplectic form $\omega = \sum_i dp_i \wedge dq_i$ with the tautological form $\alpha = \sum_i q_i dp_i$. Take $L = M \times \mathbb{C}$. Consider now $f = p_j$ and $g = q^j$. Then we get that*

$$X_f = \frac{\partial}{\partial q_j},\ X_g = -\frac{\partial}{\partial p_j}.$$

Thus

$$P_j := q(f) = -i\nabla_{X_f} + f = -i\left(\frac{\partial}{\partial q_j} + i\alpha\left(\frac{\partial}{\partial q_j}\right)\right) + p_j = -i\frac{\partial}{\partial q_j} + p_j,$$

and similarly,

$$Q_j := q(g) = -i\nabla_{X_g} + g = -i\left(-\frac{\partial}{\partial p_j} + i\alpha\left(-\frac{\partial}{\partial p_j}\right)\right) + q_j = -i\left(-\frac{\partial}{\partial p_j} - iq_j\right) + q_j = -i\frac{\partial}{\partial p_j}.$$

It follows from the above theorem that the conditions are satisfied for $\Gamma(M, L) \cong C^\infty(M)$, and also on the space $\mathbb{H} = L^2(T^*Q)$.
However, looking from the point of view of quantum mechanics, we see that the resulting Hilbert space is too big: the wave function on $\mathbb{H}$ should only depend on n variables, rather than $2n$ variables we obtained. From a mathematical point of view, the representation of Q_j, P_j is not irreducible, that is, there is a generalized subspace of $\mathbb{H}$ for which the action of Q_j, P_j is invariant. That generalized subspace is

$$\mathbb{H}_0 = \{f \in \mathbb{H} :\ f = g \circ \pi\},$$

where π is the usual projection of T^*Q into Q. So a natural candidate for our representation space is $\mathbb{H}_0$, in which we obtain:

$$P_j = p_j,\ Q_j = -i\frac{\partial}{\partial p_j}.$$

Chapter 7

Polarizations

In the last chapter we saw the first step into quantization, called prequantization. In particular, in example 6.3.2 we saw that, in general, the Hilbert space we obtain from this process is too big. However in that particular case we could deduce a way to reduce it. In fact, we can generalize the procedure for general symplectic manifolds through the use of so called polarizations. As such, this chapter focuses mainly on these objects. In particular we are going to see three main types of polarization: real polarizations, complex polarization and Kähler polarizations.

7.1 Real Polarizations

Definition 7.1.1. *Let* (M, ω) *be a symplectic manifold. A* ***real polarization*** *on* M *is a foliation* $D \subset TM$ *on* M*, if it is maximal isotropic, that is, for all* $p \in M$*:*

$$\omega_p(X, Y) = 0, \ \forall X, Y \in D_p$$

and there is no subspace of T_pM *containing* D_p *properly with the above property.*

However, there might not exist a real polarization, as can be seen in the following example.

Example 7.1.1. *Take* S^2 *with the usual symplectic form* ω*. Now,* $H^1(S^2, \mathbb{Z}/2\mathbb{Z})$ *is trivial, because* S^2 *is simply connected. It turns out that classes of this group (also known as the first Stiefel-Whitney class) uniquely determines real line bundles. Therefore it follows that all real line bundles over* S^2 *are trivial. Moreover, given a one dimensional distribution on* S^2 *would have to have a nowhere vanishing section which would in turn means that* TS^2 *would also have to have a nowhere vanishing section which contradicts the Hairy ball theorem.*

This example serves to motivate us to generalize our notion of real polarization into a complex polarization.

Note: One often uses **singular real polarizations**, where some of the leaves of the polarization are allowed to be singular. Of course, singular real polarizations exist on S^2.

7.2 Complex Polarization

Definition 7.2.1. *Let (M,ω) be a $2n$-symplectic manifold. A **complex polarization** P on M is a complex vector subspace of $TM \otimes \mathbb{C}$ of dimension n such that:*

- *For all $X, Y \in \Gamma(M,P)$ we have that $[X,Y] \in \Gamma(M,P)$ (i.e., P is involutive);*
- *For all $p \in M$, P_p is maximal isotropic (i.e., P is Lagrangian);*
- *$D_p = P_p \cap \overline{P}_p \cap T_pM$ has constant rank $k \in \{0, ..., n\}$.*

Furthermore, we say that a complex polarization is

- **Real**, if $P = \overline{P}$;
- **Pseudo Kähler**, if for all $p \in M$, $D_p = \{0\}$ and **Kähler** if the hermitian form induced is positive definite;
- **Strongly involutive**, if the distribution defined as $E_p = (P_p + \overline{P}_p) \cap T_pM$ is integrable;
- **Reducible**, if the orbit space M/D is a smooth manifold and the projection $\pi : M \to M/D$ is a submersion.

Example 7.2.1. *Take $M = T^*\mathbb{R}^n \cong \mathbb{C}^n$ using $z_j = p_j + iq_j$. Then usual symplectic form is $\omega = \sum_i dp_i \wedge dq_i = \frac{i}{2}\sum_j dz_j \wedge d\overline{z}_j$. From example 2.2.1 we have that*

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial p_j} - i\frac{\partial}{\partial q_j}\right), \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial p_j} + i\frac{\partial}{\partial q_j}\right).$$

Thus we consider the following polarization

$$P := \text{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial \overline{z}_j},\ 1 \leq j \leq n\right\}.$$

In fact, it follows trivially from Schwarz lemma that this polarization is involutive. Moreover, it follows that this distribution is Lagragian and it is maximal isotropic because dim$P = n$. Furthermore,

$$\overline{P} := \text{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_j},\ 1 \leq j \leq n\right\},$$

*and thus, $P_p \cap \overline{P}_p = \{0\}$, hence $P_p \oplus \overline{P}_p = T_pM \oplus \mathbb{C}$. Thus we conclude that P is a Kähler polarization, and it is known as the **holomorphic polarization**.*
*It also follows that $\overline{P}$ is a Kähler polarization known as the **antiholomorphic polarization**.*

Going back to our example of S^2, it follows that S^2 also has a holomorphic polarization defined locally by $\frac{\partial}{\partial \bar{z}}$ for a local complex coordinate z.

As the reader might have guessed by the name of Kähler polarization, that a having a Kähler polarization might be related to the Kähler structure on a manifold. Indeed, it turns out that a Kähler manifold always have a Kähler polarization. But perhaps more strikingly, these two notions are equivalent, as we shall see in the following theorem.

Theorem 7.2.1. *Every Kähler manifold admits a Kähler polarization. Moreover, if (M, ω) is a symplectic manifold and admits a Kähler polarization, then it has a compatible complex structure.*

Proof. Suppose that (M, J, ω) is Kähler manifold of dimension $2n$. Then it follows that dim$T_{(1,0)} =$ dim$T_{(0,1)} = n$. Moreover, given $x, y \in T_{(1,0)}$ and $z, w \in T_{(0,1)}$ it follows from the compatibility that :

$$\omega(x, y) = \omega(Jx, Jy) = \omega(ix, iy) = -\omega(x, y)$$
$$\omega(z, w) = \omega(Jz, Jw) = \omega(-iz, -iw) = -\omega(z, w)$$

From the dimensions, it follows that both $T_{(1,0)}$ and $T_{(0,1)}$ are maximal isotropic. Futhermore $T_{(1,0)} \cap T_{(0,1)} = \{0\}$ so D_p has constant rank. It follows from theorem 3.1.2 that $T_{(1,0)}$ and $T_{(0,1)}$ are involutive, and therefore Kähler polarizations.

Suppose now that (M, ω) has a Kähler polarization $\mathcal{P}$. Then $TM \otimes \mathbb{C} = \mathcal{P} \oplus \overline{\mathcal{P}}$. As such, given $v \in T_pM \otimes \mathbb{C}$ we have that $v = w_1 + w_2$, where $w_1 \in \mathcal{P}_p$ and $w_1 \in \overline{\mathcal{P}}_p$. Now take

$$J_p : T_pM \to T_pM, \qquad w \mapsto -iw_1 + iw_2$$

Thus

$$\begin{aligned}
\omega(Jx, Jy) &= \omega(-ix_1, -iy_1) + \omega(-ix_1, iy_2) + \omega(ix_2, -iy_1) + \omega(ix_2, iy_2) \\
&= -\underbrace{\omega(x_1, y_1)}_{=0} + \omega(x_1, y_2) + \omega(x_2, y_1) - \underbrace{\omega(x_2, y_2)}_{=0} \\
&= \omega(x_1, y_2) + \omega(x_2, y_1) \\
&= \omega(x, y).
\end{aligned}$$

So J is compatible with ω. Moreover, $\mathcal{P} = T_{(0,1)}$ and $\overline{\mathcal{P}} = T_{(1,0)}$. Finally, it follows that the Riemannian metric $g(x, y) := \omega(x, Jy)$ is positive definite, and so we conclude that J is a almost complex structure on M compatible with ω. By Newlander-Nirenberg theorem it follows that J is integrable, and thus (M, J, ω, g) is Kähler. ∎

Chapter 8

Spaces of Polarized Sections

We now have all the ingredients necessary, we are able to construct our representation space, that is, the Hilbert space of our quantum model. For that we will have to choose a complex polarization on our manifold, and then consider polarized section, in order to obtain our desired space. In general there is no easy way to do so, however, if our manifold is Kähler, there is indeed a general way to proceed. As such, in this chapter we will start to define what are polarized functions and sections, then we will why the Kähler is so special.

8.1 Polarized sections

Definition 8.1.1. *Let (M, ω) be a symplectic manifold and let $\mathcal{P}$ be a complex polarization on it. We say that a function $f \in C^\infty(M, \mathbb{C})$ is a **polarized function** if*

$$\mathcal{L}_X f = 0, \ \forall X \in \Gamma(M, \mathcal{P}).$$

*Similarly, given $L \xrightarrow{\pi} M$ line bundle with connection ∇, a **polarized section** is a $s \in \Gamma(M, L)$ such that*

$$\nabla_X s = 0, \ \forall X \in \Gamma(M, \mathcal{P}).$$

Intuitively, the polarized sections are sections of L which are constant along the fibers of $\mathcal{P}$.

We also see that the Hilbert space of “waves functions” we are looking for should be based on the space

$$\Gamma_{\nabla, \mathcal{P}} := \{s \in \Gamma(M, L); \ \nabla_X s = 0, \ \forall X \in \Gamma(M, \mathcal{P})\}.$$

$\Gamma_{\nabla, \mathcal{P}}$ is clearly a vector space over $\mathbb{C}$, and it is also easy to see that it a module over the ring of polarized functions. However, in general, it is not the case that if s is polarized that $\nabla_X s$ is going

to be polarized. We will deal with this problem later.
What we would like now is to construct our Hilbert space from $\Gamma_{\nabla,\mathcal{P}}$. Inevitably, we want to endow this space with a inner-product, but then one has to contemplate which inner-product is reasonable, as contrary to the prequantum space, integration along the induced volume form no longer works.

8.2 Kähler quantization

As we have seen in the last chapter, given a Kähler manifold we have a Kähler polarization $\mathcal{P}$ on M such that in local holomorphic coordinates $\mathcal{P}$ is the holomorphic polariazation.

In addiction to this, if we have a prequantum bundle (L, ∇, H) on M, then we have a unique complex structure on L compatible with the prequantum bundle. Hence, L is naturally a holomorphic line bundle, and the polarized sections are the holomorphic section.

As such, the symplectic form induces a volume form on M given by vol $= C\omega^n$, where C is a positive real constant and therefore our Hilbert space is

$$\mathbb{H}_{\mathcal{P}} := \left\{ s \in \Gamma_{Hol}(M, L) : \int_M \langle s, s \rangle d\text{vol} < \infty \right\}.$$

Example 8.2.1. *Let $M = T^*\mathbb{R}^n$ and consider the usual symplectic form $\omega = \sum_i dp_i \wedge dq_i$. Take the usual complex coordinates $z_j = p_j + iq_j$ and consider $\mathcal{P}$ the holomorphic polarization*

$$\mathcal{P} = \textit{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_j},\ 1 \leq j \leq n \right\}.$$

Consider also $L \xrightarrow{\pi} M$ to be the trivial line bundle. In this coordinates the symplectic form is given by $\omega = \frac{i}{2} \sum_i dz_i \wedge d\overline{z}_i$ and its fundamental form is given by

$$\alpha = \frac{i}{2} \sum_i \overline{z}_i dz_i,$$

which yields $\alpha(X) = 0,\ \forall X \in \Gamma(M, \mathcal{P})$.

Rather then defining the connection in terms of $s_1(p) = (p, 1),\ p \in M$, we consider the section

$$s_e(p) = \left(p, \textit{exp}\left(-\frac{\pi}{2}||z||^2\right)\right) = \textit{exp}\left(-\frac{\pi}{2}||z||^2\right) s_1(p),$$

and thus define the connection to be

$$\nabla_X f s_e := (\mathcal{L}_X f + i\alpha(X) f) s_e,\ f \in C^\infty(M).$$

The section fs_e is polarized iff $\mathcal{L}_X f = 0$ for all $X \in \Gamma(M, \mathcal{P})$. But by the choice of $\mathcal{P}$, this is equivalent to ask that f is holomorphic. Consequently

$$\Gamma_{\nabla,\mathcal{P}}(M, L) \cong \mathcal{O}(\mathbb{C}^n).$$

As a result we obtain that

$$\mathbb{H}_{\mathcal{P}} = \left\{ f \in \mathcal{O}(\mathbb{C}^n);\ \int_{\mathbb{C}^n} |f|^2 \textit{exp}(-\pi|z|^2) d\textit{vol} \leq \infty \right\},$$

where, in this case, vol is the Lebesgue measure on $\mathbb{C}^n$. $\mathbb{H}_{\mathcal{P}}$ is Hilbert space with innerproduct given by

$$\langle f, g \rangle := \int_{\mathbb{C}^n} \overline{f} g \textit{exp}(-\pi|z|^2) d\textit{vol}.$$

Let's now compare this with our previous result, that is, let's compute $q(g)$ for $g = z_j$ and for $g = \overline{z}_j$. We have that

$$X_{z_j} = -2i\frac{\partial}{\partial \overline{z}_j},\ X_{\overline{z}_j} = 2i\frac{\partial}{\partial z_j},$$

and therefore for f holomorphic

$$\nabla_{X_{z_j}} f s_1 = \left(-2i\frac{\partial f}{\partial \overline{z}_j} + i\alpha \left(-2i\frac{\partial}{\partial \overline{z}_j} f \right) \right) s_1 = 0,$$

so $q(z_j) = z_j$. Similarly,

$$\begin{aligned}
\nabla_{X_{\overline{z}_j}} f s_1 &= \left(2i\frac{\partial f}{\partial z_j} + i\alpha \left(2i\frac{\partial}{\partial z_j} f \right) \right) s_1 \\
&= \left(2i\frac{\partial}{\partial z_j} - i\overline{z}_j \right) f s_1,
\end{aligned}$$

so

$$q(\overline{z}_j) = 2\frac{\partial}{\partial z_j} - \overline{z}_j + \overline{z}_j = 2\frac{\partial}{\partial z_j},$$

which is exactly what we have obtained before, up to constants.

Let's compute ∇s_1, where $s_1 = \textit{exp}(\frac{\pi}{2}||z||^2)s_e = hs_e$,

$$\begin{aligned}
\nabla s_1 &= \left(dh + 2\pi i \frac{i}{2} \sum_i \overline{z}_i h \right) s_e \\
&= \left(\frac{\pi}{2} \sum_i (\overline{z}_j dz_j + z_j d\overline{z}_j) - \pi \overline{z}_i \right) h s_e \\
&= 2\pi i \left(\frac{i}{4} \sum_i (\overline{z}_j dz_j - z_j d\overline{z}_j) \right) s_1.
\end{aligned}$$

So it turns out that if we wanted to define the connection using s_1 we would have to use the *sympletic potential* $\beta = \frac{i}{4}\sum_i(\bar{z}_j dz_j - z_j d\bar{z}_j)$. *Indeed, if we denote this connection by* ∇' *then we obtain that*

$$\Gamma_{\nabla',\mathcal{P}}(M,L) = \{fs_1;\ f \in \mathcal{O}(\mathbb{C}^n)\}$$
$$\Gamma_{\nabla,\mathcal{P}}(M,L) = \{fs_e;\ f \in \mathcal{O}(\mathbb{C}^n)\} = \{fs_1;\ f\mathit{exp}(\frac{\pi}{2}||z||^2) \in \mathcal{O}(\mathbb{C}^n)\}.$$

8.3 Directly quantizable observables

In a nutshell, what we have now is the created a Hilbert space. However constructing this hilbert space is not enough. As such in this section we will dwell a bit into a different question. For which f will $q(f)$ be an operator on this Hilbert space? In particular, if s is a polarized section, will $q(f)s$ be polarized? In general this is not the case. As such we need to restrict our space of functions.

Definition 8.3.1. *We say that a vector field* X ***preserves*** P *if for all* $Y \in P$ *we have that* $[X, Y] \in P$. *We say that a function* f *is* ***directly quantizable with respect to*** P *if* X_f *preserves* P. *We denote the space of all directly quantizable functions to be* $\mathfrak{R}_P$.

As we may see in the following proposition, if we restrict to work with functions on $\mathfrak{R}_P$, then $q(f)s$ will be polarized whenever s is.

Proposition 8.3.1. *Let* $f \in \mathfrak{R}_P$ *and* s *polarized section of* P. *Then* $q(f)s$ *is polarized.*

Proof.

$$\begin{aligned}\nabla_X(q(f)s) &= \nabla_X\left(-i\nabla_{X_f}s + Fs\right)\\ &= -i\nabla_X\nabla_{X_f}s + (\mathcal{L}_X f)s\end{aligned}$$

Using the curvature we see that

$$\frac{1}{2\pi i}\nabla_X\nabla_{X_f}s = \frac{-1}{2\pi i}\nabla_{X_f}\nabla_X s + \omega(X, X_f)s - \frac{1}{2\pi i}\nabla_{[X,X_f]}s.$$

But because f is directly quantizable and s is polarized, the above is equal to $\omega(X, X_f)s$, which in turn is equivalent to $(-\mathcal{L}_X f)s$, which proves our claim. ■

8.4 Existence of Polarized sections

We have now seen how to use polarization in order to obtain the correct Hilbert space. However, we only saw that for Kähler Polarizations. This case is rather special, so what about the general case? As one might have guessed, by the title of this section, we might run into a problem fairly quickly, does there always exist polarized sections? In particular, does there always exist non-zero global polarized sections? The next example, or rather counterexample, give us a negative result.

Example 8.4.1. *Consider $M = T^*S^1$. The cotangent bundle is trivial so in fact $M = S^1 \times \mathbb{R}$. Take the trivial line bundle $L = M \times \mathbb{C}$ and H the induced hermitian inner product on L. Consider the connection given by the Liouville form $\alpha = -pdq$. Finally consider the polarization $\mathcal{P}$ generated by the angle variable, i.e.,*

$$\frac{\partial}{\partial q}.$$

This polarization is also known as the horizontal polarization. As such, one sees that a general section $s = fs_1,\ f \in C^\infty(M)$ is a polarized section if in local coordinates we have

$$\frac{\partial f}{\partial q} = 2\pi i p f.$$

Thus we have that the solution of this ODE is given by

$$f(p, q) = g(p)\textit{exp}(2\pi i p q),$$

where, g is an arbitrary smooth function on $\mathbb{R}$. But f must be periodic on q which implies that we must have $p \in \mathbb{Z}$. That is, there are only non-zero solutions for a fixed p. But f is continuous, so f must be zero.

There is another more general way to see why this problem raises from. Notice that the leaves of our polarization are $S_p = S^1 \times \{p\}$ for $p \in \mathbb{R}$. So considering the restriction of the line bundle to the leaves $L|_{S_p} \xrightarrow{\pi} S_p$ we obtain that the restriction of the connection $\nabla|_{S_p}$ must be a flat connection.

So consider $x = (1, p) \in S_p$ and the curve $\gamma(t) = (\textit{exp}(2\pi i t), p)$, for $t \in [0, 1]$. Then the parallel transport along γ is given by

$$Q(\gamma) = \textit{exp}\left(-i\int_\gamma pdq\right) = \textit{exp}(-ip).$$

So if we have $s \in \Gamma(M, L)$ global polarized section, then its restriction to S_p is a horizontal section and therefore determines the parallel transport given by $\nabla|_{S_p}$. So if $s(x) \neq 0$, then the parallel transport is $s(\gamma(0)) \mapsto s(\gamma(1))$ and $\gamma(0) = \gamma(1) = x$ so $Q(\gamma) = 1$, which implies that $\frac{p}{2\pi} \in \mathbb{Z}$. Therefore, we conclude that s must be zero outside the set

$$S := \bigcup \left\{S_p;\ \frac{p}{2\pi} \in \mathbb{Z}\right\} = S^1 \times 2\pi\mathbb{Z}.$$

Hence, by continuity of s, we get that s must be zero. This set is known as the ***Bohr-Sommerfeld set****.*

As it turns out, this hindrance is not exclusive to this problem, but rather arises from the fact that the leaves are not simply connected.

Given $p \in M$ and a loop γ we have seen that $Q(\gamma)$ is going to be a complex number. If we take the collection of all of these $Q(\gamma)$ we obtain a group, which we will denote by $G(p)$. This group is known as the **Holonomy group** of the connection at p, and if we assume that M is connected, we have that $G(p) \cong G(q)$, $p, q \in M$. In the case of a flat connection we get the following natural group homomorphism:

$$\begin{aligned} \text{Hol}_\nabla : \pi_1(M) &\to G(x) \\ [\gamma] &\mapsto Q(\gamma). \end{aligned}$$

By construction of $G(x)$, this homomorphism is always surjective. Consequently, if M is simply connected, we have that $G(x)$ is the trivial group.

Let (M, ω) be symplectic manifold and let (L, ∇, H) be the prequantum line bundle. Let $\mathcal{P}$ be a reducible complex polarization. Fix a leaf Λ of the distribution $D = \mathcal{P} \cap \overline{\mathcal{P}} \cap TM$, i.e., $\Lambda = \pi^{-1}(x)$ for a suitable $x \in M/D$. It follows that ∇_Λ is a flat connection.

Definition 8.4.1. *Let $G_\Lambda(x)$ denote the holonomy group of ∇_Λ. Then the **Bohr-Sommerfeld set** is the set*

$$S := \bigcup \{p \in M : \ G_\Lambda(p) = \{1\}\}.$$

As a consequence, if the leaf Λ is simply connected, we conclude that $\Lambda \subset S$, and accordingly

$$\{p \in M : \ \Lambda(p) \text{ is simply connected}\} \subset S,$$

thence, it follows that if all the leaves are simply connect, $S = M$.

So, in order to formalize our argument motivated by our example, it sufices to prove the following proposition.

Proposition 8.4.1. *Any polarized section must vanish outside the Bohr-Sommerfeld set.*

Proof. Suppose that s is polarized section s such that for a given $p \in M$ we have that $s(p) \neq 0$. Let Λ be a leaf through p. Then $s|_\Lambda \in \Gamma(\Lambda, L|_\Lambda)$ is the horizontal section with respect to ∇_Λ. In particular, consider the parallel transport of $L_p \to L_q$ of ∇_Λ. Taking $p = q$ and considering γ a loop on p, the parallel transport of $s(\gamma(0)) = s(p)$ is determined by $s(\gamma(1)) = s(p)$, and so $Q(\gamma) = 1$. Therefore G_Λ is trivial, hence, $p \in S$. ∎

So we now have all the ingredients to answer our question

Theorem 8.4.1 ([11])**.** *Given a complex reducible polarization, there is a global non-zero polarized section only if the Bohr-Sommerfeld set has non-empty interior. In particular, if all leaves are simply connected.*

In general, this problem can be circumvented by considering distributional sections of L.

Chapter 9

Half-form quantization

The main goal for this chapter is to introduce, in a elementary way, a common technique in quantization, called the half-form quantization or the half-form correction. This technique is related to a method used in physics for the quantization of the harmonic oscillator. We will also introduce the concept of pairing maps, used to compare different quantizations.

9.1 Half-form quantization

As we have seen, in the case for the real polarizations, the prequantum Hilbert space may be zero. Indeed this follows from the fact that the polarized sections may have infinity norm. In order to work around this, we present now the so called “half form correction”. The ideia is rather simple: Consider the $M = T^*\mathbb{R} \cong \mathbb{R}^2$ with the vertical polarization. Then the polarized sections are the ones that do not depend on the momentum. As such it makes no sense to integrate over the “momentum” variables. Of course, in this case this can be done without introducing any new machinery. Unfortunately, this is not so for the general case.

Consider the leaf space induced by the polarization. We will now assume that the leaf space has a smooth manifold structure. Notice that it may be that the leaf space is not orientable. Even if it is, there is no canonical “volume measure” on it. We will assume for sake of simplicity that the leaf space is orientable. In the not orientable case, one has to introduce the notion of densites, that allow us to integrate on non orientable manifolds (see more about densities, for example, in [9] and about half-form quantization on the more general case on [11]). The ideia is the following: We will construct a new Hilbert space called the half-form hilbert space such that the elements are such that, pointwise, they are n-forms on the leaf space. We will follow the approach given in [12].

Let Ξ be the leaf space of P and let $\pi : M \to \Xi$ be quotient map, where we assume it is smooth submersion.

Definition 9.1.1. *The canonical bundle of P, written as $\mathcal{K}_p$ is the real line bundle whose sections are n forms such that:*

$$\iota_X \alpha = 0, \ \forall X \in P.$$

A section is said to be polarized if

$$\iota_X d\alpha = 0, \ \forall X \in P.$$

From this it follows that any n-form satisfying the above condition implies that $\alpha(X_1, ..., X_n) = 0$, whenever any of the $X_j \in P$. As such it follows that any given point p, we may look at α as an n-linear, alternating functional on the quotient of T_pM by the intersection of P_p with the real tangent space $P_p^{\mathbb{R}}$. Thus, at each point the space of possible values for α is one dimensional. On the other hand, if α is polarized, then by the exact same reasoning we see that it is equivalent to saying that $d\alpha = 0$.

Proposition 9.1.1. *Let α be a polarized section of $\mathcal{K}_P$. Then, there is a unique n-form $\tilde{\alpha}$ on Ξ such that :*

$$\alpha = \pi^* \tilde{\alpha}.$$

Conversely, if β is an n-form on Ξ, then $\alpha := \pi^* \beta$ is a polarized section of $\mathcal{K}_P$.

Proof. $\Longleftarrow$
Let β be an n-form on Ξ and define $\alpha := \pi^*\beta$. Then it follows that α is a section because P is in the kernel of $d\pi$. Moreover, because the differential commutes with the pullback it follows that $d\alpha = 0$. So it is indeed a polarized section of $\mathcal{K}_P$.

$\Longrightarrow$
By the Flow-box theorem, we know that locally the polarization is going to look like the vertical polarization in $\mathbb{R}^{2n}$. Let $U \times V$ be one of those neighborhoods. Therefore we only have to prove this in that case. Using the observation above, we see that if α is a section of $\mathcal{K}_P$, then locally it must be of the form

$$\alpha = f(x, y) dx_1 \wedge ... \wedge dx_n,$$

for some $f \in C^\infty(M)$. But α is polarized section, which implies that f cannot depend on the "momentum variable" y. Hence

$$\alpha = f(x) dx_1 \wedge ... \wedge dx_n.$$

Then we conclude that α determines a n-form $\hat{\alpha}$ on $U \times V$ using the pullback of the projection $U \times V \to U$ and thus, using now the quotient map, it follows the result. ∎

Proposition 9.1.2. *Let α be a section of $\mathcal{K}_P$. Let X be a vector field that preserves P. Then $\mathcal{L}_X\alpha$ is also a section of $\mathcal{K}_P$. Moreover, if α is polarized, then so is $\mathcal{L}_X\alpha$.*

Proof. The first part follows from the following observation:
Let $X_1, ..., X_n$ vector fields on M such that $X_1 \in P$. Then, by using the formula in proposition 12.32(d) in [9] we have:

$$\mathcal{L}_X\alpha(X_1, ..., X_n) = X\alpha(X_1, ..., X_n) - \sum_{j=2}^{n} \alpha(X1, ..., X_{j-1}, [X, X_1], X_{j+1}, ..., X_n) -$$
$$- \alpha([X, X_1], X_2, ..., X_n).$$

Thus it follows that $\forall X_1 \in P$, $\iota_X\mathcal{L}\alpha = 0$. The second observation follows trivially form Cartan magic formula. ∎

Proposition 9.1.3. *Let X be a vector field that preseves P. Then there is a unique vector filed on Ξ such that for all $p \in P$:*

$$d\pi_p(X) = Y.$$

If $\alpha = \pi^*\beta$ is a polarized section, then

$$\mathcal{L}_X(\pi^*\beta) = \pi^*(\mathcal{L}_Y(\beta)).$$

It follows that

$$\mathcal{L}_X(\pi^*\beta) = (\textit{div}_\beta Y \circ \pi)\pi^*\beta.$$

Proof. See proposition 23.39 in [12]. ∎

Intuitively, what this result is saying is that when we identify the polarized sections of the canonical bundle with the n-forms on the leaf space, the operator $\mathcal{L}_X$ corresponds to the lie derivative on the leaf space in the direction of Y.

Henceforth we assume that Ξ is orintable.

Definition 9.1.2. *choose a nowhere vanishing oriented n-form β so that $\alpha := \pi^*\beta$ is a nowhere vanishing section of $\mathcal{K}_P$. Then we say that a section of $\mathcal{K}_P$ is **non-negative** if at each point it is a non-negative multiple of α.*

By the fact that Ξ is orientable, tell us that α is globally non-vanishing section and therefore $\mathcal{K}_P$ is trivial. This allow us to consider its square root δ_P, that is, a line bundle such that $\delta_P \otimes \delta_P \cong \mathcal{K}_P$. For instances we may take δ_P to be the trivial bundle. We assume that the above isomorphism was choosen to be such that for any section s of δ_P we have that $s \otimes s$ is non-negative.

Let α be a section of $\mathcal{K}_P$ and X a vector field in P. Then we may define the following n-form:

$$\nabla_X \alpha = \iota_X d\alpha.$$

By Cartan's magic formula we see that for section of $\mathcal{K}_P$ that $\nabla_X \alpha = \mathcal{L}_X \alpha$. Notice that as $X \in P$ implies it preserves P, we conclude by proposition 9.1.2 that $\nabla_X \alpha$ is a section $\mathcal{K}_P$. Notice also that this operator satisfies all the properties of a connection, except it is only defined along the directions of P, as reader should have guessed by the use of the suggestive notation. We call this operator the natural partial connection on $\mathcal{K}_P$.

All of the above construction was done for real vector bundles. We can extend this construction to complex bundles. As such we, Let $\delta_P^{\mathbb{C}}$ be the complex square root of $\mathcal{K}_P^{\mathbb{C}}$. Then we consider the line bundle $L \otimes \delta_P^{\mathbb{C}}$. If $s \in L \otimes \delta_P^{\mathbb{C}}$, then we may decompose as $s = \mu \otimes \nu$, where μ is nonvanishing section of L and ν a section of $\delta_P^{\mathbb{C}}$.

We may then consider the induced connection on $L \otimes \delta_P^{\mathbb{C}}$ given by:

$$\nabla_X s = (\nabla_X \mu) \otimes \nu + \mu \otimes \nabla_X \nu.$$

Given two sections s_1 and s_2 of $L \otimes \delta_P^{\mathbb{C}}$, we may combine them using the hermitian product of L, in the following way:

$$(s_1, s_2) := \langle \mu_1, \mu_2 \rangle \overline{\nu}_1 \otimes \nu_2.$$

This yields a section of $\mathcal{K}_P^{\mathbb{C}}$. Consider now the following inner-product on the space of polarized sections of $L \otimes \delta_P^{\mathbb{C}}$:

$$\langle s_1, s_2 \rangle := \int_\Xi \widetilde{(s_1, s_2)},$$

where $\widetilde{(s_1, s_2)}$ is the form obtained from the proposition 9.1.1 .

Definition 9.1.3. *The* ***half-form Hilbert space*** *is the completion of the space of smooth polarized sections of $L \otimes \delta_P^{\mathbb{C}}$ whose normed induced by the above inner product is finite.*

Now we have to adjust our definition of quantum operator. In particular, we are going to use the prequantum operator for the L (denoted now by q_{pre} as to not rise extra confusion). Now all we need is to define how it should act on the $\delta_P^{\mathbb{C}}$.

Definition 9.1.4. *Let f be a smooth function such that X_f preserves P. Then, we define the quantum operator $q(f)$ as follows:*

$$q(f)s := (q_{pre}(f)\mu) \otimes \nu + \mu \otimes \mathcal{L}_{X_f} \nu.$$

We now need to see that it satisfies the properties that we have set. In particular, the only one we have to check is the one about the lie and poisson brackects.

Proposition 9.1.4. *Let f and g such that both X_f and X_g preserve P. Then:*

$$i[q(f), q(g)] = q(\{f, g\}).$$

Proof. It suffices to prove the result locally. As such let ν_0 be a local nonvanishing section of $\delta_P^{\mathbb{C}}$ such that any other section s of $L \otimes \delta_P^{\mathbb{C}}$ may be decomposed as $s = \mu \otimes \nu_0$. If X is any vector field preserving P, then there is a function $\gamma(X)$ such that $\mathcal{L}_X(\nu_0) = \gamma(X)\nu_0$ and as such we have

$$q(f)(\mu \otimes \nu_0) = [q_{pre}(f) + \gamma(X_f)]\mu \otimes \nu_0.$$

As such, form a very simple computation we can see that the result follows if we can justify that

$$X_f(\gamma(X_g)) - X_g(\gamma(X_f)) = -\gamma(X_{\{f,g\}}).$$

In turn this follows from:

$$\begin{aligned}
\mathcal{L}_{[X_f,X_g]}\nu_0 &= \gamma([X_f, X_g])\nu_0 \\
[\mathcal{L}_{X_f}(\nu_0), \mathcal{L}_{X_g}(\nu_0)] &= -\gamma(X_{\{f,g\}})\nu_0 \\
\mathcal{L}_{X_f}(\mathcal{L}_{X_g}(\nu_0)) - \mathcal{L}_{X_g}(\mathcal{L}_{X_f}(\nu_0)) &= -\gamma(X_{\{f,g\}})\nu_0 \\
(X_f(\gamma(X_g)) + X_g(\gamma(X_f)))\nu_0 &= -\gamma(X_{\{f,g\}})\nu_0.
\end{aligned}$$

∎

Chapter 10

Quantization of toric manifolds

In this chapter, we will first give a very short and elementary introduction to toric manifolds. These geometric objects have been around for 50 years and are native to algebraic geometry. They are of special interest for symplectic geometers, as they provide a rich class of objects with large symmetries and are completely integrable hamiltonian spaces. Afterwards, we will see explicitly how quantization may be achieved in this objects.

10.1 Toric manifolds

In this section we will introduce some elementary facts about toric manifolds. Our main goal is to arrive at a theorem due to Miguel Abreu.

Definition 10.1.1. *A **Toric manifold** is a compact connected $2n$-symplectic manifold (M, ω) equipped with an effective hamiltonian action of a n-dimensional torus $\mathbb{T}^n$ and with a choice of corresponding moment map.*

The 2-sphere is a toric manifold, where here our torus is simply S^1 and the action is given by rotations around the z-axis. Then, the moment map is simply going to be the height function, and its image is the interval $[-1, 1]$.
The complex projective spaces are also toric manifolds. For instance, $\mathbb{CP}^2$ equipped with the Fubini-Study form, the action of $\mathbb{T}^2$ on $\mathbb{CP}^2$ is given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0; z_1; z_2] = [z_0; e^{i\theta_1} z_1, e^{i\theta_2} z_2]$$

and the corresponding moment map is

$$\mu([z_0; z_1; z_2]) = \frac{-1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\right)$$

It is easy to see that the its image is a triangle with vertices at the $(0,0), (\frac{-1}{2}, 0), (0, \frac{-1}{2})$.

With this two examples in mind, one might wonder if it is a coincidence that the images of this two manifolds under their moment map are polytopes. The answer is no, it has to be! This was first proved (for the case that the action was not effective) by Atiyah (see [4] theorem 27.1). In fact, when we imposed that the action is effective, then we get an even stronger theorem.

Definition 10.1.2. *A **Delzant polytope** P is a polytope in $\mathbb{R}^n$ satisfying:*

- ***simplicity**, that is, at each vertex there are exactly n edges meeting there.*
- ***rationality**, that is, at the vertex p, the edges meeting there are of the form $p + tu_i$, where $u_i \in \mathbb{Z}^n$.*
- ***smoothness**, that is, at each vertex, the u_i can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$*

It is quite easy to see that the polytopes of the above examples are indeed Delzant. As such one might think that for toric manifolds, the moment polytope are Delzant polytopes. This is indeed the case. But wait, there is more to it. The conserve is also true! That is, there is a one to one correspondence between delzant polytopes and toric monifolds.

Theorem 10.1.1 (Delzant)**.** *Toric manifolds are classified by Delzant polytopes. More specifically, there is a bijective correspondence between these two sets is given by the moment map.*

$$\{\textit{toric manifolds}\} \xrightarrow{"1-to-1"} \{\textit{Delzant polytope}\}$$
$$(M, \omega, \mathbb{T}^n, \mu) \mapsto P$$

Proof. See section 28.3 and onwards in [4] for the proof. ∎

One important thing to keep in mind is that there is a generic way to obtain the manifold using the Delzant polytope. One simply has to consider $\triangle \times \mathbb{T}^n$ and then collapse the tori along the boundary in an appropriate way. For more information regarding this, please see [4].

A lattice vector $v \in \mathbb{Z}^n$ is said to be **primitive** if it cannot be written as ku, where, $|k| > 1,\ k \in \mathbb{Z}$ and $u \in \mathbb{Z}^n$.

Take now v_i to be the primitives of the d-faces of the P. Then, we may describe P as the set of:

$$P = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \leq \lambda_i,\ i \in \{1, ..., d\}\},\ \text{for some } \lambda_i \in \mathbb{R}.$$

Definition 10.1.3. *A toric kahler manifold is a toric manifold that is also kahler and such that the effective hamiltonian action is also holomorphic.*

10.2 Symplectic potentials

Let P° be the interior of P. Then we have that $\check{X}_P := \mu^{-1}(P^\circ)$, known as the **open orbit**, is an open dense of M consisting of the points in which the action is free. It is then known that

$$\check{X}_P \cong \mathbb{C}^n/2\pi i\mathbb{Z} = \mathbb{R}^n \times i\mathbb{T}^n \cong (\mathbb{C}^*)^n.$$

As such, in the **"complex" coordinates** $z = u + iv$ the $\mathbb{T}^n$ action is given by

$$\theta \cdot (u + iv) = u + i(v + \theta),$$

and the complex structure which is multiplication by i is then given by

$$\begin{bmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix}$$

Now, in these coordinates we see that because ω must be invariant by the action of torus, then the Kähler potential must only depend of the u coordinate. Let $f \in C^\infty(\check{X}_P)$ be that potential. Thus the matrix that represents the symplectic form is given by

$$\begin{bmatrix} 0 & F \\ -F & 0 \end{bmatrix}$$

Where F is the hessian of f in u coordinates. Moreover, a simple computation shows that the compatible riemannian metric must be of the form:

$$\begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}$$

For this to be a metric, then we see that f must be strictly convex. This was from a point of view of complex coordinates. However, our manifold is also symplectic. As such we may consider coordinates given by the relation

$$M^\circ \cong P^\circ \times \mathbb{T}^n,$$

called the **"symplectic" coordinates** or the **angle coordinates** (x, y), as with this coordinates the matrix associated to the symplectic form becames the usual one.

$$\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}$$

In this coordinates the action of the torus is given by

$$\theta \cdot (x, y) = (x, y + \theta)$$

As it turns out, the complex structure is then given by the hessian of a potential $g \in C^\infty(P^\circ)$ (denoted by G), and the associated matrix is

$$J = \begin{bmatrix} 0 & -G^{-1} \\ G & 0 \end{bmatrix} \tag{10.1}$$

As a consequence of this, is that the riemannian metric compatible is given by

$$\begin{bmatrix} G & 0 \\ 0 & G^{-1} \end{bmatrix}$$

In order to change between these two coordinates, one simply considers the Legendre transform, as follows

$$x = \frac{\partial f}{\partial u} \text{ and } y = v.$$

Consider now

$$\ell_r(x) = \langle x, v_r \rangle - \lambda_r,$$

where v_r are taken to be inward pointing. Then it is easy to see that $x \in P^\circ$ iff $\ell_r(x) > 0$ for all r. Then we may consider the following smooth function $g_P : P^\circ \to \mathbb{R}$

$$g_P(x) = \sum_{r=1}^{d} \frac{1}{2} \ell_r(x) \log(\ell_r(x)) \tag{10.2}$$

Theorem 10.2.1 (Guillemin)**.** *The "canonical" compatible toric complex structure J_P on (M_P, ω_P) is given in the (x, y) symplectic coordinates of $\check{X}_P \cong P^\circ \times \mathbb{T}^n$ by*

$$J_p = \begin{bmatrix} 0 & -G_P^{-1} \\ G_P & 0 \end{bmatrix},$$

with $G_p = \text{Hess}_x(g_p)$.

Proof. See the original paper [13]. ■

Theorem 10.2.2 (Abreu)**.** *Let (M_P, ω_P, τ_P) be the toric symplectic manifold associated to a Delzant polytope $P \subset \mathbb{R}^n$, and J any compatible toric complex structure. Then J is determined, using 10.1* *by a "potential" $g \in C^\infty(P^\circ)$ of the form*

$$g = g_P + h,$$

where g_P is given by 10.2, h is smooth on the whole P, and the matrix $G =$ Hess$_x(g)$ is positive definite on P° and has determinant of the form

$$\det(G) = \left[\delta(x) \prod_{r=1}^{d} \ell_r(x)\right]^{-1},$$

with δ being a smooth and strictly positive function on the whole P.
Conversely, any such g determines a compatible toric complex structure J on (M_P, ω_P), which in the (x, y) symplectic coordinates of $\check{X}_P^\circ \cong P^\circ \times \mathbb{T}^n$ has the form 10.1.

Proof. The proof can be found in [14]. ■

This symplectic potential allow us to define a diffeomorphism between P° and $\mathbb{R}^n$ in the following way: for each $x \in P^\circ$ associate to $y := \frac{\partial g}{\partial x} \in \mathbb{R}^n$. This, in turn, allow us to define a $\mathbb{T}^n$ equivariant biholomorphism between $P^\circ \times \mathbb{T}^n$ and $(\mathbb{C}^*)^n$, defined by assigning $(x, \theta) \in P^\circ \times \mathbb{R}^n$ to a w given by:

$$w := (e^{y_1 + i\theta_1}, ..., e^{y_n + i\theta_n}).$$

The w are therefore a coordinate system for M°. The inverse of this transformation is then given by

$$x := \frac{\partial h}{\partial y},\ h(y) = x(y) \cdot y - g(x(y)).$$

In a analogous way, we can define coordinates around the vertices of the polytope is the following way: We first notice that a given vertex v is completely defined as the intersection of n faces. We may assume that it is the first n faces (if not, we may need to do some reordering) $l_1(v) = ... = l_n(v) = 0$. We may therefore define a n by n matrix A_v with integer coefficients such that:.

$$(A_v)_{ij} = v_i^j.$$

We further define the domain of the chart to be

$$U_v = \mu^{-1}\left(\{v\} \cup \bigcup_{\text{faces adjacent to } v} F\right)$$

By letting $\lambda_v = (\lambda_1, ..., \lambda_n)$ we set

$$x_v = A_v x - \lambda_v, \qquad \theta_v = (A_v^{-1})^t \theta.$$

We may then obtain $w_v = e^{y_v + i\theta_v}$, where $y_v = \frac{\partial g}{\partial x_v}$. In the coordinates, we have that $\omega = \sum dx_v \wedge d\theta_v$. These ω's are going to permit us to describe the holomorphic sections of L explicitly, as we will see latter.

10.3 Divisors and Fans

Following [15], in broad terms, a **Divisor** $D \subset M$ in a complex manifold is a finite linear combination of irreducible complex hypersurfaces. So in particular a divisor D is given by

$$D = \sum_j m_j V_j, \quad m_j \in \mathbb{Z}, \text{ for hypersurfaces } V_j.$$

A divisor is said to be **effective** if $m_j \geq 0,\ \forall j$. By **hypersurfaces**, we mean that there is an open cover U_α of M and non-constant holomorphic functions $f_\alpha : U_\alpha \to \mathbb{C}$ such that $f_\alpha^{-1}(0) = V \cap U_\alpha$. By an **irreducible hypersurface** we mean that we cannot write it as the union of two non-empty hypersurfaces.

Consider now $\pi : L \to M$ a holomorphic line bundle. We say that s is a **meromorphic section** if in a local holomorphic trivializations it is a meromorphic function. Thus, the **order of vanishing of** s **at** x, where $s(x) = 0$, is the lowest natural number m such that locally s has a non-zero partial derivative of order m at x. The **zero divisor** of s is then simply the linear combination of irreducible components of $s^{-1}(0)$:

$$Z(s) := \sum m_j V_j, \quad m_j \in \mathbb{N},$$

where m_j are the order of vanishing of s along V_j. Similarly, we say that s has **a pole of order** m **at** x if $1/s$ has a zero of order m at x. Moreover, the polar divisor of s is the linear combination of the irreducible components of $s^{-1}(\infty)$:

$$P(s) = \sum n_j U_j, \quad n_j \in \mathbb{N},$$

where n_j are the order of the pole at U_j. Finally, the **divisor of** s is simply:

$$div(s) := Z(s) - P(s).$$

Some author denote $Z(s)$ by $s^{-1}(0)$ and $P(s) = s^{-1}(\infty)$.

Let now V by an irreducible hypersurface of M. Then, on the intersection of two elements of the open cover, U_α and U_β, we have the corresponding holomorphic functions that define locally V, $f_i : U_i \to \mathbb{C}$ such that $f_i^{-1}(0) = V \cap U_i$, $i = \alpha, \beta$. Moreover, we also have the transition functions ϕ_{ij} and $\phi_{ij} \neq 0$. Then, on U_{ij} we have that $f_i = \phi_{ij} f_j$, and so the zero sets coincide, and therefore divisors are globally defined. Moreover, because the zeroes coincide, we can define the nonvanishing holomorphic function $f_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}$ as $f_{\alpha\beta} = f_\alpha / f_\beta$. Moreover, $f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha} = 1$. Thus, we may use these function as transition functions for some **holomorphic line bundle denoted by L_S.**

Using the same reasoning as above, we may define the **line bundle L_D of a divisor** $D = \sum_{j=1}^k m_j V_j$ as

$$L_D := \bigotimes_{j=1}^k L_{V_j}^{m_j}.$$

If s is a meromorphic section of a line bundle L, then $L_{div(s)} \cong L$ and $D \cong \text{div}(s)$, where $\cong$ means linear equivalence, that is, if there exist a meromorphic function ψ such that $D - div(s) = div(\psi)$.

Toric manifolds are native to algebraic geometry. There they are given by an object called fan. These fans give us a tool to compute the divisors.

Definition 10.3.1. *A **convex polyhedral cone** in $\mathbb{R}^n$ is a set of the form*

$$C = \left\{ \sum_{i=1}^k a_i v_i \in \mathbb{R}^n \quad ; \quad a_i \geq 0 \right\},$$

where v_i are vectors called the generators of C.
The dual of a cone is given by

$$C^\star := \{ f \in (\mathbb{R}^n)^\star \, ; \, f(x) \geq 0 \, \forall x \in C \}.$$

A cone is **rational** if the set of generators are in $\mathbb{Z}^n$ and it is said to be **smooth** if the set of generators form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Farkas' theorem states that the dual of a rational cone is a rational cone. A **supporting hyperplane** for a cone C is a hyperplane of the form

$$H_f := \{ x \in \mathbb{R}^n \, ; \, f(x) = 0 \},$$

where $f \in C^\star \backslash \{0\}$. A **face** of a cone is either itself (non proper face) or the intersection of C with any supporting hyperplanes (proper face).

Definition 10.3.2 ([16])**.** *A **fan** $\triangle$ is a nonempty finite collection of strongly convex rational cones such that every face of every cone belongs in the fan; and the intersection of any two cones is a face of both them.*

Using this, we may define toric varieties through the usage of the spectrum of a ring. We will not provide any more details for this here. However, more information can be found in [17] and [16].

We will now see how to obtain a fan from a Polytope P. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear function. Then we denote by $\text{supp}_P f$ the collection of points in the polytope where f achieves its minimum. This set is known as the supporting face of f in P.

Definition 10.3.3. *Let F be a face of a polytope P. The cone associated to F is the closure of the subset of $(\mathbb{R}^n)^\star$ consisting of all linear functions $f : \mathbb{R}^n \to \mathbb{R}$ such that supp$_P f = F$. this subset is generally denoted by $C_{F,P}$. Then, the* ***fan of the polytope*** *P, denoted by $\triangle_P$, is simply the collection of cones $C_{F,P}$ for all faces $F of P$.*

Suppose now that the origin in the interior of the polytope, Then $\triangle_P$ coincides with the fan spanned by the faces of the dual polytope:

$$P^\star := \{f \in (\mathbb{R}^n)^\star \,;\, f(v) \geq -1, \, \forall v \in P\}.$$

What this means is that the rays from the origin through the proper faces of $P^\star$ and the origin can be used to form the cones. To better understand how to obtain the fan from a polytope, we present the following example.

Example 10.3.1. *Consider a triangle, as our polytope. Then the associated fan can easily be seen to be given by the collection of the following cones:*

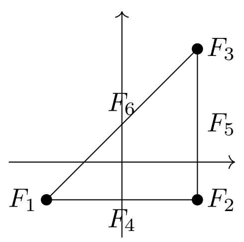

Figure 10.1: Polytope.

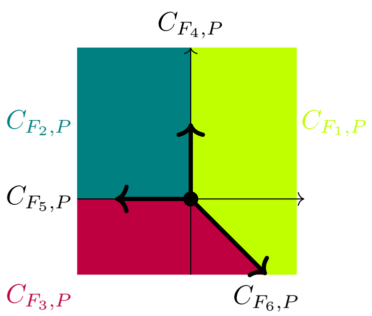

Figure 10.2: The associated cones.

Notice that the whole polytope corresponds to the zero dimension cone, the origin.

In the toric case, these cones play an important role. Each of these cones corresponds to orbits of the torus action on the variety. Moreover, each of these cones are then associated to a unique irreducible invariant divisor (called the irreducible torus-invariant divisors). Let $\triangle^1$ denoted the set of 1-cones. As it turns out, there is a one-to-one correspondence between irreducible torus-invariant divisors and the elements of $\triangle^1$. Following the example above, we see that elements of $\triangle^1$ are then the primitive integral vectors ν_j which are normal to the j-face the Polytope. So it then follows that the irreducible divisors are:

$$D_j = \mu^{-1} \left(\{x \in P \,;\, l_j(x) := \langle \nu_j, x \rangle + \lambda_j = 0\}\right).$$

So given a divisor $D^L = \sum_{j=1}^r \lambda_j^L D_j$, $\lambda_j^L \in \mathbb{Z}$, we define the Line bundle $L = \mathcal{O}(D^L)$. Let σ_{D^L}

be the unique up to a constant meromorphic section of L, with corresponding divisor D^L. Then, following Proposition 4.1.2 in [17], we obtain that for any meromorphic function $w^m,\ m \in \mathbb{Z}^n$ on the open orbit, its divisor is given by

$$\text{div}(w^m) = \sum_{j=1}^{r} \langle \nu_j, m \rangle D_j,$$

and so we have that the space of holomorphic sections is

$$H^0(M, L) = \text{span}_{\mathbb{C}}\{w^m \sigma_{D^L}\ ;\ m \in \mathbb{Z}^n,\ \text{div}(w_0^m \sigma_{D^L}) = \langle \nu_j, m \rangle + \lambda_i^L \geq 0\}.$$

Now, looking at this, we see there is a bijection between the basis of $H^0(M, L)$ and the integral points of the Delzant polytope with integral vertices.

10.4 Complex line bundle, holomorphirc polarization and the Hilbert space for Kähler toric manifolds

It is not hard to see that given a complex line bundle L, there is a canonical isomorphism given by $(|\cdot|, arg)$ such that $L \cong |L| \otimes L^{U(1)}$. This isomorphism, will induce a split on the connection. It is straightforward to see that the connections form are, respectfully, $\alpha^{|L|} = \text{Re}\alpha$ and $\alpha^{L^{U(1)}} = i\text{Im}\alpha$. Using this ideia and the coordinates we have defined above, we may define a Hermitian structure on L, by setting $||1_0|| = e^{-h(x)}$ and $||1_v|| = e^{-h_v(x)}$, where $h_m(x) = (x - m)^t \frac{\partial g}{\partial x} - g(x)$. So one may define a system of normalized sections as follows:

$$1_0^{U(1)} = \frac{1_0}{||1_0||}, \qquad 1_v^{U(1)} = \frac{1_v}{||1_v||},\ v \text{ vertex}$$

Moreover, this allow us to define a connection with curvature $-i\omega$

$$\nabla 1_0^{U(1)} = -ixd\theta 1_0^{U(1)}, \qquad \nabla 1_v^{U(1)} = -ix_v d\theta_v 1_v^{U(1)},\ v \text{ vertex} \tag{10.3}$$

Using the Liouville measure, we are able to consider the injection of smooth in distributional sections:

$$\iota : C^{\infty}(L_\omega|_U) \to C^{-\infty}(L_\omega|_U) := (C_c^{\infty}(L_\omega^{-1}|_U))^*$$
$$s \mapsto \iota s(\phi) = \int_U s\phi \frac{\omega^n}{n!},$$

where U is any open subset of $\check{X}_P$ and L_ω is the prequantum line bundle on P°. Consider now the following family of symplectic potentials:

$$g_s = g_P + s\psi,$$

where ψ is a smooth uniformly convex on P and consider the associated holomorphic polarization

$$\mathcal{P}_s = \text{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial w_s^i},\ i = 1, ..., n \right\}. \tag{10.4}$$

We then define the limit Polarization

$$\mathcal{P}_\infty := \lim_{s\to\infty} \mathcal{P}_s. \tag{10.5}$$

Consider also the real polarization

$$\mathcal{P}_{\mathbb{R}} = \text{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial \theta^i},\ i = 1, ..., n \right\}. \tag{10.6}$$

Proposition 10.4.1. *On $\check{X}_P$, $\mathcal{P}_\infty = \mathcal{P}_{\mathbb{R}}$.*

Proof. This follows from the fact that the hessian is positive definite and

$$\frac{\partial}{\partial y_s^i} = \sum_j (G_s^{-1})_{ij} \frac{\partial}{\partial x_j},$$

and so because $(G_s^{-1})_{ij} \to 0$ as $s \to \infty$, we get that :

$$\text{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial w_s^i} \right\} = \text{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial y_s^i} - i\frac{\partial}{\partial \theta^i} \right\} \xrightarrow{s\to\infty} \text{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \theta^i} \right\} = \mathcal{P}_{\mathbb{R}}$$

∎

So this result says that the holomorphic polarization at infinity collapses to a real polarization. This is an interesting way to see real polarization, due to the fact that this allow us to study them through complex polarization, as the former are easier to work with.

Now over the boundary of P, it follows that w_s^i can only be zero as y_v^i goes to $-\infty$, which may only happen at the boundary. Therefore, if over a face we have that $w_s^i \neq 0$, then it follows that $\frac{\partial}{\partial w_s^i} \to \frac{\partial}{\partial \theta_s^i}$, and then, we arrive at the following result

Proposition 10.4.2. *On X_P, we have that:*

$$\mathcal{P}_\infty = \mathcal{P}_{\mathbb{R}} \oplus \textit{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial w^i};\ w^i = 0 \right\}.$$

Proof. See [1] theorem 3.4. ∎

From which follows that

Theorem 10.4.1.

$$C^\infty\left(\lim_{s\to\infty}\mathcal{P}_s\right) = C^\infty(\mathcal{P}_\mathbb{R}).$$

Proof. See [1] theorem 1.2. ■

This result tells us that the considered family of Kähler polarizations converges to the real polarization. Now it is a consequence that the norm of a polarized holomorphic section σ^m is given by $e^{-h_m\circ\mu}$. As such, in order to study the norm for a given $x \in P^\circ$, we only have to look at the function

$$f_m(x) = (m-x)\frac{\partial\psi}{\partial x} - \psi(x).$$

As it turns out, for ψ strictly convex, this function has a global minimum at m, which yields the following result

Proposition 10.4.3.

$$\frac{e^{-sf_m(x)}}{||e^{-sf_m}||_1} \xrightarrow{s\to\infty} \delta(x-m),$$

in the sense of distributions.

Proof. See [1] lemma 3.7. ■

Consider now $W \subset \check{X}_P$ to a open set that is invariant by the action of $\mathbb{T}^n$. Then, following [1] we define

$$\delta^m(\tau) = \int_{\mu_P^{-1}(m)} e^{i\ell(m)\theta}\tau = \hat{\tau}(x=m, \theta=-m),\ \forall\tau \in C_c^\infty(L_\omega^{-1}|_U),$$

where $\hat{\tau}$ represents the Fourier transform of τ. Moreover, the holomorphic sections are given as follows:

$$\sigma_s^m := e^{-h_s(x)}w_s^m\mathbb{1}.$$

This proposition then let us prove the following theorem, which describes what happens to the sections of the line bundle.

Theorem 10.4.2. *For $n \in P\cap\mathbb{Z}$, consider the family of L^1-normalized J_s-holomorphic sections*

$$\mathbb{R}^+ \ni s \mapsto \xi_s^m := \frac{\sigma_s^n}{||\sigma_s^n||_1} \in C^\infty(L_\omega) \xrightarrow{\iota} (C_c^\infty(L_\omega^{-1}|_U))^*.$$

Then, as $s\to\infty$, $\iota(\xi_s^n)$ converges to δ^m in $(C_c^\infty(L_\omega^{-1}|_U))^*$.

Proof. See [1] Theorem 1.3. ■

This result does not use the half form-correction. In order to implement it, we will consider the split $\mathcal{K}_P \cong |\mathcal{K}_P| \oplus \mathcal{K}_P^{U(1)}$. In particular, notice that $|\mathcal{K}_P|$ is always trivial, and therefore admits a square root, denoted by $|\mathcal{K}_P|^{\frac{1}{2}}$. For instances, if we consider $\mathcal{P}_{\mathbb{R}}$, then the fibers are generated by $dX = dx_1 \wedge ... \wedge dx_n$, and as such we may define $|dX|$ as:

$$|dX| : \mathcal{X}(M^\circ)^n \to C^0(M^\circ), \qquad (X_1, ..., X_n) \mapsto |dX(X_1, ..., X_n)|$$

This gives a better picture of what the sections of $|\mathcal{K}_P|$ are. As a consequence, we may define $\sqrt{|dX|}(X_1, ..., X_n) = |dX(X_1, ..., X_n)|^{\frac{1}{2}}$. We also define $dZ_s := dz_s^1 \wedge ... \wedge dz_s^n$, as the generators of the fibers defined by the g_s. A global trivializing section of $|\mathcal{P}_s|^{\frac{1}{2}}$ is then $\frac{\sqrt{|dZ_s|}}{||dZ_s||^{\frac{1}{2}}}$. Therefore, we obtain the following half-form Hilbert space for each s

$$\mathcal{H}_s := \left\{ \sigma \otimes \frac{\sqrt{|dZ_s|}}{||dZ_s||^{\frac{1}{2}}};\ \sigma \text{ is a polarized section of } L \right\}.$$

The following propositions shows us how this objects behave.

Proposition 10.4.4.

$$\frac{\sqrt{|dZ_s|}}{||dZ_s||^{\frac{1}{2}}} \xrightarrow{s\to\infty} \sqrt{|dX|}.$$

Proof. See lemma 4.14 in [2] ∎

Proposition 10.4.5.

$$\frac{\sigma_s^m}{||\sigma_s^m||_2}||dZ_s||^{\frac{1}{2}} \xrightarrow{s\to\infty} 2^{\frac{n}{2}}\pi^{\frac{n}{4}}\delta^m.$$

Proof. See theorem 4.13 in [2] ∎

Proposition 10.4.6. *For large values of s we have that:*

$$||\sigma_s^m||_2 \sim \pi^{\frac{n}{4}} e^{g_s(m)}.$$

Proof. See lemma 4.12 in [2] ∎

Taking $\tilde{\sigma}_s^m = \sigma_s^m \otimes \frac{\sqrt{|dZ_s|}}{||dZ_s||^{\frac{1}{2}}}$ we obtain:

Proposition 10.4.7.

$$\frac{\tilde{\sigma}_s^m}{||\sigma_s^m||_2} \xrightarrow{s\to\infty} 2^{\frac{n}{2}}\pi^{\frac{n}{4}}\delta^m \otimes \sqrt{|dX|}.$$

Proof. See theorem 4.15 in [2] ∎

Has it was shown in [2], we have that

$$\mathcal{H}_{\mathbb{R}} = \left\{ \sigma \otimes \frac{\sqrt{|dZ_s|}}{||dZ_s||^{\frac{1}{2}}};\ \sigma \in \bigcap_{i=1}^{n} \text{Ker}\nabla_{\frac{\partial}{\partial\theta_i}} \right\},$$

From which it follows that $\mathcal{H}_{\mathbb{R}}$ has a basis given by $\{\delta^m \otimes \sqrt{|dX|}\}$. Thus we see that the above result is also valid for the half form space.

Example 10.4.1. *Take the sphere S^2. Then its moment polytope is the interval $P = [-\frac{1}{2}, N + \frac{1}{2}]$, for N natural. Consider also the inequalities:*

$$\ell_1(x) = x + \frac{1}{2} > 0 \qquad \ell_2(x) = N + \frac{1}{2} - x > 0.$$

We consider now the following strictly convex function $\psi = \frac{x^2}{2}$ on the moment polytope. Therefore the symplectic potential is given by:

$$g_s = \frac{1}{2}\Big(\ell_1(x)\log(\ell_1(x)) + \ell_2(x)\log(\ell_2(x))\Big) + \frac{sx^2}{2}.$$

The associated coordinates are:

$$y_s = \log\left(\sqrt{\frac{x+\frac{1}{2}}{N+\frac{1}{2}-x}}\right) - sx, \qquad w_s = \sqrt{\frac{x+\frac{1}{2}}{N+\frac{1}{2}-x}}e^{sx+i\theta}.$$

Let now $h_m(x)$ be:

$$h_m(x) = (x-m)y_s - g_s(x).$$

For s considerably large, it follows that $h_m(x) \sim \frac{s(x-m)^2}{2} - \frac{sm^2}{2}$. Let the Hessian of g_s be H. Then a basis for the space of holomorphic sections is given by:

$$\sigma_s^m = e^{-h_m(x)+im\theta}\,\text{det}H^{\frac{1}{4}}1_0^{U(1)},$$

and for large s we get

$$\sigma_s^m \sim s^{\frac{1}{4}}e^{\frac{sm^2}{2}}e^{\frac{s(x-m)^2}{2}}1_0^{U(1)}.$$

It also follows that $||\sigma_s^m||_2 \sim e^{\frac{sm^2}{2}}\pi^{\frac{1}{4}}$. Looking now at the z_s coordinates, we see that for large values of s, $z_s \sim \log w_s$. Therefore:

$$dz_s \sim sdx + id\theta \text{ and } ||dz_s|| = \text{det}H^{\frac{1}{4}} \sim s^{\frac{1}{4}}.$$

This allows us to explicity see that

$$\frac{\sigma_s^m}{||\sigma_s^m||_2}||dZ_s||^{\frac{1}{2}} \sim \frac{s^{\frac{1}{2}}}{\pi^{\frac{1}{4}}}e^{\frac{s(x-m)^2}{2}+im\theta}1_0^{U(1)}$$

which converges to $\delta(x-m)$. Therefore we have shown a particular case of proposition 10.4.5.

10.5 Relationship with complex time Hamiltonian flow

We may reformulate this using Hamiltonian complex time flow. We will be using the notation establish in chapter 5. This method is based on the work of Thiemann and known as the Thiemann complexifier method. For this, let ψ be the strongly convex function on P. Let $\mathcal{P}_g$ be Kähler polarization of (X_P, ω) given by

$$\mathcal{P}_g = \text{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \overline{z}_j}, j = 1, ..., n \right\},$$

where $z_j = \frac{\partial g}{\partial x_j} + i\theta_j$.

Proposition 10.5.1. *Let $s > 0$. Then:*

- *As distributions, $\mathcal{P}_s = e^{is\mathcal{L}_{X_\psi}} \mathcal{P}_g$.*
- *In the pointwise sense as a power series in s, $dZ_s = e^{is\mathcal{L}_{X_\psi}} dZ_0$.*

Proof. See [3] theorem 3.4. ∎

Consider now the **Kostant-Souriau prequantum operator** associated to a smooth function h is defined by $\hat{h} := i\nabla_{X_h} + h$. Therefore, we may consider the t-time flow of the lifted vector field using $e^{-it\hat{h}} : \Gamma(L) \to \Gamma(L)$. We would like to extend this to imaginary time.

Recall from 10.3 that we defined the connection on L to be

$$\nabla 1^{U(1)} = -ixd\theta 1^{U(1)}.$$

The prequantum operator associated to ψ is:

$$\hat{\psi} = iX_\psi - x \cdot \frac{\partial \psi}{\partial x} + \psi.$$

We now need to know how the operator will behave with half-forms. In other to do this in a consistant manner, we set

$$e^{is\mathcal{L}_\psi} \sqrt{dZ} = \sqrt{dZ_s}$$

Proposition 10.5.2 ([3]). *For any $s > 0$, the operator $e^{s\hat{\psi}} \otimes e^{is\mathcal{L}_\psi} : \mathbb{H}_{\mathcal{P}_0} \to \mathbb{H}_{\mathcal{P}_s}$ is an isomorphism and*

$$e^{s\hat{\psi}} \otimes e^{is\mathcal{L}_\psi} \sigma_0^m = \sigma_s^m,$$

For all $m \in P$ integral.

Proof. This result follows from using the above observations, in particular the ones about the basis of $\mathbb{H}_{\mathcal{P}_s}$ and how we defined the action for half-forms. As such we only need to make the following effortless calculation:

$$e^{s\hat{\psi}}\left(w^n e^{-h_0}\right) = e^{-s(x\cdot\frac{\partial\psi}{\partial x}-\psi)-h_0}e^{isX_\psi}(w_0^m) = e^{-h_s}w_s^m.$$

∎

As $\mathcal{P}_\mathbb{R}$ is preserved by the flow of ψ we may consider the natural quantization of ψ on $\mathbb{H}_{\mathcal{P}_s}$ given by the operator:

$$\hat{\psi}_\mathbb{R} : \mathbb{H}_{\mathcal{P}_\mathbb{R}} \to \mathbb{H}_{\mathcal{P}_\mathbb{R}}, \qquad \delta^m \otimes \sqrt{dX} \mapsto \psi(m)\delta^m \otimes \sqrt{dX},$$

which is well defined, as the support of $\delta^m \otimes \sqrt{dX}$ is $\mu^{-1}(m)$. This operator may be extended in the following way:

$$\hat{\psi}_\mathbb{R} : \mathbb{H}_{\mathcal{P}_g} \to \mathbb{H}_{\mathcal{P}_g}, \qquad \sigma^m \mapsto \psi(m)\sigma^m.$$

Define an operator $A^\psi_{g,s} : \mathbb{H}_{\mathcal{P}_g} \to \mathbb{H}_{\mathcal{P}_s}$:

$$A^\psi_{g,s} := \left(e^{s\hat{\psi}} \otimes e^{is\mathcal{L}_\psi}\right) \circ e^{-s\hat{\psi}_\mathbb{R}}$$

We may consider the operator $A^\psi_{g,\infty} : \mathcal{H}_{\mathcal{P}_g} \to \mathcal{H}_{\mathcal{P}_\mathbb{R}}$determined by:

$$A^\psi_{g,\infty}\left(\frac{\sigma_0^m}{||\sigma_0^m||_2}\right) := \frac{(2\pi)^{\frac{n}{2}}e^{g(m)}}{||\sigma_0^m||_2}\delta^m \otimes \sqrt{dX}$$

Theorem 10.5.1 ([3])**.**

$$\lim_{s\to\infty} A^\psi_{g,s} = A^\psi_{g,\infty}.$$

Proof. Notice that

$$\begin{aligned} e^{-s\hat{\psi}_\mathbb{R}}\left(\frac{\sigma_0^m}{||\sigma_0^m||_2}\right) &= \sum_{j=1}^{\infty}\frac{(-s)^j}{j!}\hat{\psi}^j_\mathbb{R}(\sigma_0^m) \\ &= \sum_{j=1}^{\infty}\frac{(-s)^j}{j!}\psi(m)^j\sigma_0^m \\ &= e^{-s\psi(m)}\sigma_0^m \end{aligned}$$

Therefore

$$A_{g,s}^{\psi}\left(\frac{\sigma_0^m}{||\sigma_0^m||_2}\right) = \left(e^{s\hat{\psi}} \otimes e^{is\mathcal{L}_\psi}\right) \circ e^{-s\hat{\psi}_{\mathbb{R}}}\left(\frac{\sigma_0^m}{||\sigma_0^m||_2}\right)$$
$$= \frac{e^{-s\psi(m)}\left(e^{s\hat{\psi}} \otimes e^{is\mathcal{L}_\psi}\right)(\sigma_0^m)}{||\sigma_0^m||_2}$$
$$= \frac{e^{-s\psi(m)}\sigma_s^m}{||\sigma_0^m||_2}. \tag{10.7}$$

Which, using proposition 10.4.6 and the theorem 10.4.7, the result follows. ■

Chapter 11

New Toric Polarizations on $\mathbb{CP}^1$

As we have seen in the last chapter, the choice of a strictly convex function ψ in the moment polytope allows for the degeneration of the Kähler polarizations into the vertical polarization. Moreover, under this degeneration, it was shown that holormorphic sections converge to the Dirac delta distributional sections, with support on the fibers corresponding to the integral points of the moment polytope. This was seen using three different techniques: L^1-normalized sections as in [1]; L^2-normalized and half-form corrected sections as in [2]; and using the "Hamiltonian flow" with complex time is, as in [3]. Taking the special case of $S^2 \cong \mathbb{CP}^1$, this translates into the collapse of the sphere into a infinite rod from the metric point of view.

In this chapter, we are going to generalize these results for the special case of S^2. In particular, we are going to study what happens to theorem 10.4.2 when we consider a special class of function on the moment polytope of S^2. Here we are considering the sphere with the momentum map given by the height function, whose moment polytope corresponds to a interval in $\mathbb{R}$. The class of functions that we are interested in is the class of functions on the moment polytope whose second derivative is a bump function. As such, our main goal is to study the consequences of this choice and to reformulate theorem 10.4.1, theorem 10.4.2, and theorem 10.5.1 for these types of functions, which had not been considered previously.

11.1 L^1-normalized sections

Let $P = [-\frac{1}{2}, N + \frac{1}{2}]$ [1], where $N \in \mathbb{N}$ be moment polytope of S^2. For the rest of this section, we will fix a $m \in P$ and consider ψ to be a function on P such that its second derivative is a bump function with support $supp\, \psi'' = [m - \alpha, m + \alpha]$, like shown below:

[1] For simplicity we will use this polytope P throughout chapter 11. In fact, for L^1-normalized sections, without half-forms, $P = [0, N]$ would be more appropriate. This simplification does not change the final results and conclusions.

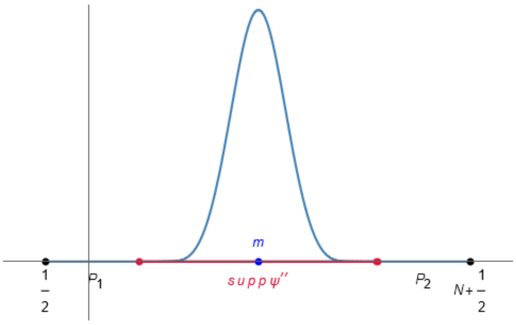

Figure 11.1: ψ''

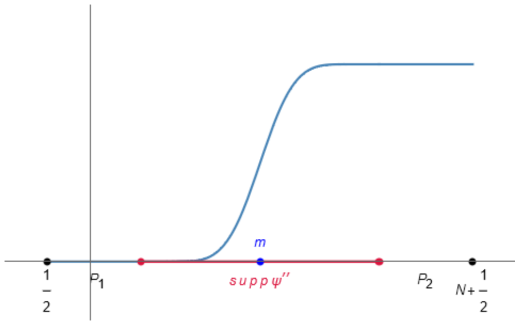

Figure 11.2: ψ'

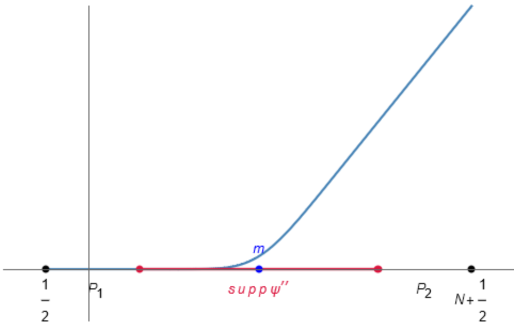

Figure 11.3: ψ

Where $P_1 = \left[-\frac{1}{2}, m-\alpha\right]$, and $P_2 = \left[m+\alpha, N+\frac{1}{2}\right]$. We also assume, without loss of generality, that

$$\int_{supp\,\psi''} \psi'' dx = 1.$$

This is so that $\psi'(m+\alpha) = 1$. We shall first see what happens to the polarizations, along the family of this polarizations given by $g_p + s\psi,\ s \geq 0$. Recall the definition of $\mathcal{P}_s$ in 10.4, of $\mathcal{P}_\infty$ in 10.5 and of $\mathcal{P}_\mathbb{R}$ in 10.6.

Lemma 11.1.1. *On $\check{X}_{\text{supp}\,\psi''} := \mu^{-1}(\text{supp}\,\psi''^{\circ})$, $\mathcal{P}_\infty = \mathcal{P}_\mathbb{R}$. On the remaining part of the open orbit, the polarization remains unchanged, that is equal to $\mathcal{P}_0$.*

Proof. The proof of this lemma follows exactly as the proof of proposition 10.4.1 ■

We now have two cases: either supp ψ'' does not contain the boundary of P; or it does contain it. In either of these cases we obtain the following result:

Proposition 11.1.1. *1. If supp $\psi'' \cap \partial P = \emptyset$, then on $\mu^{-1}(\text{supp}\,\psi'')$ with have $\mathcal{P}_\infty = \mathcal{P}_\mathbb{R}$ and on $\mu^{-1}(P \backslash \text{supp}\,\psi'')$, $\mathcal{P}_\infty = \mathcal{P}_0$.*

2. If supp $\psi'' \cap \partial P \neq \emptyset$, then on $\mu^{-1}(\text{supp}\,\psi'')$

$$\mathcal{P}_\infty = \mathcal{P}_\mathbb{R} \oplus \text{span}_\mathbb{C}\left\{\frac{\partial}{\partial w_j};\ \omega_j = 0\right\},$$

and on $\mu^{-1}(P\backslash \textit{supp}\,\psi'')$, $\mathcal{P}_\infty = \mathcal{P}_0$.

Proof. The first case follows immediately from lemma 11.1.1. For the second case, we have to show:

$$\text{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial w_k^s}\right\} \to \text{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial \theta_k}\right\}$$

on any occasion that $w_k \neq 0$. Let F be any face in the coordinate chart. We will write that $j \in F$ whenever $w_j = 0$ in F. Then it follows that

$$\begin{aligned}
\text{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial w_k^s}\right\} &= \text{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial w_j};\ j \in F\right\} \oplus \text{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial w_j^s};\ j \notin F\right\} \\
&= \text{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial w_j};\ j \in F\right\} \oplus \text{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial y_j^s} - i\frac{\partial}{\partial \theta_j};\ j \notin F\right\} \\
&= \text{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial w_j};\ j \in F\right\} \oplus \text{span}_{\mathbb{C}}\left\{(\text{Hess}g_s)^{-1}\frac{\partial}{\partial l_j} - i\frac{\partial}{\partial \theta_j};\ j \notin F\right\}
\end{aligned}$$

Which yields the desired result as $s \to \infty$. ■

As a consequence, we have that

Theorem 11.1.2. *On* $\mu^{-1}(\textit{supp}\,\psi'')$:

$$C^\infty(\mathcal{P}_\infty) = C^\infty(\mathcal{P}_{\mathbb{R}}).$$

Proof. We have two cases, either if supp $\psi'' \cap \partial P = \emptyset$ or if supp $\psi'' \cap \partial P \neq \emptyset$. The first case follows from 11.1.1. For the other case, using proposition 11.1.1, we have to show that:

$$C^\infty\left(\mathcal{P}^\infty = \mathcal{P}_{\mathbb{R}} \oplus \text{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial w_j};\ \omega_j = 0\right\}\right) = C^\infty(\mathcal{P}_{\mathbb{R}}).$$

But this follows from the following observation: any complexified vector field ξ such that when restricts to a section of $\mathcal{P}_{\mathbb{R}}$ on an open dense subset must be such that $\overline{\xi} = \xi$, but this implies that it cannot have components along the holomorphic direction, i.e, $\text{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial w_j};\ \omega_j = 0\right\}$. ■

Lemma 11.1.3. *Let* ψ *be as before. Then for* $x \geq m + \alpha$, $\psi(x) = x - m$.

Proof. Let

$$c := \int_{m-\alpha}^{m+\alpha} \psi'(x)dx = \psi(m+\alpha).$$

This result follows from the fact that

$$((x-m)\psi')' = (x-m)\psi'' + \psi',$$

and so

$$\begin{aligned}
\int_{m-\alpha}^{m+\alpha} \left((x-m)\psi'(x)\right)' dx = \int_{m-\alpha}^{m+\alpha} [(x-m)\psi''(x) + \psi'(x)]dx & \iff \\
\alpha = c + \int_{m-\alpha}^{m+\alpha} (x-m)\psi''(x)dx & \iff \\
\alpha = c + \int_{-\alpha}^{\alpha} y\psi''(y+m)dy & \iff \\
\alpha = c & \iff \\
\alpha = \int_{m-\alpha}^{m+\alpha} \psi'(x)dx & \iff \\
\alpha = \psi(m+\alpha). &
\end{aligned}$$

The fourth equivalence sign comes from the fact that the integrand function is odd.
As such, for all $x \geq m+\alpha,\ \psi(x) = x - m$. ■

Consider now the following function

$$f_n(x) = (x-n)\frac{\partial \psi}{\partial x} - \psi(x).$$

In the following lemmas, we will study the behavior of the wave functions, as we take $s \to \infty$

Lemma 11.1.4. *For $n \in$ supp ψ'', f_n has a global minimum at $x = n$, and in the sense of distributions:*

$$\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} \delta(x-n).$$

Proof. Notice that

$$f_n'(x) = (x-n)\psi''(x).$$

Thus it follows that

$$\begin{aligned}
f_n(x) &= f_n(n) + \int_0^1 \frac{d}{dt} f_n(n + t(x-n))dt \\
&= -\psi(n) + \int_0^1 t(x-n)^2\psi''(n+t(x-n))dt \\
&> -\psi(n),
\end{aligned}$$

where the last observation comes from the fact that the integral is always greater than zero. So $x = n$ is a global minimum. Notice also that whenever $m - \alpha < x < n,\ f_n$ is decreasing and whenever $m+\alpha > x > n$ f_n is increasing. In a neighborhood of n, $f_n(x) < 0$. Take $\tilde{\varepsilon} > 0$, then

$$\begin{aligned}||e^{-sf_n}||_1 &= \int_P e^{-sf_n(x)}dx \\ &\geq \int_{B_{\tilde{\varepsilon}}(n)} e^{-sf_n(x)}dx \\ &\geq \int_{B_{\tilde{\varepsilon}}(n)} e^{-s(-\psi(n)+\frac{x^2}{2}\psi''(n))}dx \\ &\geq \text{Vol}(B_{\tilde{\varepsilon}}(n))e^{s\psi(n)-s\frac{\tilde{\varepsilon}^2}{2}\psi''(n)}.\end{aligned}$$

Consider now the following cases:

- $x \notin \text{supp}\psi''$. Notice that f_n is zero in P_1. Moreover, in P_2, $f_n(x) = m-n$. As we have seen, $-\psi(n)$ is the global minimum of f_n, so we conclude that $m-n+\psi(n) > 0$. Thus, choosing $\tilde{\varepsilon}$ such that $\psi(n) > \frac{\tilde{\varepsilon}^2}{2}\psi''(n)$, we have

$$\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} 0,\ \forall x \notin \text{supp}\,\psi''$$

- $x \in \text{supp}\psi''$. Consider now, $0 < \varepsilon < \alpha$. Thus:

$$\begin{aligned}\int_{P\setminus B_\varepsilon(n)} e^{-sf_n(x)}dx &= \int_{[-1/2,m-\alpha]} e^{-sf_n(x)}dx + \int_{[m-\alpha,n-\varepsilon]} e^{-sf_n(x)}dx + \\ &+ \int_{[n+\varepsilon,m+\alpha]} e^{-sf_n(x)}dx + \int_{[m+\alpha,N+1/2]} e^{-sf_n(x)}dx \\ &\leq (m-\alpha+\frac{1}{2}) + (n-\varepsilon-m+\alpha)e^{-sf_n(n-\varepsilon)} + \\ &+ (m+\alpha-n-\varepsilon)e^{-sf_n(n+\varepsilon)} + (N+\frac{1}{2}-m-\alpha)e^{-s(m-n)} \\ &\leq (m-\alpha+\frac{1}{2}) + (n-\varepsilon-m+\alpha)e^{s\psi(n)-s\frac{\varepsilon^2}{2}\psi''(n-\varepsilon)} + \\ &+ (m+\alpha-n-\varepsilon)e^{s\psi(n)-s\frac{\varepsilon^2}{2}\psi''(n+\varepsilon)} + (N+\frac{1}{2}-m-\alpha)e^{-s(m-n)}.\end{aligned}$$

  Thus, choosing $\tilde{\varepsilon}$ such that $\psi(n)+m-n > \frac{\tilde{\varepsilon}^2}{2}\psi''(n)$ and $\frac{\varepsilon^2}{2}\psi''(n\pm\varepsilon) \geq \frac{\tilde{\varepsilon}^2}{2}\psi''(n)$, we conclude that

$$\int_{P\setminus B_\varepsilon(n)} \frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} 0,$$

which proves our claim.

■

Lemma 11.1.5. *For $n < m$ and $n \notin$ supp ψ'',*

$$\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} \frac{1}{m-\alpha+\frac{1}{2}}\chi_{P_1}.$$

Proof. For $n < m$ and $n \notin \operatorname{supp} \psi''$, it then follows that f_n' is positive in $\operatorname{supp} \psi''$ and it is zero in the remaining parts of the polytope. Also $f_n(x) = 0$ on P_1 and $f_n(m+\alpha) = m - n > 0$, thus f_n is a non-negative bounded function.

Notice now that by the dominated convergence theorem, we have that

$$\begin{aligned}
||e^{-sf_n}||_1 &= \int_P e^{-sf_n(x)}dx \\
&= \int_{[-1/2,m-\alpha]} e^{-sf_n(x)}dx + \int_{[m-\alpha,m+\alpha]} e^{-sf_n(x)}dx + \int_{[m+\alpha,N+1/2]} e^{-sf_n(x)}dx \\
&= m - \alpha + \frac{1}{2} + \int_{[m-\alpha,m+\alpha]} e^{-sf_n(x)}dx + (N + \frac{1}{2} - m - \alpha)e^{-s(m-n)} \\
&\xrightarrow{s\to\infty} m - \alpha + \frac{1}{2}
\end{aligned}$$

Which immediately implies that

$$\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} \frac{1}{m - \alpha + \frac{1}{2}}\chi_{P_1}.$$

■

Lemma 11.1.6. *For $n > m$ and $n \notin \operatorname{supp} \psi''$,*

$$\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} \frac{1}{N + 1/2 - m - \alpha}\chi_{P_2}.$$

Proof. For $n > m$ and $n \notin \operatorname{supp} \psi''$, we now have that f_n' is negative in $\operatorname{supp} \psi''$ and zero in the remaining parts of the polytope. Also $f_n|_{P_1} = 0$ and $f_n(m+\alpha) = m - n < 0$. We then have that

$$||e^{-sf_n}||_1 \geq \int_{P_2} e^{-sf_n(x)}dx = (N + \frac{1}{2} - m - \alpha)e^{-s(m-n)},$$

which in turn implies that

$$\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} 0, \ \forall x \in P_1 \cup \text{supp}\psi''.$$

Moreover, notice that

$$\int_P \frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1}dx = 1,$$

and for any $\varepsilon > 0$

$$\begin{aligned}
\int_{[-1/2,m+\alpha-\varepsilon)} \frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1}dx &\leq (m + \alpha - \varepsilon + \frac{1}{2})(N + \frac{1}{2} - m - \alpha)^{-1}e^{s(-f_n(m+\alpha-\varepsilon)+(m-n))} \\
&\xrightarrow{s\to\infty} 0
\end{aligned}$$

Which yields the desired result. ■

The following theorem is simply the combination of the three previous lemmas.

Theorem 11.1.7. *Let ψ be as above. Consider also the function*

$$f_n(x) = (x-n)\frac{\partial \psi}{\partial x} - \psi(x),$$

where $n \in P \cap \mathbb{Z}$. Then

1. *For $n \in$ supp ψ'', f_n has a global minimum at $x = n$, and in the sense of distributions:*

$$\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} \delta(x-n).$$

2. *For $n < m$ and $n \notin$ supp ψ'',*

$$\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} \frac{1}{m-\alpha+\frac{1}{2}}\chi_{P_1}.$$

3. *For $n > m$ and $n \notin$ supp ψ'',*

$$\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} \frac{1}{N+1/2-m-\alpha}\chi_{P_2}.$$

The following table condenses some facts regarding the function $\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1}$, which are going to be useful later on.

	$f_n(x)$	$e^{-sf_n(x)}$	$\frac{e^{-sf_n(x)}}{\|\|e^{-sf_n}\|\|_1}$
for $n < m, n \notin$ suppψ''	is positive	is decreasing	is bounded by $\frac{1}{m-\alpha+1/2}$
for $n > m, n \notin$ suppψ''	is negative	is increasing	is bounded by $\frac{1}{N+1/2-m-\alpha}$

Table 11.1: Some facts about the functions of theorem 11.1.7.

Theorem 11.1.8. *For $n \in P \cap \mathbb{Z}$, consider the family of L^1-normalized J_s-holomorphic sections*

$$\mathbb{R}^+ \ni s \mapsto \xi_s^n := \frac{\sigma_s^n}{||\sigma_s^n||_1} \in C^\infty(L_\omega) \xhookrightarrow{\iota} (C_c^\infty(L_\omega^{-1}|_U))^*.$$

Then,

1. *For $n \in$ supp ψ'', as $s \to \infty$, $\iota(\xi_s^n)$ converges to δ^n in $(C_c^\infty(L_\omega^{-1}|_U))^*$.*

2. *For $n \in P_j$ as $s \to \infty$, $\iota(\xi_s^n)$ converges to $\frac{1}{||\sigma_0^n||_{L^1(P_j)}} \int_{P_j} \sigma_0^n \hat{\tau}(\cdot, -n)dx$ in $(C_c^\infty(L_\omega^{-1}|_U))^*$.*

Proof. The proof of this result follows exactly the same as in 1.3. In short, we can consider a partition of unity $\{\rho_v\}$ subordinated to the covering by vertex charts $\{\check{\mathrm{P}}_v\}$, and therefore we only have to check the result in each chart. Thus choosing a test section $\tau \in C^\infty(L_\omega^{-1})$, we may define

$$h_n^s(x) = (x-n)^t\psi' - g_s = h_n^0(x) - sf_n(x).$$

Hence, following the computations in 1.3

$$(\iota(\xi_s^n))(\tau) = \frac{1}{||\sigma_s^n||_1}\int_P e^{-h_n^s(x)}\hat{\tau}(x,-n)dx.$$

Also

$$||\sigma_s^n||_1 = \int_{M^\circ} e^{-h_n^s\circ\mu_P}\omega^n = (2\pi)^n\int_P e^{-h_n^s}dx.$$

Then we see

$$\frac{||e^{-h_n^0 - sf_n}||_1}{||e^{-sf_n}||_1} = \int_P \frac{e^{-sf_n}}{||e^{-sf_n}||_1}e^{-h_n^0}dx.$$

Now, using theorem 11.1.7 we obtain

- For $n \in \operatorname{supp}\psi''$,

$$\frac{||e^{-h_n^0 - sf_n}||_1}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} e^{-h_n^0(n)};$$

- For $n < m$ and $n \notin \operatorname{supp}\psi''$, using the dominated convergence theorem, we have

$$\frac{||e^{-h_n^0 - sf_n}||_1}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} \frac{1}{m-\alpha+1/2}\int_{P_1} e^{-h_n^0(x)}dx;$$

- For $n > m$ and $n \notin \operatorname{supp}\psi''$, using the dominated convergence theorem, we have

$$\frac{||e^{-h_n^0 - sf_n}||_1}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} \frac{1}{N+1/2-m-\alpha}\int_{P_2} e^{-h_n^0(x)}dx.$$

Which then implies that:

- for $n \in \operatorname{supp}\psi''$,

$$\begin{aligned}
\iota(\xi_s^n)(\tau) &= \int_P \frac{e^{-h_n^0 - sf_n}}{||e^{-h_n^0 - sf_n}||_1}\hat{\tau}(\cdot,-n)dx \\
&= \int_P \frac{e^{-h_n^0 - sf_n}}{||e^{-h_n^0 - sf_n}||_1}\frac{||e^{-sf_n}||_1}{||e^{-sf_n}||_1}\hat{\tau}(\cdot,-n)dx \\
&\xrightarrow{s\to\infty} \int_P e^{-h_n^0}e^{h_n^0}\delta^n\hat{\tau}(\cdot,-n)dx \\
&= \delta^n(\tau);
\end{aligned}$$

- For $n < m$ and $n \notin \operatorname{supp}\psi''$, using the dominated convergence theorem:

$$\begin{aligned}
\iota(\xi_s^n)(\tau) &= \int_P \frac{e^{-h_n^0+sf_n}}{||e^{-h_n^0+sf_n}||_1}\hat{\tau}(\cdot,-n)dx \\
&= \int_P \frac{e^{-sf_n}}{||e^{-sf_n}||_1}\frac{||e^{-sf_n}||_1}{||e^{-h_n^0+sf_n}||_1}e^{-h_n^0}\hat{\tau}(\cdot,-n)dx \\
&\xrightarrow{s\to\infty} \int_P \frac{1}{m-\alpha+1/2}\chi_{P_1}\frac{m-\alpha+1/2}{||\sigma_0^n||_1}e^{-h_n^0}\hat{\tau}(\cdot,-n)dx \\
&= \frac{1}{||\sigma_0^n||_{L^1(P_1)}}\int_{P_1}\sigma_0^n\hat{\tau}(\cdot,-n)dx
\end{aligned}$$

- For $n > m$ and $n \notin \operatorname{supp}\psi''$, using the dominated convergence theorem:

$$\begin{aligned}
\iota(\xi_s^n)(\tau) &= \int_P \frac{e^{-h_n^0+sf_n}}{||e^{-h_n^0+sf_n}||_1}\hat{\tau}(\cdot,-n)dx \\
&= \int_P \frac{e^{-sf_n}}{||e^{-sf_n}||_1}\frac{||e^{-sf_n}||_1}{||e^{-h_n^0+sf_n}||_1}e^{-h_n^0}\hat{\tau}(\cdot,-n)dx \\
&\xrightarrow{s\to\infty} \int_P \frac{1}{N+1/2-m-\alpha}\chi_{P_2}\frac{N+1/2-m-\alpha}{||\sigma_0^n||_1}e^{-h_n^0}\hat{\tau}(\cdot,-n)dx \\
&= \frac{1}{||\sigma_0^n||_{L^1(P_2)}}\int_{P_2}\sigma_0^n\hat{\tau}(\cdot,-n)dx
\end{aligned}$$

■

Remark 11.1.1. *These Theorems shows that these degenerations allow us to "split" the phase space.*

For instance, suppose that there are no integral points on the support of ψ. In this case, then the sections σ^n converge to their normalized-restriction on the corresponding part of the polytope, i.e. P_1 if $n < m$ or P_2 if $n > m$. Therefore, the polytope is broken up into three pieces

$$P = P_1 \cup P_2 \cup \mathit{supp}\psi'',$$

where suppψ'' does not support any sections. Moreover, as these sections only have support on their respective P_i, this result says that the quantization on the whole S^2, corresponds to a sum of the contributions from each part. We then observe that the Hilbert space of holomorphic quantization breaks into two parts corresponding to regions in the phase space separated by ∞ Riemanninan distance, generated by the corresponding sections.

If the support does contain at least one integral point, we now have three pieces which support sections. In particular, the the sections supported in supp ψ'' converge to distributional sections. And yet again, doing quantization in each pieces, or the sphere, will lead to the same results. In this case, we see that the phase space breaks into three pieces, with the new extra piece corresponding to the ∞ segment "connecting" the two parts from above.

This result is quite interesting as, in general, there is no way of "decomposing" a phase space into subsets, in some geometrically natural way, in such a way that the quantization of the symplectic manifold also "decomposes" as a sum of the quantizations of those subsets.

Remark 11.1.2. *Notice that all the above results are still valid on the plane, where the moment polytope is of the form* $\left[\frac{-1}{2}, +\infty\right]$*.*

Remark 11.1.3. *One may generalize straightforwardly for larger dimension, for instance for* $\mathbb{P}^2$*, where one can divide the polytope by codimension 1 "walls".*

11.2 More bump functions

Let us now further generalize the results of the previous section and suppose that the second derivative of ψ now is given by two bump functions with disjoint supports, say $supp_1 = [m_1 - \alpha, m_1 + \alpha]$ and $supp_2 = [m_2 - \beta, m_2 + \beta]$, where $m_1 < m_2$. Furthermore, we assume that each of these bump functions has area equal to 1. We will see that the same behaviour as described above will prevail. Indeed, one can clearly see this simply by observing the following:

Let $P_1 = [-\frac{1}{2}, m_1 - \alpha]$, $P_2 = [m_1 + \alpha, m_2 - \beta]$ and $P_3 = [m_2 + \beta, N + \frac{1}{2}]$. Then the addition of the second bump function will only affect the expression of ψ and ψ' on $supp_2 \cup P_3$. In fact, on P_3, ψ' will be 2 rather than 1, which now is the value it takes on P_2. For ψ, the following can be said: on P_1, $\psi = 0$, on P_2, $\psi(x) = x - m_1$. Now on P_3:

$$
\begin{aligned}
\int_{m_2-\beta}^{m_2+\beta} ((x-m_2)\psi'(x))'\,dx &= \int_{m_2-\beta}^{m_2+\beta} (x-m_2)\psi''(x) + \psi'(x)dx && \iff \\
\beta\psi'(m_2+\beta) + \beta\psi'(m_2-\beta) &= \int_{m_2-\beta}^{m_2+\beta} \psi'(x)dx + \int_{m-\beta}^{m_2+\beta} (x-m_2)\psi''(x)dx && \iff \\
3\beta &= \int_{m_2-\beta}^{m_2+\beta} \psi'(x)dx + \int_{-\beta}^{\beta} y\psi''(y+m_2)dy && \iff \\
3\beta &= \psi(m_2+\beta) - \psi(m_2-\beta) && \iff \\
3\beta &= \psi(m_2+\beta) - (m_2-\beta-m_1) && \iff \\
2\beta + m_2 - m_1 &= \psi(m_2+\beta).
\end{aligned}
$$

As such, for all $x \in P_3,\ \psi(x) = 2x - m_2 - m_1$.

In fact, we can clearly see that this result can be further generalized if we had N bump functions with disjoint supports, each with area 1. Let supp$_j := [m_j - \alpha_j, m_j + \alpha_j],\ m_j \in P,\ \alpha_j > 0,$ $j = 1, ..., N$ be the supports of the N bump functions. Then, following the same naming convention as before, let

$$P_1 = [-1/2, m_1 - \alpha_1],\ P_{N+1} = [m_N + \alpha_N, N + 1/2],\ P_j = [m_{j-1} + \alpha_{j-1}, m_j - \alpha_j], j = 2, ..., N.$$

then

$$\psi(x) = jx - \sum_{k=1}^{j} m_k, \ \forall x \in P_j.$$

As such, the inclusion of another bump function only changes the behaviour of f_n on the support of that bump function and on the P_j immediately before it and the P_j after it. This leads us to the following generalization of the theorem 11.1.7:

Theorem 11.2.1. *Let ψ be as describe above. Consider also the function*

$$f_n(x) = (x-n)\frac{\partial \psi}{\partial x} - \psi(x),$$

where $n \in P \cap \mathbb{Z}$. Then

1. *For $n \in$ suppψ'', f_n has a global minimum at $x = n$, and in the sense of distributions:*

$$\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} \delta(x-n).$$

2. *For $n \in P_j$*

$$\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} \frac{1}{\textit{Vol}(P_j)}\chi_{P_j},$$

Proof. We will prove this theorem will be done by induction on the number of bump functions K.

For $K = 1$, this case is the same as theorem 11.1.7.

For $K \implies K+1$. We assume without loss of generality that $m_{K+1} > m_K > ... > m_1$. Thus by the induction hypothesis, we only have to take care whenever $n \in P_{K+1} \cup \text{supp}_{K+1} \cup P_{K+2}$. As usual, we will divide the proof in cases:

1. $n \in \text{supp}_{K+1}$

   (a) $n = m_{K+1}$. In this case it follows that $f_n' = 0$ on $P_1 \cup ... \cup P_{K+1} \cup \{m_{K+1}\} \cup P_{K+2}$, and $f_n' < 0$ on $\text{supp}_1 \cup ... \cup \text{supp}_K \cup (m_{K+1} - \alpha_{K+1}, m_{K+1})$ and $f_n' > 0$ on $(m_{K+1}, m_{K+1} + \alpha_{K+1})$. Thus, using the exact same argument as before,

$$\begin{aligned} f_{m_{K+1}}(x) &= -\psi(m_{K+1}) + \int_0^1 t(x-m_{K+1})^2\psi''(m_{K+1} + t(x-m_{K+1}))dt \\ &> -\psi(m_{K+1}). \end{aligned}$$

   So $x = m_{K+1}$ is a global minimum of $f_{m_{K+1}}$.

Moreover,

$$f_{m_{K+1}}(m_{K+1}+\alpha_{K+1}) = (K+1)\alpha_{K+1} - ((K+1)(m_{K+1}+\alpha_{K+1}) - m_K - m_{K-1}$$
$$= m_K + m_{K-1} - (K+1)m_{K+1} < 0.$$

Following the proof of theorem 11.1.7, we take $\tilde{\varepsilon} > 0$, then

$$||e^{-sf_{m_{K+1}}}||_1 \geq \int_{B_{\tilde{\varepsilon}}(m_{K+1})} e^{-sf_{m_{K+1}}(x)}dx \geq \text{Vol}(B_{\tilde{\varepsilon}}(m_{K+1}))e^{s\psi(m_{K+1})-s\frac{\tilde{\varepsilon}^2}{2}\psi''(m_{K+1})}.$$

Thus, choosing $\tilde{\varepsilon}$ such that $\psi(m_{K+1}) > \frac{\tilde{\varepsilon}^2}{2}\psi''(m_{K+1})$, we have that for $x \notin \text{supp}_{K+1}$:

$$\frac{e^{-sf_m(x)}}{||e^{-sf_m}||_1} \xrightarrow{s\to\infty} 0$$

Consider now, $0 < \varepsilon < \alpha$. Thus:

$$\int_{P\setminus B_\varepsilon(m_{K+1})} e^{-sf_{m_{K+1}}(x)}dx = \sum_{j=1}^{K+2}\int_{P_j} e^{-sf_{m_{K+1}}(x)}dx + \sum_{j=1}^{K}\int_{\text{supp}_j} e^{-sf_{m_{K+1}}(x)}dx +$$
$$+ \int_{[m_{K+1}-\alpha_{K+1}, m_{K+1}-\varepsilon]} e^{-sf_{m_{K+1}}(x)}dx + \int_{[m_{K+1}+\varepsilon, m_{K+1}+\alpha_{K+1}]} e^{-sf_{m_{K+1}}(x)}dx$$
$$\leq \text{Vol}(P_1) + \sum_{j=2}^{K+1}\text{Vol}(P_j)e^{-sf_{m_{K+1}}(m_j-\alpha_j)} + \sum_{j=1}^{K}\alpha_j e^{-sf_{m_{K+1}}(m_j+\alpha_j)} +$$
$$+ 2(\alpha_{K+1}-\varepsilon)e^{-sf_{m_{K+1}}(m_{K+1}\pm\varepsilon)} + \text{Vol}(P_{K+2})e^{-sf_{m_{K+1}}(m_{K+1}+\alpha_{K+1})}$$

Thus, choosing $\tilde{\varepsilon}$ such that $\psi(m_{K+1}) > \frac{\tilde{\varepsilon}^2}{2}\psi''(m_{K+1})$ and $\frac{\varepsilon^2}{2}\psi''(m_{K+1}\pm\varepsilon) \geq \frac{\tilde{\varepsilon}^2}{2}\psi''(m_{K+1})$, we obtain

$$\int_{P\setminus B_\varepsilon(m_{K+1})} \frac{e^{-sf_{m_{K+1}}(x)}}{||e^{-sf_{m_{K+1}}}||_1}dx \xrightarrow{s\to\infty} 0,$$

which proves our first claim.

(b) $n \neq m_{K+1}$ This follows easily using the same argument as above. In this case it follows that $f_n' = 0$ on $P_1 \cup ... \cup P_{K+1} \cup \{n\} \cup P_{K+2}$, and $f_n' < 0$ on $\text{supp}_1 \cup ... \cup \text{supp}_K \cup (m_{K+1} - \alpha_{K+1}, n)$ and $f_n' > 0$ on $(n, m_{K+1}+\alpha_{K+1})$. Thus, using the exact same argument as before,

$$f_n(x) = -\psi(n) + \int_0^1 t(x-n)^2\psi''(n+t(x-n))dt$$
$$> -\psi(n).$$

So $x = n$ is a global minimum of f_n.It also follows that in neighborhood of n, $f_n(x) =$

$-\psi(n) < 0$. Now, as before, we consider $\tilde{\varepsilon} > 0$:

$$||e^{-sf_n}||_1 \geq \text{Vol}(B_{\tilde{\varepsilon}}(n))e^{s\psi(n)-s\frac{\tilde{\varepsilon}^2}{2}\psi''(n)}.$$

Thus, choosing $\tilde{\varepsilon}$ such that $\psi(n) > \frac{\tilde{\varepsilon}^2}{2}\psi''(n)$, we have that for $x \notin \text{supp}\,\psi''$:

$$\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} 0.$$

Let now $0 < \varepsilon < \alpha$. Thus:

$$\int_{P\setminus B_\varepsilon(n)} e^{-sf_n(x)}dx \leq \text{Vol}(P_1) + \sum_{j=2}^{K+1} \text{Vol}(P_j)e^{-sf_{m_{K+1}}(m_j-\alpha_j)} + \sum_{j=1}^{K} \alpha_j e^{-sf_{m_{K+1}}(m_j+\alpha_j)} +$$
$$+ (n - \varepsilon - m_{K+1} + \alpha_{K+1})e^{-sf_{m_{K+1}}(m_{K+1}-\varepsilon)} +$$
$$+ (m_{K+1} + \alpha_{K+1} - n - \varepsilon)e^{-sf_{m_{K+1}}(m_{K+1}+\varepsilon)} + \text{Vol}(P_{K+2})e^{-sf_{m_{K+1}}(m_{K+1}+\alpha_{K+1})}$$

So we choose $\tilde{\varepsilon}$ such that $\psi(n) > \frac{\tilde{\varepsilon}^2}{2}\psi''(n)$ and $\frac{\varepsilon^2}{2}\psi''(n \pm \varepsilon) \geq \frac{\tilde{\varepsilon}^2}{2}\psi''(n)$. We now just have to look at $\psi(m_{K+1} + \alpha_{K+1}) + \psi(n)$, which is positive regardless if $n < m_{K+1}$ or $n > m_{K+1}$. So it follows that

$$\int_{P\setminus B_\varepsilon(m_{K+1})} \frac{e^{-sf_{m_{K+1}}(x)}}{||e^{-sf_{m_{K+1}}}||_1}dx \xrightarrow{s\to\infty} 0,$$

which proves our claim.

2. $P_{K+1} \cup P_{K+2}$

(a) $n \in P_{K+1}$ we have that f'_n is positive in supp_{K+1}, negative on the others $\text{supp}'_k s$ and it is zero in the remaining parts of the polytope. In particular, it follows that f_n is constant on P_{K+1}, where its value is $-\psi(n)$, which is the minimum of the function.

We then have that

$$||e^{-sf_n}||_1 \geq \int_{P_{K+1}} e^{-sf_n(x)}dx = \text{Vol}(P_{K+1})e^{s\psi(n)},$$

which in turn implies that

$$\frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1} \xrightarrow{s\to\infty} 0,\ \forall x \notin P_{K+1}.$$

Moreover, notice that

$$\int_P \frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1}dx = 1,$$

and

$$\int_{P\setminus P_{K+1}} \frac{e^{-sf_n(x)}}{||e^{-sf_n}||_1}dx \leq \sum_{j=1}^{K} \frac{\mathsf{Vol}(P_j)}{\mathsf{Vol}(P_{K+1})} e^{-s(f_n(m_j-\alpha_j)+\psi(n))} + \frac{\mathsf{Vol}(P_{K+2})}{\mathsf{Vol}(P_{K+1})} e^{-s(f_n(N+1/2)+\psi(n))} +$$
$$+ \sum_{j=1}^{K+1} \frac{\mathsf{Vol}(\mathsf{supp}_j)}{\mathsf{Vol}(P_{K+1})} e^{-s(f_n(m_j+\alpha_j)+\psi(n))}$$
$$\xrightarrow{s\to\infty} 0$$

Which yields the desired result.

(b) $n \in P_{K+2}$. This case follows exactly as the above, but instead of P_{K+1} we exchange that for P_{K+2}.

■

Notice furthermore, that the theorems 11.1.1, 11.1.2 and 11.1.8 , are still valid for these types of functions, and in particular, their proofs are essential the same.

Remark 11.2.1. *Remarks 11.1.1, 11.1.2, and 11.1.3 are still valid when we consider more bump functions, with the appropriate adaptations.*

11.3 Complex time Hamiltonian flow approach for half-form corrected sections

In the previous two sections, we have studied the effect of the imaginary time flow generated by ψ on the holomorphic L^1-normalized sections. We will now deduce the same results following the approach given in section 10.5. In order to accomplish this, we notice that we have already seen that the polarization converges. Moreover, proposition 10.5.1 is easily seen to be valid in this case. As such, all that is left to do is to prove the analogue of Theorem 10.5.1.

Theorem 11.3.1. *Recall that the operator $A^{\psi}_{g,s} : \mathbb{H}_{\mathcal{P}_g} \to \mathbb{H}_{\mathcal{P}_s}$: is defined by*

$$A^{\psi}_{g,s} := \left(e^{s\hat{\psi}} \otimes e^{is\mathcal{L}_{\psi}}\right) \circ e^{-s\hat{\psi}_{\mathbb{R}}}.$$

Then the operator $A^{\psi}_{g,\infty} : \mathcal{H}_{\mathcal{P}_g} \to \mathcal{H}_{\mathcal{P}_{\mathbb{R}}}$ is determined by:

1.

$$A^{\psi}_{g,\infty}\left(\frac{\sigma_0^n}{||\sigma_0^n||_2}\right) := \frac{(2\pi)^{\frac{n}{2}} e^{g(n)}}{||\sigma_0^n||_2} \delta^n \otimes \sqrt{dX},$$

if $n \in supp\, \psi''$;

2.

$$A^{\psi}_{g,\infty}\left(\frac{\sigma_0^n}{||\sigma_0^n||_2}\right) := \frac{1}{||\sigma_0^n||_2}\chi_{P_i}\sigma_0^n \otimes \sqrt{dX},$$

if $n \in P_i$,

is such that

$$\lim_{s\to\infty} A^{\psi}_{g,s} = A^{\psi}_{g,\infty}.$$

Proof. Notice that the first case follows exactly like theorem 10.5.1. Therefore we only have to worry about whenever $n \in P_i$. From 10.7 we have that

$$A^{\psi}_{g,s}\left(\frac{\sigma_0^n}{||\sigma_0^n||_2}\right) = \frac{e^{-s\psi(n)}\sigma_s^n}{||\sigma_0^n||_2}.$$

Assume now that $n \in P_1$, then $\psi(n) = 0$. It then follows that by the dominated convergence theorem,

$$\begin{aligned}\int_P e^{-sf_n(x)}e^{-((x-n)g_0'(x)-g_0(x))}dx &= \int_{P_1} e^{-((x-n)g_0'(x)-g_0(x))}dx+\\ &+\int_{P\setminus P_1} e^{-sf_n(x)}e^{-((x-n)g_0'(x)-g_0(x))}dx\\ &\xrightarrow{s\to\infty}\int_{P_1} e^{-((x-n)g_0'(x)-g_0(x))}dx\end{aligned}$$

which implies our result.

Lastly, assume now that $n \in P_2$. Then $\psi(n) = (n-m)$. For any $\varepsilon > 0$, we obtain that

$$\begin{aligned}\int_{[-1/2,m+\alpha-\varepsilon]} e^{-s(n-m)}e^{-sf_n(x)}e^{-((x-n)g_0'(x)-g_0(x))}dx &\leq Ae^{-s(n-m)}e^{-sf_n(m+\alpha-\varepsilon)}\\ &\xrightarrow{s\to\infty} 0,\end{aligned}$$

where $A = (m+\alpha-\varepsilon+\frac{1}{2})e^{-((\xi-n)g_0'(\xi)-g_0(\xi))}$, where ξ is the maximum of the function on this interval. The convergence follows from the fact that $(n-m) = -f_n(m+\alpha)$ and that f_n is decreasing non-positive function. Moreover, notice that on P_2, we have that

$$\begin{aligned}\int_{P_2} e^{-s(n-m)}e^{-sf_n(x)}e^{-((x-n)g_0'(x)-g_0(x))}dx &= \int_{P_2} e^{-s(n-m)}e^{-s(m-n)}e^{-((x-n)g_0'(x)-g_0(x))}dx\\ &= \int_{P_2} e^{-((x-n)g_0'(x)-g_0(x))}dx\end{aligned}$$

which proves our claim. ∎

Bibliography

[1] T. Baier, C. Florentino, J. Mourão, and J. Nunes. Toric kähler metrics seen from infinity, quantization and compact tropical amoebas. *Journal of Differential Geometry - J DIFFEREN GEOM*, 89:411–454, 11 2011.

[2] W. Kirwin, J. Mourão, and J. Nunes. Degeneration of kähler structures and half-form quantization of toric varieties. *Journal of Symplectic Geometry*, 11:603–643, 2013. doi: 10.4310/JSG.2013.v11.n4.a4.

[3] W. D. Kirwin, J. M. Mourão, and J. P. Nunes. Complex symplectomorphisms and pseudo-kähler islands in the quantization of toric manifolds. *Mathematische Annalen*, 364, Apr. 2015. ISSN 0025-5831. doi: 10.1007/s00208-015-1205-0.

[4] A. C. da Silva. *Lectures on Symplectic Geometry*. Springer, 2008.

[5] A. Newlander and L. Nirenberg. Complex analytic coordinates in almost complex manifolds. *Annals of Mathematics*, 65(3):391–404, 1957. ISSN 0003486X. URL `http://www.jstor.org/stable/1970051`.

[6] P. A. Griffiths and J. Harris. *Principles of algebraic geometry*. Wiley classics library. Wiley, New York, NY, 1994. URL `https://cds.cern.ch/record/433962`.

[7] A. Moroianu. *Lectures on Kähler Geometry*. London Mathematical Society Student Texts. Cambridge University Press, 2007. doi: 10.1017/CBO9780511618666.

[8] M. B. Pereira. Complexified symplectomorphisms and geometric quantization. MSc Thesis in Mathematics and Applications, Instituto Superior Técnico, Lisboa, Portugal, November 2017.

[9] J. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer, 2003. ISBN 9780387954486. URL `https://books.google.pt/books?id=eqfgZtjQceYC`.

[10] J. M. Mourão and J. P. Nunes. On complexified analytic hamiltonian flows and geodesics on the space of kähler metrics. *International Mathematics Research Notices*, 2015(20):10624–10656, 01 2015. ISSN 1073-7928. doi: 10.1093/imrn/rnv004. URL `https://doi.org/10.1093/imrn/rnv004`.

[11] M. Schottenloher. *Lecture Notes on Geometric Quantization*. Mathematisches Institut der Universitat Munchen, 2021.

[12] B. Hall. *Quantum Theory for Mathematicians*. Graduate Texts in Mathematics. Springer New York, 2013. ISBN 9781461471165. URL https://books.google.pt/books?id=bYJDAAAAQBAJ.

[13] V. W. Guillemin. Kaehler structures on toric varieties. *Journal of Differential Geometry*, 40: 285–309, 1994.

[14] M. Abreu. Kahler geometry of toric manifolds in symplectic coordinates. *Fields Inst. Commun.*, 35, 08 2000.

[15] J. Fine. *A rapid introduction to Kähler geometry*. URL https://joel.fine.web.ulb.be/preprints/Kahler_intro_ch2.pdf.

[16] A. C. da Silva. *Symplectic Toric Manifolds*. 2001. URL https://people.math.ethz.ch/~acannas/Papers/toric.pdf.

[17] D. Cox, J. Little, and H. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, United States, 2011. ISBN 978-0-8218-4819-7. doi: 10.1090/gsm/124.